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DIPLOMARBEIT

“Hardy spaces and robustness of linear systems”

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Zusammenfassung

Die Beschreibung eines linearen Modells eines technischen oder physikalischen Systems geschieht oft durch eine sogenannte Transfermatrix, die als Element eines geeigneten Hardy-Raumes betrachtet wird. Es wird auch oft angenommen, dass Eingang und Ausgang des modellierten Systems Elemente eines weiteren Hardy-Raumes sind. In dieser Arbeit wird eine detaillierte mathematische Einführung in die Theorie der Hardy-Räume gegeben. Eigenschaften der Hardy-Räume, die wichtig für die Beschreibung von technischen oder physikalischen Systemen sind, werden betont.
Eine wichtige Frage ist, wie robust ein System unter Eingangsstörungen ist. Die Frage kann beantwortet werden, wenn der Verstärkungsfaktor eines Systems bekannt ist. Dieser ist das maximale Verhältnis zwischen Ausgang und Eingang und kann deshalb nicht leicht bestimmt werden, da jeder mögliche Eingang betrachtet werden muss. Das Hauptergebnis dieser Arbeit ist, dass man unter gewissen Voraussetzungen an das Systemmodell das Verstärkungsverhältnis in einer viel einfacheren Weise berechnen kann. Dieses Ergebnis ist in der Literatur bekannt, aber nach Wissen des Autors sind keine überzeugenden Beweise in der Literatur verfügbar. Die Lücken existierender Beweise werden diskutiert und ein vollständiger Beweis wird angegeben.

Abstract

A linear model of a technical or physical system is often described by a so called transfer matrix, which is considered as an element of an appropriate Hardy space. It is also often assumed that the input and output of the modelled system are elements of another Hardy space. In this work a detailed mathematical introduction to the theory of Hardy spaces is given. Properties of Hardy spaces which are important for the description of technical or physical systems are highlighted.
An important question is how robust a system is under input disturbances. This question can be answered if the gain of the system is known. The gain of a system is the maximum ratio between output and input and is therefore not easy to determine (since every possible input must be considered). The main result of this work is, that, under certain assumption on the model of the system, one can calculate the gain in a much simpler way. This result is well known in the literature, but to the author's best knowledge a convincing proof is not available. The gaps in existing proofs are discussed and a complete proof is given.

List of symbols

$\mathbb{N}, \mathbb{R}, \mathbb{C}$	the natural, real and complex numbers, resp.
$\mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$	the non-negative and strictly positive real numbers, resp.
$\mathbb{C}_{\mathrm{Re}>0}$	the open right half plane, i.e. the set $\{\ s \in \mathbb{C}\ \mid \mathrm{Re}\, s > 0\ \}$
$i\mathbb{R}$	the imaginary axis, i.e. the set $\{\ s \in \mathbb{C}\ \mid\ s = i\omega,\ \omega \in \mathbb{R}\ \}$
$\mathbb{B}_r(s_0)$	$:= \{\ s \in \mathbb{C}\ \mid\ \lvert s - s_0\rvert < r\ \}$, an open ball with center $s_0 \in \mathbb{C}$ and radius $r > 0$
$A \cong B$	the metric spaces A and B are isometrically isomorphic
$V,\ \lVert\cdot\rVert_V$	finite dimensional Banach space with its norm
$\lVert\cdot\rVert_{V,p}$	the p-norm on V, $p \in [1, \infty]$, see Remark 6
$L_p(M, V)$	the Lebesgue spaces, see Definition 3 for $M \subseteq \mathbb{R}$ and Corollary 5 for $M = i\mathbb{R}$
$\lVert\cdot\rVert_{L_p(M)}$	the norm of the Lebesgue spaces $L_p(M, V)$, see Definition 3 or Corollary 5
$\mathcal{F}$ $(\overline{\mathcal{F}})$	the (extended) Fourier transform, see Definition 7 and Proposition 9 (Theorem 10),
$\mathcal{L}$ $(\overline{\mathcal{L}})$	the (extended) Laplace transform, see Definition 20 and Proposition 23 (Theorem 24)
$\chi_M : \mathbb{R} \to \{0, 1\}$	the characteristic function for $M \subseteq \mathbb{R}$
$\lambda(M)$	the Lebesgue measure of some measurable $M \subseteq \mathbb{R}$
$\operatorname*{ess\text{-}sup}_{t\in\mathbb{R}} h(t)$	$:= \inf \big\{\ \sup_{t\in\mathbb{R}\setminus N} h(t)\ \mid\ N \subseteq \mathbb{R},\ \lambda(N) = 0\ \big\}$, the essential supremum of $h : \mathbb{R} \to \mathbb{R}$
∂M	the boundary of some set $M \subseteq \mathbb{C}$
$H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$	the Hardy spaces on the right half plane, see Definition 13
$\mathcal{B}_p^+\{F\}$	the boundary function $i\mathbb{R} \to V$ of some $F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$, see Proposition 17
$H_p^+(i\mathbb{R}, V)$	the space of boundary functions $\mathcal{B}_p^+(F)$ for $F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$, see Definition 18,
$\sigma_{\max}[M]$	the maximal singular value of some matrix $M \in \mathbb{C}^{n\times m}$
$\lVert\cdot\rVert_n$	$:= \lVert\cdot\rVert_{\mathbb{C}^n,2}$, the Euclidian norm on $\mathbb{C}^n$

$\mathbb{S}^k$	$:= \left\{ s \in \mathbb{C}^{k+1} \mid \|s\|_{k+1} = 1 \right\}$, the k-dimensional (Euclidian) unit sphere
$\|\cdot\|_{n\times m}$	the induced Euclidian norm on $\mathbb{C}^{n\times m}$, i.e. $\|M\|_{n\times m} := \sigma_{\max}[M]$
v^T, v^*	the transpose or complex conjugate transpose of $v \in \mathbb{C}^n$, resp.
$\mathfrak{P}(M)$	the power set of some set M
H_p^n	$:= H_p(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C}^n)$
$H_p^{n\times m}$	$:= H_p(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C}^{n\times m})$
L_p^n	$:= L_p(i\mathbb{R}, \mathbb{C}^n)$
$L_p^{n\times m}$	$:= H_p(i\mathbb{R}, \mathbb{C}^{n\times m})$
$\mathcal{R}H_p^n$, $\mathcal{R}H_p^{n\times m}$	the real rational subspace of H_p^n, $H_p^{n\times m}$, resp.
$\mathcal{R}L_p^n$, $\mathcal{R}L_p^{n\times m}$	the real rational subspace of L_p^n, $L_p^{n\times m}$, resp.
$\mathcal{M}_G^{H_p}, \mathcal{M}_G^{L_p}$	the multiplication operators, see Proposition 27
$\|\cdot\|_{\mathrm{op}}$	norm of the multiplication operators, see Proposition 27

Small letters, e.g. f and g, are mainly used for functions in the time domain $L_p(\mathbb{R}, V)$. Capital letters, e.g. F and G, denote functions in Hardy spaces. Capital letters with a tilde, e.g. $\tilde{F}$ and $\tilde{G}$, are used for functions in the frequency domain $L_p(i\mathbb{R}, V)$.

1 Introduction

Many problems in engineering and physics are modelled as dynamical systems with m real input signals $u(\cdot) = \big(u_1(\cdot), \ldots, u_m(\cdot)\big)^T$, n real output signals $y(\cdot) = \big(y_1(\cdot), \ldots, y_n(\cdot)\big)^T$ and an operator $\mathcal{G} : \mathcal{U} \to \mathcal{Y}, u \mapsto \mathcal{G}(u) = y$, which maps any input u from some suitable input space $\mathcal{U}$ to the output y of an output space $\mathcal{Y}$ (see Figure 1).

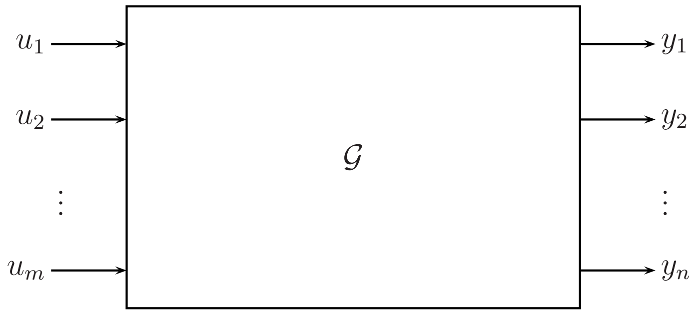

Figure 1: A system with m inputs $u_1, \ldots, u_m$ and n outputs $y_1, \ldots, y_n$

In many applications the real system is approximated by a linear system, i.e. $\mathcal{G}$ is a linear operator:

$$\mathcal{G}(\lambda_1 u^1 + \lambda_2 u^2) = \lambda_1 \mathcal{G}(u^1) + \lambda_2 \mathcal{G}(u^2) \quad \forall \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \forall u^1, u^2 \in \mathcal{U}.$$

Then it is often possible to describe the operator $\mathcal{G}$ by linear differential equations:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x^0, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

where A, B, C, D are suitable real matrices, x is the "state space" variable and x^0 the initial value.

Example 1 *Consider two masses m_1 and m_2 which are connected with a spring and are pulled with the two forces F_1 and F_2 in opposite directions (see Figure 2).*

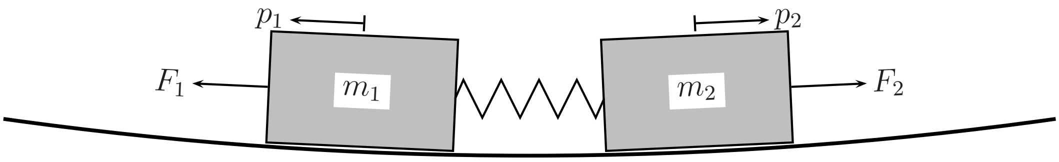

Figure 2: A physical example

The input is $u^T = (u_1, u_2) = (F_1, F_2)$ and the output consists of the two positions of the masses, i.e. $y^T = (y_1, y_2) = (p_1, p_2)$. Let $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$

denote the linear friction coefficients of the masses m_1, m_2, resp., let $d > 0$ be the spring constant of the spring between m_1 and m_2 and let $g > 0$ be the gravitation constant. Using Newton laws one can easily derive the following approximative description of the physical system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{g+d}{m_1} & -\frac{\gamma_1}{m_1} & -\frac{d}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{d}{m_2} & 0 & -\frac{g+d}{m_2} & -\frac{\gamma_2}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} u, \quad x(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u.$$

The state variables $x^T = (x_1, x_2, x_3, x_4) = (p_1, v_1, p_2, v_2)$ represent the position p_1 respectively the velocity v_1 of the mass m_1 and the position p_2 respectively the velocity v_2 of the mass m_2. Note that this model does not capture the case that the two masses get too close to each other.

As seen in the example, it is often very natural to describe a system by differential equations and state space variables. But the operator $\mathcal{G}$ which maps the input u to the output y is not explicit. It is also not straightforward to see how noise in the input signal affects the output signal.

Another description of the operator $\mathcal{G}$ is obtained if one considers the so called "frequency domain". The main idea is to transform the input u to the frequency domain space, to multiply it with a transfer matrix G to get the function Y and to transform it back to the output y (see Figure 3).

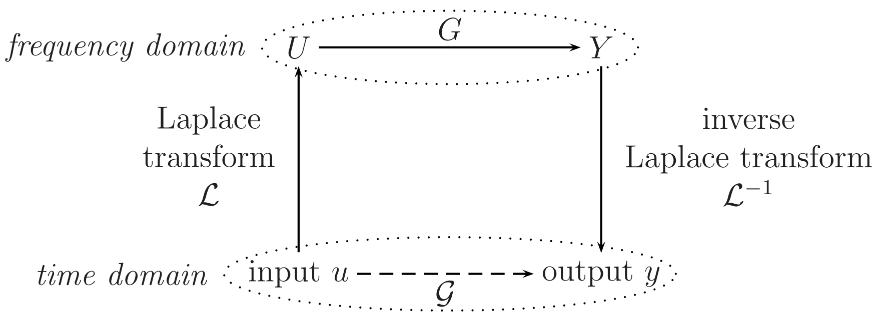

Figure 3: The frequency domain

Given a transfer matrix

$$G : \mathbb{C} \to \mathbb{C}^{n \times m}, \quad s \mapsto G(s)$$

and a (transformed) input function

$$U : \mathbb{C} \to \mathbb{C}^m, \quad s \mapsto U(s)$$

the function $Y : \mathbb{C} \to \mathbb{C}^n$ is simply

$$Y(s) = G(s)U(s) \quad \forall\, s \in \mathbb{C}.$$

The action of the operator $\mathcal{G}$ is reduced to a simple multiplication in the frequency domain. One advantage of this representation is for example that it is very easy to concatenate different systems, e.g. if two systems with transfer matrices G_1 and G_2 are considered, where the output y_1 of the first system is the input u_2 of the second, then the transfer matrix G of the concatenated system is simply described by $G(s) = G_2(s)G_1(s)$ for $s \in \mathbb{C}$.

If an operator $\mathcal{G}$ is given by a state space representation (1) with zero initial values, then the transfer matrix is

$$G(s) = C(sI - A)^{-1}B + D$$

for all $s \in \mathbb{C}$ where it is defined. Note that for a given transfer matrix there are many different state space representations with the same input-output-behaviour.

Example 2 *The transfer matrix of Example 1 is given by*

$$G(s) = \begin{bmatrix} \frac{m_2 s^2 + \gamma_2 s + d + g}{q(s)} & \frac{-d}{q(s)} \\ \frac{-d}{q(s)} & \frac{m_1 s^2 + \gamma_1 s + d + g}{q(s)} \end{bmatrix} \quad \textit{for all } s \in \mathbb{C} \textit{ with } q(s) \neq 0,$$

where, for $s \in \mathbb{C}$,

$$\begin{aligned} q(s) = {} & m_1 m_2 s^4 + (m_1\gamma_2 + m_2\gamma_1)s^3 + \big((m_1 + m_2)(d + g) + \gamma_1\gamma_2\big)s^2 \\ & + (\gamma_1 d + \gamma_2 d)(g + d)s + g^2 + 2dg. \end{aligned}$$

In applications the possible inputs are often restricted, for example one could consider only continuous inputs u. It turns out that this space of input functions is often too restrictive and instead of the space of continuous functions the more general Lebesgue spaces L_p for $1 \leq p \leq \infty$ are considered. The special case $p = 2$ plays an important role, because if an input function is an L_2-function, it can be interpreted as an input signal with bounded energy. It will turn out that the Laplace transform of an L_2-function is again an L_2-function, but in the frequency domain. It is obvious that the subspace of L_2-functions u, such that $u(t) = 0$ for all $t < 0$, is of particular interest. The Laplace transform of such functions is exactly the Hardy space H_2, which is a special case of the Hardy spaces H_p for $1 \leq p \leq \infty$.

An important question one can ask is now:
How robust is a given system to input disturbances?

More precisely this means that one would like to know how uncertainties or noise in the input u affect the output y. By linearity of the Laplace transform it is sufficient to consider this question in the frequency domain only. Let

$$\tilde{U} = U + N$$

be the disturbed input, were U is the nominal input and N is some noise, which both are considered as H_2-functions (in the frequency domain). It is easy to see that the new output $\tilde{Y}$ fulfils

$$\tilde{Y} = Y + GN$$

where $Y = GU$ and G is the transfer matrix of the nominal system. Since the noise N belongs to the same function space as the input it is not necessary to distinguish between nominal input and noise. With respect to robustness, one essential property of the system is its gain. The value of the gain will then also be a measure for the robustness of the system $\mathcal{G}$ under input disturbances.

The gain for a single input U can be measured by

$$\frac{\|GU\|}{\|U\|} \quad \text{for some suitable norms.}$$

Of particular interest is the "worst case" gain over all possible inputs, i.e.

$$\sup_{U \in H_2 \setminus \{0\}} \frac{\|GU\|}{\|U\|}. \tag{2}$$

The main aim of this work is to show that under certain assumptions on the transfer matrix G the value (2) is identical to

$$\sup_{\omega \in \mathbb{R}} \|G(i\omega)\|. \tag{3}$$

This result is often stated as a fact in the (engineering) literature, but in most cases a proof is not given. If proofs are given, they are not convincing and have many gaps. Therefore the second aim of this work is to give a convincing mathematical presentation of this topic.

This work is structured as follows: In the first section, the Lebesgue spaces and the Fourier transform are introduced. It is important to note that many results in the literature are given only for the scalar case and therefore the proofs have to be adjusted. The next section introduces the Hardy spaces and important properties of them. Unlike Lebesgue spaces, Hardy spaces can not be found in the literature easily, especially the properties of the Hardy spaces which are used in this work. The Laplace transform and its properties are presented in Section 3, in particular the relationship between the Laplace transform and the Hardy spaces is highlighted. In the last section the equality between (2) and (3) is stated and proved. Furthermore, two proofs in well established textbooks are discussed.

2 Lebesgue spaces and Fourier transforms

Throughout this section let V be a finite dimensional $\mathbb{C}$-Banach space with norm $\|\cdot\|_V$.

Definition 3
For $p \in [1, \infty)$ define

$$\hat{L}_p(\mathbb{R}, V) := \left\{ f : \mathbb{R} \to V \;\middle|\; f \text{ Lebesgue-integrable}, \int_{\mathbb{R}} \|f(t)\|_V^p \,\mathrm{d}t < \infty \right\}$$

and

$$\|\cdot\|_{\hat{L}_p(\mathbb{R})} : \hat{L}_p(\mathbb{R}, V) \to \mathbb{R}_{\geq 0}, \quad f \mapsto \|f\|_{L_p(\mathbb{R})} := \left(\int_{\mathbb{R}} \|f(t)\|_V^p \,\mathrm{d}t \right)^{1/p}.$$

For $p = \infty$ define

$$\hat{L}_\infty(\mathbb{R}, V) := \left\{ f : \mathbb{R} \to V \;\middle|\; f \text{ Lebesgue-integrable}, \operatorname*{ess-sup}_{t \in \mathbb{R}} \|f(t)\|_V < \infty \right\}$$

and

$$\|f\|_{\hat{L}_\infty(\mathbb{R})} := \operatorname*{ess-sup}_{t \in \mathbb{R}} \|f(t)\|_V.$$

Let for $f \in \hat{L}_p(\mathbb{R}, V)$, $1 \leq p \leq \infty$,

$$[f]_p := \left\{ g \in \hat{L}_p(\mathbb{R}, V) \;\middle|\; \|f - g\|_{L_p(\mathbb{R})} = 0 \right\},$$

then the *Lebesgue spaces* for $1 \leq p \leq \infty$ are defined as

$$L_p(\mathbb{R}, V) := \left\{ [f]_p \;\middle|\; f \in \hat{L}_p(\mathbb{R}, V) \right\}$$

and, for $[f] \in L_p(\mathbb{R}, V)$,

$$\|[f]\|_{L_p(\mathbb{R})} := \|f\|_{\hat{L}_p(\mathbb{R})}.$$

For $M \subseteq \mathbb{R}$ define

$$L_p(M, V) := \{ [f] \in L_p(\mathbb{R}, V) \mid \forall x \notin M : f(x) = 0 \} \subseteq L_p(\mathbb{R}, V). \tag{4}$$

Note that $\|\cdot\|_{\hat{L}_p(\mathbb{R})}$ is only a semi norm on $\hat{L}_p(\mathbb{R}, V)$. To get a vector space with norm it is therefore necessary to define $L_p(\mathbb{R}, V)$ as set of equivalence classes. For $[f] \in L_p(\mathbb{R}, V)$ it is not possible to speak of $[f](t)$ at some point $t \in \mathbb{R}$, since different functions $f_1, f_2 \in [f]$ can have different values $f_1(t) \neq f_2(t)$. For the sake of presentation, the notation is abused and in the following every equivalence class $[f]$ will be identified with a representant f and every $g \in [f]$ will also be identified with f. The value of $f(t)$ for *some $t \in \mathbb{R}$ is then the value of the specific representant and it will not be distinguished between $\hat{L}_p(\mathbb{R}, V)$ and $L_p(\mathbb{R}, V)$ (for more details see (Amann and Escher, 2001*b*), p. 121-122).*

It is important to note that $L_p(M, V) \subseteq L_p(\mathbb{R}, V)$ for every $M \subseteq \mathbb{R}$, in particular every function $f \in L_p(M, V)$ is a function defined on the whole of $\mathbb{R}$ and not only on M. Since $L_p(M, V) \subseteq L_p(\mathbb{R}, V)$ it is not necessary to define a new norm for $L_p(M, V)$.

It is an interesting fact, that for every COMPACT *M the Lebesgue spaces are a decreasing sequence of sub-spaces, i.e.*

$$L_1(M, V) \supset L_2(M, V) \supset \ldots \supset L_\infty(M, V).$$

But, for $p, q \in [1, \infty]$ with $p \neq q$,

$$L_p(\mathbb{R}, V) \not\subseteq L_q(\mathbb{R}, V) \quad \textit{and} \quad L_p(\mathbb{R}, V) \not\supseteq L_q(\mathbb{R}, V).$$

For example let

$$f_1 : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} t^{-1/2}, & \textit{for } t \in (0, 1], \\ 0, & \textit{otherwise}, \end{cases}$$

$$f_2 : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} t^{-1}, & \textit{for } t \geq 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Then it is easy to see that

$$f_1 \in L_1(\mathbb{R}, \mathbb{R}) \quad \textit{but} \quad f_1 \notin L_2(\mathbb{R}, \mathbb{R}) \quad \textit{and}$$

$$f_2 \in L_2(\mathbb{R}, \mathbb{R}) \quad \textit{but} \quad f_2 \notin L_1(\mathbb{R}, \mathbb{R}).$$

*The following Proposition is a standard result for the Lebesgue spaces L_p and a proof can be found in (Amann and Escher, 2001*b*), Theorem X.4.10.*

Proposition 4
The space $L_p(\mathbb{R}, V)$ with norm $\|\cdot\|_{L_2(\mathbb{R})}$ is a Banach space for any $p \in [1, \infty]$. If V is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_V$, then $L_2(\mathbb{R}, V)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{L_2(\mathbb{R})} := \int_{\mathbb{R}} \big\langle f(t), g(t) \big\rangle_V \mathrm{d}t\,.$$

Corollary 5
The set

$$L_p(i\mathbb{R}, V) := \left\{ \tilde{F} : i\mathbb{R} \to V \;\middle|\; \big(\omega \mapsto \tilde{F}(i\omega)\big) \in L_p(\mathbb{R}, V) \right\}$$

is a Banach space.
If V is a Hilbert space, then $L_2(i\mathbb{R}, V)$ is a Hilbert space with inner product

$$\langle \tilde{F}, \tilde{G} \rangle_{L_2(i\mathbb{R})} := \frac{1}{2\pi} \int_{\mathbb{R}} \left\langle \tilde{F}(i\omega), \tilde{G}(i\omega) \right\rangle_V \mathrm{d}\omega$$

and norm

$$\|\tilde{F}\|_{L_2(i\mathbb{R})} := \sqrt{\langle \tilde{F}, \tilde{F} \rangle_{L_2(i\mathbb{R})}}.$$

Note that the inner product on $L_2(i\mathbb{R}, V)$ differs from the inner product on $L_2(\mathbb{R}, V)$ by the constant factor 2π, the reason for that will become clear in Proposition 9 and Theorem 10. It is therefore important to distinguish between the different norms $\| \cdot \|_{L_2(\mathbb{R})}$ and $\| \cdot \|_{L_2(i\mathbb{R})}$.

Remark 6
For a basis $e_1, e_2, \dots, e_n$ of V and $1 \leq p < \infty$ define the p-norm by

$$\| \cdot \|_{V,p} : V \to \mathbb{R}_{\geq 0}, \quad v = \sum_{i=1}^{n} \xi_i e_i \mapsto \|v\|_{V,p} := \left(\sum_{i=1}^{n} |\xi_i|^p \right)^{1/p}.$$

For $f \in L_p(\mathbb{R}, V)$ there exist $f^1, \dots, f^n \in L_p(\mathbb{R}, \mathbb{C})$ such that $f(t) = \sum_{i=1}^{n} f^i(t) e_i$ for all $t \in \mathbb{R}$. If V and $\mathbb{C}^n$ are equipped with the p-norm, then

$$\begin{aligned} \|f\|_{L_p(\mathbb{R},V)}^p &= \int_{\mathbb{R}} \|f(t)\|_{V,p}^p \mathrm{d}t = \int_{\mathbb{R}} \sum_{i=1}^{n} |f^i(t)|^p \mathrm{d}t = \sum_{i=1}^{n} \|f^i\|_{L_p(\mathbb{R},\mathbb{C})}^p \\ &= \left\| (f^1, \dots, f^n)^T \right\|_{(L_p(\mathbb{R},\mathbb{C}))^n}^p. \end{aligned}$$

For $p = \infty$ define

$$\| \cdot \|_{V,\infty} : V \to \mathbb{R}_{\geq 0}, \quad v = \sum_{i=1}^{n} \xi_i e_i \mapsto \|v\|_{V,\infty} := \max_{i=1\dots n} |\xi_i|,$$

then

$$\begin{aligned} \|f\|_{L_\infty(\mathbb{R},V)} &= \operatorname*{ess-sup}_{t\in\mathbb{R}} \|f(t)\|_{V,\infty} = \operatorname*{ess-sup}_{t\in\mathbb{R}} \max_{i=1\dots n} |f_i(t)| = \max_{i=1\dots n} \operatorname*{ess-sup}_{t\in\mathbb{R}} |f_i(t)| \\ &= \left\| (f^1, \dots, f^n)^T \right\|_{(L_\infty(\mathbb{R},\mathbb{C}))^n}. \end{aligned}$$

Therefore the spaces $L_p(\mathbb{R}, V)$ and $\big(L_p(\mathbb{R}, \mathbb{C})\big)^n$ can be identified and in the remainder of this work $f = (f^1, \dots, f^n)^T$ will be written.
Note that, if V is a Hilbert space with induced norm, then there exists a basis such that this norm is a 2-norm.

Definition 7 (The Fourier integral)
Let $f : \mathbb{R} \to V$ be a Lebesgue integrable function, then, for $i\omega \in i\mathbb{R}$,

$$\mathcal{F}\{f\}(i\omega) := \int_{\mathbb{R}} e^{-i\omega t} f(t) \mathrm{d}t$$

is called the *Fourier integral*, if it exists.

It might seem artificial to consider the Fourier integral with the domain $i\mathbb{R}$, and not just $\mathbb{R}$, but in the context of Hardy spaces this yields more consistency.

In the remainder of this section important properties of the Fourier transform are highlighted. Although the next Proposition is not used elsewhere, it shows that the Fourier transform operating on L_1-functions has some remarkable "good" properties.

Proposition 8
If $f \in L_1(\mathbb{R}, V)$, then $\tilde{F} : i\mathbb{R} \to V, i\omega \mapsto \mathcal{F}\{f\}(i\omega)$, is uniformly continuous and bounded on the whole of $i\mathbb{R}$.

Proof. In (Doetsch, 1970), Satz 24.1, the statement is shown for $V = \mathbb{C}$. By Remark 6 it can be assumed that $f = (f^1, \ldots, f^n)^T \in \big(L_1(\mathbb{R}, \mathbb{C})\big)^n$ and one has

$$\tilde{F} := \mathcal{F}\{f\} = (\tilde{F}^1, \ldots, \tilde{F}^n) := (\mathcal{F}\{f^1\}, \ldots, \mathcal{F}\{f^n\})$$

and therefore $\tilde{F}$ is uniformly continuous by uniform continuity of $\tilde{F}^1, \ldots, \tilde{F}^n$. Without restriction suppose that V is equipped with the ∞-norm, then boundedness of $\tilde{F}$ follows from

$$\|\tilde{F}\|_{L_\infty(i\mathbb{R},V)} = \left\| \big(\|\tilde{F}^1\|_{L_\infty(i\mathbb{R},\mathbb{C})}, \ldots, \|\tilde{F}^n\|_{L_\infty(i\mathbb{R},\mathbb{C})}\big)^T \right\|_{\mathbb{C}^n,\infty} < \infty.$$

qed

Proposition 9
The Fourier-transform

$$\mathcal{F} : L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V) \to L_2(i\mathbb{R}, V)$$

is well defined and a linear bounded operator. If V is equipped with the 2-norm then $\mathcal{F}$ is isometric, i.e.

$$\forall\, f \in L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V) : \quad \|f\|_{L_2(\mathbb{R},V)} = \|\mathcal{F}\{f\}\|_{L_2(i\mathbb{R},V)}. \tag{5}$$

Proof. Lemma X.9.17 in (Amann and Escher, 2001*b*) gives

$$\forall\, f \in L_1(\mathbb{R}, \mathbb{C}) \cap L_2(\mathbb{R}, \mathbb{C}) : \quad \mathcal{F}\{f\} \in L_2(i\mathbb{R}, \mathbb{C}) \quad \text{and} \quad \|f\|_{L_2(\mathbb{R},\mathbb{C})} = \|\mathcal{F}\{f\}\|_{L_2(i\mathbb{R},\mathbb{C})}.$$

By Remark 6

$$f \in L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V) \quad \Longleftrightarrow \quad f = (f^1, \ldots, f^n) \in \big(L_1(\mathbb{R}, \mathbb{C}) \cap L_2(\mathbb{R}, \mathbb{C})\big)^n$$

and therefore

$$\mathcal{F}\{f\} = \big(\mathcal{F}\{f^1\}, \ldots, \mathcal{F}\{f^n\}\big)$$

which shows that $\mathcal{F}$ is well defined. Linearity of $\mathcal{F}$ follows from the definition of the Fourier integral.

If V is equipped with the 2-norm, then

$$\begin{aligned}\|\mathcal{F}\{f\}\|_{L_2(i\mathbb{R})} &= \left\|\left(\|\mathcal{F}\{f^1\}\|_{L_2(i\mathbb{R})}, \ldots, \|\mathcal{F}\{f^n\}\|_{L_2(i\mathbb{R})}\right)\right\|_{\mathbb{C}^n,2} \\ &= \left\|\left(\|f^1\|_{L_2(\mathbb{R})}, \ldots, \|f^n\|_{L_2(\mathbb{R})}\right)\right\|_{\mathbb{C}^n,2} = \|f\|_{L_2(\mathbb{R})},\end{aligned}$$

in particular $\mathcal{F}$ is bounded (by equivalence of all norms on V this is true even if V is not equipped with the 2-norm). qed

There exist functions $f \in L_2(\mathbb{R}, V)$ such that the Fourier integral $\mathcal{F}\{f\}$ does not exists for all $i\omega \in i\mathbb{R}$. Consider for example again

$$f_2 : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} t^{-1}, & \text{for } t \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

which fulfils

$$f_2 \in L_2(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad f_2 \notin L_1(\mathbb{R}, \mathbb{R}).$$

The Fourier integral considered for example at the frequency $\omega = 0$ is

$$\mathcal{F}\{f_2\}(i\omega) = \int_1^\infty \frac{1}{t} \mathrm{d}t,$$

which is not a finite value.
Nevertheless the following theorem shows that it is possible to extend the Fourier transform to the whole of $L_2(\mathbb{R}, V)$, and that this extended Fourier transform is even an isomorphism between $L_2(\mathbb{R}, V)$ and $L_2(i\mathbb{R}, V)$. The drawback of this result is, that it is in general not possible to give a pointwise description of $\mathcal{F}\{f\}$, but only a description as an L_2-limit. The same is true for the so called inverse Fourier transform given implicitly in Assertion (iii) of the following theorem, however, analogous as in Proposition 9, for every $F \in L_1(i\mathbb{R}, V) \cap L_2(i\mathbb{R}, V)$ the inverse Fourier transform exists pointwise.

Theorem 10 (Plancherel)
Let V be a Hilbert space and consider the Fourier transform $\mathcal{F} : L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V) \to L_2(i\mathbb{R})$. Then there exists a bijective linear bounded operator

$$\overline{\mathcal{F}} : L_2(\mathbb{R}, V) \to L_2(i\mathbb{R}, V), \ f \mapsto \overline{\mathcal{F}}\{f\}$$

with the following properties:

(i) $\overline{\mathcal{F}}\big|_{L_1(\mathbb{R},V)\cap L_2(\mathbb{R},V)} = \mathcal{F}$

(ii) $\forall\, f, g \in L_2(\mathbb{R}, V) : \quad \langle f, g\rangle_{L_2(\mathbb{R})} = \left\langle \overline{\mathcal{F}}\{f\}\, , \overline{\mathcal{F}}\{g\} \right\rangle_{L_2(i\mathbb{R})}$

(iii) For all $f \in L_2(\mathbb{R}, V)$ and $\tilde{F} := \overline{\mathcal{F}}\{f\}$:

$$\lim_{n\to\infty} \left\| i\omega \mapsto \left(\tilde{F}(i\omega) - \int_{-n}^{n} e^{-i\omega t} f(t) \mathrm{d}t \right) \right\|_{L_2(i\mathbb{R})} = 0$$

and

$$\lim_{n\to\infty} \left\| t \mapsto \left(f(t) - \frac{1}{2\pi} \int_{-n}^{n} e^{i\omega t} \tilde{F}(i\omega) \mathrm{d}\omega \right) \right\|_{L_2(\mathbb{R})} = 0.$$

Although Theorem 10 is well known and proofs are available in literature, a proof is given in this work. There are two reasons for this: Firstly, in the literature only the case $V = \mathbb{C}$ is considered and therefore the existing proofs have to be adjusted anyway. Secondly, Theorem 10 is essential for the following sections of this work and the proof should help understanding the underlying structures. Before proving Theorem 10 an interesting fact used in the proof is emphasised as the following lemma.

Lemma 11
Let X be a Banach space, Y a normed space and $A : X \to Y$ a linear isometric mapping. Then $\operatorname{im}(A) := \{ y \in Y \mid \exists x \in X : A(x) = y \} \subseteq Y$ is a Banach space, in particular $\operatorname{im}(A) \subseteq Y$ is closed.

Proof. It is clear that $\operatorname{im}(A)$ is a linear subspace. Consider a sequence $(y_n) \in \left(\operatorname{im}(A)\right)^{\mathbb{N}}$ with $(x_n) \in X^{\mathbb{N}}$ such that $A(x_n) = y_n$ for all $n \in \mathbb{N}$. Then linear isometry yields

$$\|x_n - x_m\| = \|A(x_n - x_m)\| = \|y_n - y_m\| \quad \forall\, n, m \in \mathbb{N}.$$

This shows that (y_n) is a Cauchy sequence if, and only if, (x_n) is a Cauchy sequence. Assume that (y_n) is a Cauchy sequence and let $x := \lim_{n\to\infty} x_n$. Now $y := A(x) \in \operatorname{im}(A)$ fulfils, by continuity of A, $y = \lim_{n\to\infty} y_n$ and therefore the Cauchy sequence (y_n) has a limit in $\operatorname{im}(A)$, hence $\operatorname{im}(A)$ is a Banach space. qed

Proof of Theorem 10.
The proof is separated into several steps. The first step highlights the fact, that $L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V)$ is dense in $L_2(\mathbb{R}, V)$, which is essential for the existence of $\overline{\mathcal{F}}$. Bijectivity of $\overline{\mathcal{F}}$ follows from injectivity and surjectivity of $\overline{\mathcal{F}}$, whilst the former is a direct consequence of linearity and isometry of $\overline{\mathcal{F}}$ the latter is not trivial. The main idea is to show that the image of $\mathcal{F}$ is dense *and* closed in the Banach space $L_2(i\mathbb{R}, V)$ and must therefore be identical with it.

STEP 1: It is shown that the set $L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V)$ is a dense subset of $L_2(\mathbb{R}, V)$.

Define, for $f \in L_2(\mathbb{R}, V)$,

$$f_n := \chi_{[-n,n]} f \quad \text{for } n \in \mathbb{N},$$

which obviously fulfils

$$f_n \in L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V) \quad \forall\, n \in \mathbb{N}$$

and

$$\|f - f_n\|_{L_2(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$

STEP 2: Existence of a linear bounded isometric operator $\overline{\mathcal{F}}$.

Proposition 9 yields that $\mathcal{F}$ is a linear bounded operator and since $L_2(\mathbb{R}, V)$ is a Banach space by Proposition 4 an application of Theorem VI.2.6 in (Amann and Escher, 2001*a*) (Continuous extension of a linear bounded operator) ensures the existence of a linear bounded operator $\overline{\mathcal{F}} : L_2(\mathbb{R}, V) \to L_2(i\mathbb{R}, V)$ which extends $\mathcal{F} : L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V) \to L_2(i\mathbb{R}, V)$. Isometry of $\mathcal{F}$ carries over to $\overline{\mathcal{F}}$ by continuity of the norms $\|\cdot\|_{L_2(\mathbb{R})}$ and $\|\cdot\|_{L_2(i\mathbb{R})}$. This proves Assertion (i).

STEP 3: Injectivity of $\overline{\mathcal{F}}$.

Isometry of $\overline{\mathcal{F}}$ yields

$$\forall\, f \in L_2(\mathbb{R}, V): \quad \Big[\ \|\overline{\mathcal{F}}\{f\}\|_{L_2(i\mathbb{R})} = 0 \quad \Longleftrightarrow \quad \|f\|_{L_2(\mathbb{R})} = 0\ \Big]$$

and therefore the kernel $\ker(\mathcal{F})$ consists merely of the zero-function in $L_2(i\mathbb{R}, V)$ which is for linear functions equivalent to injectivity. It should be remembered that the zero function in $L_2(i\mathbb{R}, V)$ is the equivalence class of all functions which are only almost everywhere zero. The Fourier transform considered on $\hat{L}_2(\mathbb{R}, V)$ as introduced in Definition 3 is therefore NOT injective.

STEP 4: Surjectivity of $\overline{\mathcal{F}}$.

First consider $V = \mathbb{C}$, then in Theorem X.9.3 in (Amann and Escher, 2001*b*) it is shown that the Schwartz space

$$\mathcal{S}(\mathbb{R}) := \left\{\ f \in C^\infty(\mathbb{R}, \mathbb{C})\ \middle|\ \forall\, k, m \in \mathbb{N}: \sup_{x \in \mathbb{R}} (1 + x^2)^{k/2} \|f^{(n)}(x)\| < \infty\ \right\}$$

is dense in $L_1(\mathbb{R}, \mathbb{C})$ and in $L_2(\mathbb{R}, \mathbb{C})$, in particular $\mathcal{S}(\mathbb{R}) \subseteq L_1(\mathbb{R}, \mathbb{C}) \cap L_2(\mathbb{R}, \mathbb{C})$. Corollary X.9.13 in (Amann and Escher, 2001*b*) implies that $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ surjectively onto

$$\mathcal{S}(i\mathbb{R}) := \left\{\ \tilde{F} : i\mathbb{R} \to \mathbb{C}\ \middle|\ \tilde{F}(i\,\cdot\,) \in \mathcal{S}(\mathbb{R})\ \right\},$$

which is, again by Theorem X.9.3, dense in $L_2(i\mathbb{R}, \mathbb{C})$.
Obviously $\mathcal{S}(\mathbb{R})^n$ is then dense in $\big(L_1(\mathbb{R}, \mathbb{C}) \cap L_2(\mathbb{R}, \mathbb{C})\big)^n$ which can as in Proposition 9 be identified with $L_1(\mathbb{R}, V) \cap L_2(\mathbb{R}, V)$ and $\mathcal{S}(i\mathbb{R})^n$ is dense in $L_2(i\mathbb{R}, \mathbb{C})^n$ which can identified with $L_2(i\mathbb{R}, V)$. Writing $\overline{\mathcal{F}}\{M\} := \{\ \overline{\mathcal{F}}\{f\} \mid f \in M\ \}$ for some $M \subseteq L_2(\mathbb{R}, V)$, Corollary X.9.13 in (Amann and Escher, 2001*b*) implies therefore $\mathcal{F}\{\mathcal{S}(\mathbb{R})^n\} = \mathcal{S}(i\mathbb{R})^n$. Now

$$L_2(i\mathbb{R}, V) \supseteq \operatorname{im}(\overline{\mathcal{F}}) = \overline{\mathcal{F}}\{L_2(\mathbb{R}, V)\} \supseteq \overline{\mathcal{F}}\{\mathcal{S}(\mathbb{R})^n\} = \mathcal{F}\{\mathcal{S}(\mathbb{R})^n\} = \mathcal{S}(i\mathbb{R})^n$$

shows that the image $\mathrm{im}(\overline{\mathcal{F}})$ is dense in $L_2(i\mathbb{R}, V)$, because $\mathcal{S}(i\mathbb{R})^n$ is dense in $L_2(i\mathbb{R}, V)$. On the other hand Lemma 11 ensures that $\mathrm{im}(\overline{\mathcal{F}})$ is a closed subspace of $L_2(i\mathbb{R}, V)$. This is only possible if $L_2(i\mathbb{R}, V) = \mathrm{im}(\overline{\mathcal{F}})$.

STEP 5: Inner product equality, i.e. Assertion (ii).

Let W be an arbitrary $\mathbb{C}$-vector space with inner product $\langle \cdot, \cdot \rangle$ and with the induced norm $\| \cdot \|$, then

$$4\langle a, b\rangle = \|a + b\|^2 - \|a - b\|^2 + i\|a + ib\|^2 - i\|a - ib\|^2 \quad \forall a, b \in W.$$

Assertion (ii) follows therefore from isometry of $\overline{\mathcal{F}}$.

STEP 6: Convergence of the integrals in Assertion (iii).

The proof of this for the case $V = \mathbb{C}$ can be found in (Rudin, 1974), Theorem 9.13d (taking care of the different constants). Assertion (iii) follows now from the identification of $L_2(\mathbb{R}, V)$ with $L_2(\mathbb{R}, \mathbb{C})^n$ and $L_2(i\mathbb{R}, V)$ with $L_2(i\mathbb{R}, \mathbb{C})^n$ (see Remark 6). qed

Corollary 12
If V is a Hilbert space, then the extended Fourier transform $\overline{\mathcal{F}} : L_2(\mathbb{R}, V) \to L_2(i\mathbb{R}, V)$ as in Theorem 10 is an isometric isomorphism, i.e.

$$L_2(\mathbb{R}, V) \cong L_2(i\mathbb{R}, V).$$

This result shows that there is no essential difference between the time domain function space $L_2(\mathbb{R}, V)$ and the frequency domain space $L_2(i\mathbb{R}, V)$. In fact, in most engineering literature these different function spaces are not distinguished or the time domain is not considered at all.

3 Hardy spaces

Throughout this section let V be a finite dimensional $\mathbb{C}$-Banach space with norm $\|\cdot\|_V$. A function $F : \mathbb{C} \to V$ is said to be holomorphic if, and only if, every component function of $F = (F_1, F_2, \dots, F_n)$ is holomorphic (see Remark 6 and note that the definition is independent of the chosen basis of V).

Definition 13 (Hardy spaces)
For $p > 0$ define the *Hardy space* on the open right half plane $\mathbb{C}_{\mathrm{Re}>0}$

$$H_p(\mathbb{C}_{\mathrm{Re}>0}, V) := \left\{ F : \mathbb{C}_{\mathrm{Re}>0} \to V \;\middle|\; \begin{array}{c} F \text{ is holomorphic and} \\ \sup_{x>0} \int_{\mathbb{R}} \|F(x+i\omega)\|_V^p \,\mathrm{d}\omega \;<\; \infty \end{array} \right\}$$

and

$$\begin{aligned} \|\cdot\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})} &: H_p(\mathbb{C}_{\mathrm{Re}>0}, V) \to \mathbb{R}_{\geq 0}, \\ & F \mapsto \|F\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})} := \left(\sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \|F(x+i\omega)\|_V^p \,\mathrm{d}\omega \right)^{1/p}. \end{aligned} \tag{6}$$

Of interest are also the Hardy spaces on the left half plane $\mathbb{C}_{\mathrm{Re}<0}$

$$H_p(\mathbb{C}_{\mathrm{Re}<0}, V) := \left\{ F : \mathbb{C}_{\mathrm{Re}<0} \to V \;\middle|\; \begin{array}{c} F \text{ is holomorphic and} \\ \sup_{x<0} \int_{\mathbb{R}} \|F(x+i\omega)\|_V^p \,\mathrm{d}\omega \;<\; \infty \end{array} \right\}$$

with

$$\|F\|_{H_p(\mathbb{C}_{\mathrm{Re}<0})} := \left(\sup_{x<0} \frac{1}{2\pi} \int_{\mathbb{R}} \|F(x+i\omega)\|_V^p \,\mathrm{d}\omega \right)^{1/p} \qquad \forall\, F \in H_p(\mathbb{C}_{\mathrm{Re}<0}, V).$$

For $p = \infty$ define

$$H_\infty(\mathbb{C}_{\mathrm{Re}>0}, V) := \left\{ F : \mathbb{C}_{\mathrm{Re}>0} \to V \;\middle|\; \begin{array}{c} F \text{ is holomorphic and} \\ \sup_{\mathrm{Re}\, s>0} \|F(s)\|_V < \infty \end{array} \right\}$$

with

$$\|\cdot\|_{H_\infty(\mathbb{C}_{\mathrm{Re}>0})} : \mathbb{C}_{\mathrm{Re}>0} \to V, \quad F \mapsto \|F\|_{H_\infty(\mathbb{C}_{\mathrm{Re}>0})} := \sup_{\mathrm{Re}\, s>0} \|F(s)\|_V$$

and $H_\infty(\mathbb{C}_{\mathrm{Re}<0}, V)$ with $\|\cdot\|_{H_\infty(\mathbb{C}_{\mathrm{Re}<0})}$ analogously.

In most (mathematical) literature the Hardy spaces are defined on the open unit disc $\mathbb{B}_1(0)$:

$$H_p(\mathbb{B}_1(0), V) := \left\{ F : \mathbb{B}_1(0) \to V \;\middle|\; \begin{array}{c} F \textit{ is holomorphic and} \\ \sup_{0\leq r<1} \int_0^{2\pi} \|F(re^{i\theta})\|_V^p \,\mathrm{d}\theta \;<\; \infty \end{array} \right\}.$$

and, for $F \in H_p(\mathbb{B}_1(0), V)$,

$$\|F\|_{H_p(\mathbb{B}_1(0))} := \left(\sup_{0\leq r<1} \int_0^{2\pi} \|F(re^{i\theta})\|_V^p \,\mathrm{d}\theta \right)^{1/p}$$

Although there exists an isomorphism between $\mathbb{B}_1(0)$ *and* $\mathbb{C}_{\text{Re}>0}$

$$\Psi : \mathbb{B}_1(0) \to \mathbb{C}_{\text{Re}>0}, \quad z \mapsto s = \frac{1-z}{1+z}$$

with inverse

$$\Psi^{-1} : \mathbb{C}_{\text{Re}>0} \to \mathbb{B}_1(0), \quad s \mapsto z = \frac{1-s}{1+s},$$

it is not clear whether all properties of $H_p(\mathbb{B}_1(0), V)$ *are preserved in the Hardy space on the right-half plane* $H_p(\mathbb{C}_{\text{Re}>0}, V)$ *and vice versa. Essential for the definition of* $H_p(\mathbb{B}_1(0), V)$ *are integrals on circles with radius* $0 < r < 1$*, for* $H_p(\mathbb{C}_{\text{Re}>0}, V)$ *the translated imaginary axis* $x + i\mathbb{R}$*, for some* $x > 0$*, plays this role. As can be seen in Figure 4 there is no simple mapping between these important curves. Furthermore* $\mathbb{B}_1(0)$ *is a bounded region, whilst* $\mathbb{C}_{\text{Re}>0}$ *is not. In particular the Lebesgue measure is in the former finite in contrast to the situation for the latter.*

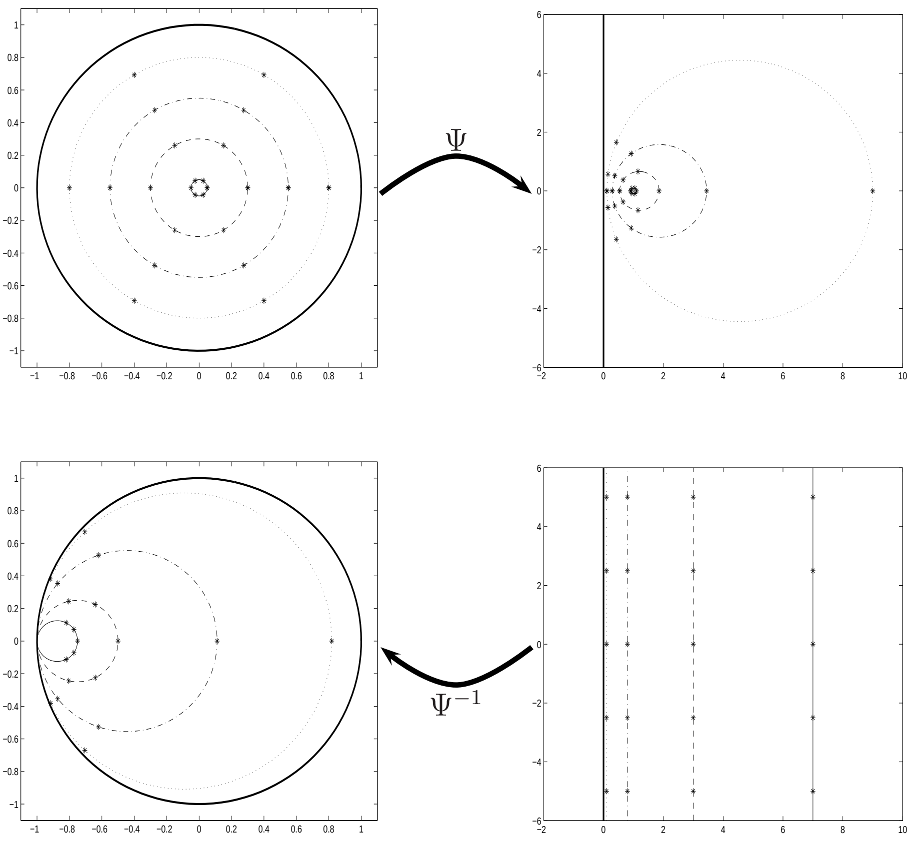

Figure 4: The isomorphism Ψ between the unit disc $\mathbb{B}_1(0)$ and the right half plane $\mathbb{C}_{\text{Re}>0}$

Nevertheless it will turn out (Proposition 17) that H_p*-functions are fully characterized by their so called boundary functions on* $\partial\mathbb{B}_1(0)$ *or* $i\mathbb{R}$*, resp. Therefore, it is possible to define an isomorphism between* $H_p(\mathbb{B}_1(0), V)$ *and* $H_p(\mathbb{C}_{\text{Re}>0}, V)$ *using the isomorphism* Ψ *(see for example Theorem 1.2.5 in (Partington, 2004) for* $p = 2$*).*

The reason for considering $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ instead of $H_p(\mathbb{B}_1(0), V)$ is simply, that, as mentioned in the introduction, the (extended) Laplace transform is an isometric isomorphism between $L_2(\mathbb{R}_{\geq 0}, V)$ and $H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ (Theorem 24). It is obvious that this result does not hold for the Hardy space on the unit disc, since in general the Laplace integral of $L_2(\mathbb{R}_{\geq 0}, V)$ does not exist on the left half plane $\mathbb{C}_{\mathrm{Re}<0}$. Another advantage of the Hardy spaces on the half plane is the existence of a very simple isometric isomorphism between $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ and $H_p(\mathbb{C}_{\mathrm{Re}<0}, V)$ described in the following remark.

Remark 14
The mapping

$$H_p(\mathbb{C}_{\mathrm{Re}>0}, V) \to H_p(\mathbb{C}_{\mathrm{Re}<0}, V), \quad F \mapsto \big(s \mapsto F(-s)\big)$$

is a linear isometric isomorphism between $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ and $H_p(\mathbb{C}_{\mathrm{Re}<0}, V)$. Therefore in the remainder of this section the properties of the Hardy spaces are only formulated for the Hardy space on the right half plane. The properties in an analogous formulation are then also true for the Hardy spaces on the left half plane.

The following Proposition highlights the structure of the H_p-spaces.

Proposition 15
For $1 \leq p \leq \infty$ the Hardy space $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ is a Banach space.

There are proofs available in (Hille, 1962), Theorem 19.1.6, for the case $1 < p < \infty$, in (Duren, 2000), Corollary 1 to Theorem 3.3, for the Hardy space on the disc, and in (Rosenblum and Rovnyak, 1985), Theorem D of Section 4.7. All proofs are only indirect, they use Fatou's Theorem (see Proposition 17) and the properties of the boundary functions. In this work a direct proof is given.

Proof. That the space $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ is a linear space is clear. It is also obvious that $\|\cdot\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})}$ fulfils $\|\lambda F\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})} = |\lambda| \|F\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})}$ for every $\lambda \in \mathbb{C}$ and for every $F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$. Every $F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ is continuous and therefore

$$\|F\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})} = 0 \Longleftrightarrow F \equiv 0.$$

To show the triangle inequality for $\|\cdot\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})}$, consider $F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ and define

$$F_x : i\mathbb{R} \to V, \quad i\omega \mapsto F(x + i\omega),$$

then, by definition,

$$\|F\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})} = \sup_{x>0} \|F_x\|_{L_p(i\mathbb{R})}.$$

Now the triangle inequality follows from the corresponding fact for the norm $\|\cdot\|_{L_p(i\mathbb{R})}$. Therefore $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ is a normed vector space.

To show completeness of $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ consider a Cauchy sequence $(F^n) \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)^{\mathbb{N}}$. Then $(F^n_x) \in L_p(i\mathbb{R}, V)^{\mathbb{N}}$ is a Cauchy sequence for every $x > 0$. By completeness of $L_p(i\mathbb{R}, V)$ (Corollary 5) this yields that for every $x > 0$ there exists $F_x \in L_p(i\mathbb{R}, V)$ such that

$$\|F^n_x - F_x\|_{L_p(i\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty.$$

It remains to show that

$$F : \mathbb{C}_{\mathrm{Re}>0} \to V, \quad s = x + i\omega \mapsto F_x(i\omega) \tag{7}$$

is holomorphic (or more precisely, that there exists a holomorphic representant of F). Without restriction it can be assumed that $V = \mathbb{C}$, because $F : \mathbb{C}_{\mathrm{Re}>0} \to V$ is holomorphic if, and only if, $F_1, F_2, \ldots, F_n : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}$ are holomorphic, where $F : \mathbb{C}_{\mathrm{Re}>0} \to V$ is identified with $(F_1, F_2, \ldots, F_n)^T : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}^n$.

To show that (7) is holomorphic (respectively has a holomorphic representant) it follows form Proposition 1.10 in (Andersson, 1997) that it suffices to show $F^n \to F$ in L^1_{loc}, i.e.

$$\forall \text{ compact } K \subseteq \mathbb{C}_{\mathrm{Re}>0} : \quad \int_K |F^n(s) - F(s)| \mathrm{d}s \to 0 \quad \text{as} \quad n \to \infty.$$

Without restriction it can be assumed that K is a rectangle:

$$K = \left\{ \; s = x + i\omega \in \mathbb{C}_{\mathrm{Re}>0} \;\mid x \in [x_l, x_r], \omega \in [\omega_d, \omega_u] \; \right\}$$

with $0 < x_l < x_r$ and $\omega_d < \omega_u$. First observe that for every $1 \leq p \leq \infty$ by application of Hölder's inequality there exists a constant $C_p > 0$ such that

$$\int_{\omega_d}^{\omega_u} |G(i\omega)| \mathrm{d}\omega < C_p \|G\|_{L_p(i\mathbb{R})} \quad \text{for all } G \in L_p(i\mathbb{R}, \mathbb{C}). \tag{8}$$

For $\varepsilon > 0$ choose $N \in \mathbb{N}$ sufficiently large, so that

$$\|F^n - F^m\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})} = \sup_{x>0} \|F^n_x - F^m_x\|_{L_p(i\mathbb{R})} < \frac{\varepsilon}{(x_r - x_l)C_p} \quad \forall\, n, m \geq N,$$

then

$$\sup_{x>0} \|F^n_x - F_x\|_{L_p(i\mathbb{R})} \leq \frac{\varepsilon}{(x_r - x_l)C_p} \quad \forall\, n \geq N. \tag{9}$$

Now

$$\begin{aligned} \int_K |F^n(s) - F(s)| \mathrm{d}s &= \int_{x_l}^{x_r} \int_{\omega_d}^{\omega_u} \big|F^n(x + i\omega) - F(x + i\omega)\big| \mathrm{d}\omega \, \mathrm{d}x \\ &\leq (x_r - x_l) \sup_{x>0} \Big(\int_{\omega_d}^{\omega_u} \big|F^n(x + i\omega) - F(x + i\omega)\big| \mathrm{d}\omega \Big) \\ &\overset{(8)}{<} (x_r - x_l) C_p \sup_{x>0} \|F^n_x - F_x\|_{L_p(i\mathbb{R})} \\ &\overset{(9)}{\leq} \varepsilon \qquad \forall\, n \geq N. \end{aligned}$$

Therefore, Proposition 1.10 in (Andersson, 1997) shows that there exists a representant of F which is holomorphic. Now

$$\|F^n - F\|_{H_p(\mathbb{C}_{\mathrm{Re}>0})} = \sup_{x>0} \|F_x^n - F_x\|_{L_p(i\mathbb{R})} \overset{(9)}{\longrightarrow} 0 \quad \text{as} \quad n \to 0,$$

which shows that every Cauchy sequence in $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ has a limit in $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$, hence $H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ is a Banach space. qed

The following two Propositions are essential for the understanding of H_p-functions. The first one shows that the supremum which is used in the definition of the H_p-spaces cannot be attained at some translated imaginary axis $x + i\mathbb{R}$ for some $x > 0$. The second one shows that every H_p-function can be identified with its boundary function on the imaginary axis $i\mathbb{R}$, which is an L_p-function.

Proposition 16
For $1 \leq p \leq \infty$ and for every $F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$ the function

$$M_p(\cdot, F) : \mathbb{R}_{>0} \to \mathbb{R}, \quad x \mapsto \|i\omega \mapsto F(x + i\omega)\|_{L_p(i\mathbb{R})}$$

is decreasing.

Proof. For the case $1 \leq p < \infty$ Theorem 19.1.4 in (Hille, 1962) shows the claim for $V = \mathbb{C}$.
A careful inspection of that proof shows that it captures the general case $V \neq \mathbb{C}$ as well (it is not necessary to assume that V is equipped with the p-Norm).

For the case $p = \infty$ observe that Equation (19.1.29) in (Hille, 1962), i.e.

$$F(x + i\omega) = \frac{x - b}{\pi} \int_{\mathbb{R}} \frac{F(b + i\varphi)}{(x - b)^2 + (\omega - \varphi)^2} \mathrm{d}\varphi \quad \forall i\omega \in i\mathbb{R}, \forall\, x > b > 0,$$

is also true for all $F \in H_\infty(\mathbb{C}_{\mathrm{Re}>0}, V)$. To prove this, the proof of Theorem 19.2.2 in (Hille, 1962) can be used almost one-to-one, the only change is, that the last step, where $b \to 0$, is left out.

For arbitrary $x > b > 0$,

$$\begin{aligned} M_\infty(x, F) &= \sup_{i\omega \in i\mathbb{R}} \|F(x + i\omega)\|_V \\ &\leq \sup_{\omega \in \mathbb{R}} \frac{x - b}{\pi} \int_{\mathbb{R}} \frac{\|F(b + i\varphi)\|_V}{(x - b)^2 + (\omega - \varphi)^2} \mathrm{d}\varphi \\ &\leq \sup_{i\varphi \in i\mathbb{R}} \|F(b + i\varphi)\|_V \sup_{\omega \in \mathbb{R}} \underbrace{\frac{x - b}{\pi} \int_{\mathbb{R}} \frac{1}{(x - b)^2 + (\omega - \varphi)^2} \mathrm{d}\varphi}_{=1} \\ &= M_\infty(b, F). \end{aligned}$$

This shows that, for fixed $F \in H_\infty(\mathbb{C}_{\mathrm{Re}>0}, V)$, the function $x \mapsto M_\infty(x, F)$ is decreasing.

qed

Proposition 17 (Fatou's Theorem)
Let $1 \le p \le \infty$. Then there exists a linear operator

$$\mathcal{B}_p^+ : H_p(\mathbb{C}_{\mathrm{Re}>0}, V) \to L_p(i\mathbb{R}, V), \quad F \mapsto \tilde{F} = \mathcal{B}_p^+\{F\}$$

with

(i) $\tilde{F}$ is the boundary function of F:

$$\tilde{F}(i\omega) = \lim_{x \searrow 0} F(x + i\omega) \quad \text{for almost all } i\omega \in i\mathbb{R}.$$

(ii) $\mathcal{B}_p^+$ is isometric:

$$\|F\|_{H_p} = \|\mathcal{B}_p^+\{F\}\|_{L_p(i\mathbb{R})} \quad \forall\, F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V).$$

(iii) F is fully described by its boundary function $\tilde{F}$:

$$F(x + i\omega_0) = \frac{x}{\pi} \int_{\mathbb{R}} \frac{\tilde{F}(i\omega)}{x^2 + (\omega_0 - \omega)^2} \mathrm{d}\omega \quad \forall \omega_0 \in \mathbb{R}, \forall\, x > 0.$$

If $p < \infty$, then:

$$F(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\tilde{F}(i\omega)}{z - i\omega} \mathrm{d}\omega \quad \text{for all } z \in \mathbb{C}_{\mathrm{Re}>0}.$$

Proof.
Showing existence of $\mathcal{B}_p^+$ and Assertion (i).

In the corollary to Theorem 11.1 in (Duren, 2000) the existence of an operator on the Hardy space on the upper half plane $\mathbb{C}_{\mathrm{Im}>0}$ with $V = \mathbb{C}$ and $1 \le p < \infty$ with an analogon of Assertion (i) is shown. Since the right half plane $\mathbb{C}_{\mathrm{Re}>0}$ and the upper half plane $\mathbb{C}_{\mathrm{Im}>0}$ are isometrically isomorphic by the rotation $z \mapsto iz$, the existence of $\mathcal{B}_p^+$ and Assertion (i) is shown for $H_p(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C})$ and $1 \le p < \infty$.

For the case $p = \infty$ consider any holomorphic bounded function $f : \mathbb{B}_1(0) \to \mathbb{C}$. In Theorem 1.2.2 in (Partington, 2004) it is shown that there exists $\tilde{f} : \partial\mathbb{B}_1(0) \to \mathbb{C}$ as radial limit of f almost everywhere , i.e.

$$\lim_{r \nearrow 1} f(re^{i\theta}) = \tilde{f}(e^{i\theta}) \quad \text{for almost all } \theta \in [0, 2\pi),$$

and $\tilde{f}$ is essentially bounded. Let $F \in H_\infty(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C})$, then, by using the isomorphism Ψ as illustrated in Figure 4,

$$f : \mathbb{B}_1(0) \to \mathbb{C}, \quad s \mapsto f(s) := F\left(\frac{1-s}{1+s}\right)$$

is well defined, holomorphic and bounded on $\mathbb{B}_1(0)$. With Theorem 1.3 in (Duren, 2000) it is easy to see that, for almost all $\omega \in \mathbb{R}$,

$$\lim_{x \searrow 0} F(x + i\omega) = \lim_{x \searrow 0} f\left(\frac{1 - x - i\omega}{1 + x + i\omega}\right) = \tilde{f}\left(\frac{1 - i\omega}{1 + i\omega}\right) =: \tilde{F}(i\omega),$$

which shows existence of $\mathcal{B}_\infty^+$ and Assertion (i) for $H_\infty(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C})$.
The general case $V \neq \mathbb{C}$ follows from considering the components $(F_1, F_2, \ldots, F_n) \in H_p(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C})^n$ of F.

Showing Assertion (ii).

Proposition 16 yields that for all $F \in H_p(\mathbb{C}_{\mathrm{Re}>0}, V)$

$$\lim_{x \searrow 0} M_p(x, F) = \|F\|_{H_p(\mathbb{C}_{\mathrm{Re}>0}, V)}.$$

Invoking the fact that $M_p(x, F) \leq \|\tilde{F}\|_{L_p(i\mathbb{R})}$ and $F \to \tilde{F}$ point-wise a.e. yields

$$\lim_{x \searrow 0} M_p(x, F) = \|\tilde{F}\|_{L_p(i\mathbb{R})}$$

and therefore $\mathcal{B}_p^+$ is isometric.

Showing Assertion (iii)

The first equation with $V = \mathbb{C}$ is proven in Theorem 11.2 in (Duren, 2000) and the second one in Theorem 11.8 in (Duren, 2000). Again (Duren, 2000) considers the upper half plane, but using the rotation from the first part of this proof one get the result for the right half plane. It is also easy to see that the result is true for the general case $V \neq \mathbb{C}$. qed

By Remark 14 there exists also an operator $\mathcal{B}_p^- : H_p(\mathbb{C}_{\mathrm{Re}<0}, V) \to L_p(i\mathbb{R}, V)$ with the analogous properties as the operator $\mathcal{B}_p^+$ in Proposition 17. In particular for every $G \in H_p(\mathbb{C}_{\mathrm{Re}<0}, V)$

$$\mathcal{B}_p^-\{G\}(i\omega) = \lim_{x \nearrow 0} G(x + i\omega) \quad \textit{for almost all } i\omega \in i\mathbb{R}$$

and

$$\|\mathcal{B}_p^-\{G\}\|_{L_p(i\mathbb{R})} = \|G\|_{H_p(\mathbb{C}_{\mathrm{Re}<0})}.$$

Definition 18

$$H_p^+(i\mathbb{R}, V) := \mathrm{im}(\mathcal{B}_p^+) \quad \text{and} \quad H_p^-(i\mathbb{R}, V) := \mathrm{im}(\mathcal{B}_p^-).$$

Corollary 19
Let $1 \leq p \leq \infty$, then the subspaces $H_p^+(i\mathbb{R}, V)$ and $H_p^-(i\mathbb{R}, V)$ of $L_p(i\mathbb{R}, V)$ are complete and

$$H_p^+(i\mathbb{R}, V) \cong H_p(\mathbb{C}_{\mathrm{Re}>0}, V) \quad \text{and} \quad H_p^-(i\mathbb{R}, V) \cong H_p(\mathbb{C}_{\mathrm{Re}<0}, V).$$

In particular, if V is a Hilbert space, $H_2^+(i\mathbb{R}, V)$ and $H_2^-(i\mathbb{R}, V)$ are Hilbert spaces with the inner product of $L_2(i\mathbb{R}, V)$.

Proof. $\mathcal{B}_p^+ : H_p(\mathbb{C}_{\mathrm{Re}>0}, V) \to H_p^+(i\mathbb{R}, V)$ and $\mathcal{B}_p^- : H_p(\mathbb{C}_{\mathrm{Re}<0}, V) \to H_p^-(i\mathbb{R}, V)$ are by definition surjective. Note that $\mathcal{B}_p^+$ is linear and since $\|\mathcal{B}_p^+\{F\}\|_{L_p(i\mathbb{R})} = 0$ if, and only if, $\|F\|_{H_p} = 0$, it is also injective (as well as $\mathcal{B}_p^-$). By isometry of $\mathcal{B}_p^+$ and $\mathcal{B}_p^-$, Lemma 11 together with Proposition 15 yields that $H_p^+(i\mathbb{R}, V)$ and $H_p^-(i\mathbb{R}, V)$ are complete subspaces of $L_p(i\mathbb{R}, V)$. qed

4 Hardy spaces and Laplace transforms

Definition 20 (The Laplace integral)
Let $f : \mathbb{R} \to V$ be a Lebesgue integrable function, then, for $\omega \in \mathbb{R}$,

$$\mathcal{L}\{f\}(s) := \int_{\mathbb{R}} e^{-st} f(t) \mathrm{d}t$$

is called the *Laplace integral*, if it exists.

In literature one can find many different definition of the Laplace integral. It is very common to define the Laplace integral only one sided, i.e. $\int_0^\infty e^{-st} f(t)\mathrm{d}t$. But this is a needless restriction since the one sided Laplace integral is just a special case where $f(t) = 0$ for all $t < 0$. On the other hand, the presented definition of the Laplace integral shows the strong relationship to the Fourier integral

$$\mathcal{F}\{f\} = \mathcal{L}\{f\}\big|_{i\mathbb{R}}.$$

In this section the properties of the Laplace transform in connection with the Hardy spaces are collected. The results are very similar to that of Section 2. The following two propositions state some general facts about the Laplace integral. The first one is on the existence of the Laplace transform and the second one on the smoothness of the Laplace integral if it exits.

Proposition 21
Let $f : \mathbb{R} \to V$ with $f(t) = 0$ for all $t < 0$ and, for some $s_0 \in \mathbb{R}$.

$$\big(t \mapsto e^{-s_0 t} f(t)\big) \in L_1(\mathbb{R}_{\geq 0}, V),$$

Then the Laplace integral

$$\mathcal{L}\{f\}(s) \in V \text{ exists for all } s \in \mathbb{C}_{\mathrm{Re} \geq s_0} := \{\ s \in \mathbb{C} \ \mid \mathrm{Re}(s) \geq s_0\ \}.$$

Proof. For arbitrary $s \in \mathbb{C}_{\mathrm{Re}\geq s_0}$ one has

$$\begin{aligned}\int_{\mathbb{R}} \left\|e^{-st} f(t)\right\|_V \mathrm{d}t = \int_0^\infty \left\|e^{-st} f(t)\right\|_V \mathrm{d}t &= \int_0^\infty |e^{-(s-s_0)t}| \left\|e^{-s_0 t} f(t)\right\|_V \mathrm{d}t \\ \leq \int_0^\infty \left\|e^{-s_0 t} f(t)\right\|_V \mathrm{d}t &< \infty\end{aligned}$$

which shows (absolute) convergence of the Laplace integral on $\mathbb{C}_{\mathrm{Re}\geq s_0}$. qed

Proposition 22
Suppose that for $f : \mathbb{R} \to V$ with $f(t) = 0$ for all $t < 0$ the Laplace integral $F(s) = \mathcal{L}\{f\}(s)$ exists on $\mathbb{C}_{\mathrm{Re}>\beta}$ for some $\beta \in \mathbb{R}$. Then $F : \mathbb{C}_{\mathrm{Re}>\beta} \to V$ is holomorphic.

Proof. In (Doetsch, 1970), Satz 6.1, the statement is shown for $V = \mathbb{C}$. Note that $F = (F^1, \dots, F^n)$ is holomorphic if, and only if, $F^1, \dots, F^n$ are holomorphic. qed

Propositions 21 (with $s_0 = 0$) and 22 yield that the Laplace transform $F = \mathcal{L}\{f\}$ of any function $f \in L_1(\mathbb{R}_{\geq 0}, V)$ is well defined on $\mathbb{C}_{\mathrm{Re}>0}$ and is holomorphic there. So one might suspect a strong relationship between L_p and Hardy spaces. Indeed the following proposition shows that for certain functions the Laplace transform is an H_2-function. This proposition is very similar to Proposition 9.

Proposition 23
The right-sided Laplace transform

$$\mathcal{L}^+ : L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V) \to H_2(\mathbb{C}_{\mathrm{Re}>0}, V), \quad f \mapsto \mathcal{L}\{f\}$$

and the left-sided Laplace transform

$$\mathcal{L}^- : L_1(\mathbb{R}_{\leq 0}, V) \cap L_2(\mathbb{R}_{\leq 0}, V) \to H_2(\mathbb{C}_{\mathrm{Re}<0}, V), \quad g \mapsto \mathcal{L}\{g\}$$

are well defined linear operators. The Laplace transform

$$\begin{aligned} &\mathcal{L} := \mathcal{L}^+ \cup \mathcal{L}^-, \\ &\mathcal{L} : \big(L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V)\big) \ \cup \ \big(L_1(\mathbb{R}_{\leq 0}, V) \cap L_2(\mathbb{R}_{\leq 0}, V)\big) \\ &\qquad\qquad \to H_2(\mathbb{C}_{\mathrm{Re}>0}, V) \cup H_2(\mathbb{C}_{\mathrm{Re}<0}, V) \end{aligned}$$

is a bounded operator.
If V is equipped with the 2-norm then $\mathcal{L}$ is isometric, i.e.

$$\begin{aligned} &\forall f \in L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V) : \quad \|\mathcal{L}\{f\}\|_{H_2(\mathbb{C}_{\mathrm{Re}>0}, V)} = \|f\|_{L_2(\mathbb{R})} \quad \text{and} \\ &\forall g \in L_1(\mathbb{R}_{\leq 0}, V) \cap L_2(\mathbb{R}_{\leq 0}, V) : \quad \|\mathcal{L}\{g\}\|_{H_2(\mathbb{C}_{\mathrm{Re}<0}, V)} = \|g\|_{L_2(\mathbb{R})}. \end{aligned}$$

Proof. Putting $s_0 = 0$ in Proposition 21 together with Proposition 22 (and taking (4) into account) yields that $\mathcal{L}^+$ is well defined provided that the boundary condition of $H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ is fulfilled. Assume first that V is equipped with the 2-norm and consider $f \in L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V)$ and $x > 0$. Define

$$f_x : \mathbb{R} \to V, \quad t \mapsto e^{-xt} f(t).$$

Then $f_x \in L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V)$ and

$$\begin{aligned} \sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \|\mathcal{L}\{f\}(x+i\omega)\|_{V,2}^2 \mathrm{d}\omega &= \sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} e^{-(x+i\omega)} f(t) \mathrm{d}t \right\|_{V,2}^2 \mathrm{d}\omega \\ &= \sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \|\mathcal{F}\{f_x\}(i\omega)\|_{V,2}^2 \, \mathrm{d}\omega \\ &= \sup_{x>0} \|\mathcal{F}\{f_x\}\|_{L_2(i\mathbb{R})}^2 \\ &\overset{(5)}{=} \sup_{x>0} \|f_x\|_{L_2(\mathbb{R})}^2 \\ &= \|f\|_{L_2(\mathbb{R})}^2. \end{aligned}$$

This shows that, for all $f \in L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V)$,

$$\mathcal{L}\{f\} \in H_2(\mathbb{C}_{\mathrm{Re}>0}, V) \quad \text{and} \quad \|\mathcal{L}\{f\}\|_{H_2(\mathbb{C}_{\mathrm{Re}>0})} = \|f\|_{L_2(\mathbb{R})}.$$

If V is not equipped with the 2-norm, then the same calculations give (invoking boudedness of the Fourier transform $\mathcal{F}$):

$$\|\mathcal{L}\{f\}\|_{H_2(\mathbb{C}_{\mathrm{Re}>0})} \leq C_V \|f\|_{L_2(\mathbb{R})},$$

where $C_V > 0$ is a constant depending on the norm of V. This shows boundedness of $\mathcal{L}^+$ and linearity follows from the definition of the Laplace integral $\mathcal{L}$.

It is easy to see that Proposition 21 and 22 are similarly also true for functions $g \in L_1(\mathbb{R}_{\leq 0}, V)$, in particular $\mathcal{L}\{g\}$ is well defined on $\mathbb{C}_{\mathrm{Re}<0}$ and is holomorphic there. Analogous arguments as above for $g \in L_1(\mathbb{R}_{\leq 0}, V) \cap L_2(\mathbb{R}_{\leq 0}, V)$ complete the proof. qed

This section ends with the important fact, that the (extended) Laplace transform is an isometric isomorphism between the time domain Lebesgue space $L_2(\mathbb{R}_{\geq 0}, V)$ and the frequency domain space $H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$. That means that there is no essential difference between the time domain and the frequency domain. In addition, the trivial orthogonal decomposition of $L_2(\mathbb{R}, V)$ into $L_2(\mathbb{R}_{\geq 0}, V) \oplus L_2(\mathbb{R}_{\leq 0}, V)$ carries over to the frequency domain in the form $L_2(i\mathbb{R}, V) = H_p^+(i\mathbb{R}, V) \oplus H_p^-(i\mathbb{R}, V)$. These interesting connections between the time domain spaces and frequency domain spaces are illustrated in Figure 5.

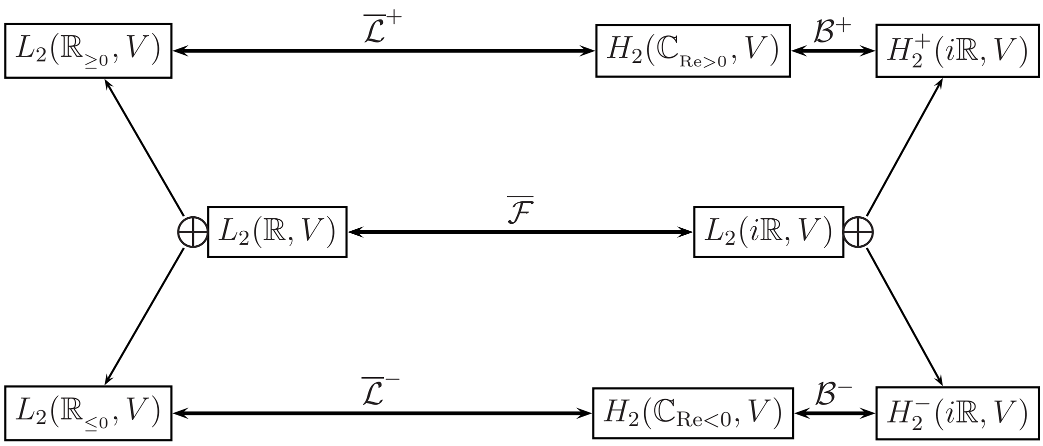

Figure 5: The connection between time and frequency domain spaces, where all "↔" denote isometric isomorphisms

Theorem 24 (Paley-Wiener)
There exist linear isomorphisms

$$\overline{\mathcal{L}}^+ : L_2(\mathbb{R}_{\geq 0}, V) \to H_2(\mathbb{C}_{\mathrm{Re}>0}, V) \quad \text{and}$$
$$\overline{\mathcal{L}}^- : L_2(\mathbb{R}_{\leq 0}, V) \to H_2(\mathbb{C}_{\mathrm{Re}<0}, V)$$

with

$$\overline{\mathcal{L}}^+\big|_{L_1(\mathbb{R}_{\geq 0})\cap L_2(\mathbb{R}_{\geq 0})} = \mathcal{L}^+ \quad \text{and}$$
$$\overline{\mathcal{L}}^-\big|_{L_1(\mathbb{R}_{\leq 0})\cap L_2(\mathbb{R}_{\leq 0})} = \mathcal{L}^-.$$

The extended Laplace transform

$$\overline{\mathcal{L}} := \overline{\mathcal{L}}^+ \cup \overline{\mathcal{L}}^-$$

is then a bounded operator.
If V is a Hilbert space, then $\overline{\mathcal{L}}$ is isometric and moreover

$$L_2(i\mathbb{R}, V) = H_2^+(i\mathbb{R}, V) \oplus H_2^-(i\mathbb{R}, V).$$

It is not easy to find a complete proof of this Theorem in literature. A starting point could be the proof of Theorem 11.9 in (Duren, 2000), but there only the case $V = \mathbb{C}$ and the upper half plane is considered. Again as for Theorem 10 a complete proof is given in this work, since this Theorem is essential. An interesting fact, which will be used in the proof, about the Laplace transform combined with the boundary operator $\mathcal{B}_2$ is highlighted as a lemma. In view of Proposition 17 it is not surprising that $\mathcal{L}^+$ combined with $\mathcal{B}^+$ is simply the Fourier transform. It is easy to see that the result of this lemma will carry over to the extended Laplace and Fourier transform.

Lemma 25

$$\forall\, f \in L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V): \quad \mathcal{B}_2^+\big(\mathcal{L}\{f\}\big) = \mathcal{F}\{f\}.$$

The assertion of Lemma 25 is illustrated by the commutative diagram in Figure 6.

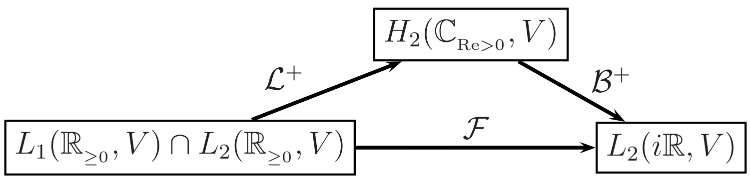

Figure 6: Illustration of Lemma 25

Proof of Lemma 25. Proposition 21 with $s_0 = 0$ yields that the function $F : \mathbb{C}_{\mathrm{Re}\geq 0} \to V$, $s \mapsto \mathcal{L}\{f\}(s)$ is well defined on the closed right-half plane. Moreover $F|_{\mathbb{C}_{\mathrm{Re}>0}} \in H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ by Proposition 23. By definition of the Fourier transform it is clear that $F|_{i\mathbb{R}} = \mathcal{F}\{f\}$.

It remains to show that $\mathcal{B}_2^+\{F|_{\mathbb{C}_{\mathrm{Re}>0}}\} = F|_{i\mathbb{R}}$, i.e.

$$F(i\omega) = \lim_{x\searrow 0} F(x + i\omega) \quad \big(= \mathcal{B}_2^+\{F\}(i\omega)\big) \quad \text{for almost all } i\omega \in i\mathbb{R}.$$

Let $i\omega \in i\mathbb{R}$ such that $\lim_{x\searrow 0} F(x+i\omega)$ exists and choose for $\varepsilon > 0$ a sufficiently large $T \in \mathbb{R}$ so that

$$\int_T^\infty \big|f(t)\big| \mathrm{d}t < \varepsilon. \tag{10}$$

Then

$$\begin{aligned}
\big|\lim_{x\searrow 0} F(x+i\omega) - F(i\omega)\big| &= \lim_{x\searrow 0}\left|\int_0^\infty e^{i\omega t} f(t)\big(e^{-xt}-1\big)\mathrm{d}t\right| \\
&\leq \lim_{x\searrow 0}\int_0^\infty \big|f(t)\big|\big(1-e^{-xt}\big)\mathrm{d}t \\
&\leq \lim_{x\searrow 0}(1-e^{-xT})\int_0^T \big|f(t)\big|\mathrm{d}t + \int_T^\infty \big|f(t)\big|\mathrm{d}t \\
&\overset{(10)}{<} \lim_{x\searrow 0}(1-e^{-xT})\|f\|_{L_1(\mathbb{R})} + \varepsilon \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ can chosen arbitrarily small the proof is complete. qed

Proof of Theorem 24
The proof concentrates only on the right-sided Laplace transform, since the proof for the left-sided Laplace transform is analogous.

Step 1: Existence of linear bounded $\overline{\mathcal{L}}^+ : L_2(\mathbb{R}_{\geq 0}, V) \to H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$.

Proposition 23 yields that $\mathcal{L}^+ : L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V) \to H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ is linear and bounded on the dense subspace $L_1(\mathbb{R}_{\geq 0}, V) \cap L_2(\mathbb{R}_{\geq 0}, V)$ of the Banach spaces $L_2(\mathbb{R}_{\geq 0}, V)$. Theorem VI.2.6 in (Amann and Escher, 2001*a*) is applicable and yields that there exists a continuous linear extension $\overline{\mathcal{L}}^+ : L_2(\mathbb{R}_{\geq 0}, V) \to H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$.

Step 2: Injectivity of $\overline{\mathcal{L}}^+$.

Consider first the case that V is equipped with the 2-norm. Then Proposition 23 yields that $\overline{\mathcal{L}}^+$ is isometric and therefore

$$\|\overline{\mathcal{L}}^+\{f\}\|_{H_2(\mathbb{C}_{\mathrm{Re}>0})} = 0 \quad \Longleftrightarrow \quad \|f\|_{L_2(\mathbb{R})} = 0.$$

By equivalence of all norms on V this is also true for any norm $\|\cdot\|_V$ of V. Together with linearity of $\overline{\mathcal{L}}^+$ this yields injectivity.

Step 3: Surjectivity of $\overline{\mathcal{L}}^+$.[1]

Let $F \in H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ and $\tilde{F} = \mathcal{B}_2^+(F) \in H_2^+(i\mathbb{R}, V) \subseteq L_2(i\mathbb{R}, V)$, then F can, by Proposition 17, be written as

$$F(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\tilde{F}(i\omega)}{z - i\omega}\mathrm{d}\omega \quad \text{for all } z \in \mathbb{C}_{\mathrm{Re}>0}.$$

[1]This step of the proof is mainly based on parts of the proof of Theorem 11.9 in (Duren, 2000).

By Theorem 10 one can define $f \in L_2(\mathbb{R}_{\geq 0}, V)$ as L_2-limit of

$$f_n : \mathbb{R}_{\geq 0} \to V,\ t \mapsto \frac{1}{2\pi} \int_{-n}^{n} e^{i\omega t} \tilde{F}(i\omega) \mathrm{d}\omega$$

as $n \to \infty$, i.e. $f = \overline{\mathcal{F}}^{-1}\{\tilde{F}\}$. Together with

$$\frac{1}{z - i\omega} = \int_0^\infty e^{-t(z-i\omega)} \mathrm{d}t$$

this yields, for all $z \in \mathbb{C}_{\mathrm{Re}>0}$,

$$\begin{aligned} F(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}(i\omega) \left(\int_0^\infty e^{-t(z-i\omega)} \mathrm{d}t \right) \mathrm{d}\omega \\ &= \int_0^\infty e^{-zt} \left(\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} \tilde{F}(i\omega) \mathrm{d}\omega \right) \mathrm{d}t \\ &= \int_0^\infty e^{-zt} \lim_{n\to\infty} f_n(t) \mathrm{d}t \\ &= \lim_{n\to\infty} \mathcal{L}^+\{f_n\}(z) \\ &= \overline{\mathcal{L}}^+\{f\}(z). \end{aligned}$$

Therefore, for every $F \in H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ exists $f \in L_2(\mathbb{R}_{\geq 0}, V)$ such that $F = \overline{\mathcal{L}}^+\{f\}$.

Step 4: V a Hilbert space.

Step 4a: Isometry of $\overline{\mathcal{L}}$.

If V is a Hilbert space then it is by Remark 6 equipped with the 2-norm. By Proposition 23 the continuous extension $\overline{\mathcal{L}}$ of $\mathcal{L}$ is isometric.

Step 4b: Showing $H_2^+(i\mathbb{R}, V) \perp H_2^-(i\mathbb{R}, V)$.

Consider $\tilde{F} \in H_2^+(i\mathbb{R}, V)$ and $\tilde{G} \in H_2^-(i\mathbb{R}, V)$ with corresponding $F \in H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ and $G \in H_2(\mathbb{C}_{\mathrm{Re}<0}, V)$ such that $\mathcal{B}_2^+\{F\} = \tilde{F}$ and $\mathcal{B}_2^-\{G\} = \tilde{G}$. Invoking Step 3 it is possible to find $f \in L_2(\mathbb{R}_{\geq 0}, V)$ and $g \in L_2(\mathbb{R}_{\leq 0}, V)$ such that $F = \overline{\mathcal{L}}\{f\}$ and $G = \overline{\mathcal{L}}\{g\}$. Observe that by continuity of $\mathcal{B}_2^+$ (linear and bounded) and Lemma 25

$$\mathcal{B}_2^+\left\{\overline{\mathcal{L}}\{f\}\right\} = \overline{\mathcal{F}}\{f\} \quad \text{and} \quad \mathcal{B}_2^-\left\{\overline{\mathcal{L}}\{g\}\right\} = \overline{\mathcal{F}}\{g\}.$$

Therefore, by invoking Theorem 10,

$$\left\langle \tilde{F}, \tilde{G} \right\rangle_{L_2(i\mathbb{R})} = \left\langle \overline{\mathcal{F}}\{f\}, \overline{\mathcal{F}}\{g\} \right\rangle_{L_2(i\mathbb{R})} = \left\langle f, g \right\rangle_{L_2(\mathbb{R})} = 0.$$

Step 4c: Showing $L_2(i\mathbb{R}, V) = H_2^+(i\mathbb{R}, V) \oplus H_2^-(i\mathbb{R}, V)$.

It remains to show that for every $\tilde{H} \in L_2(i\mathbb{R}, V)$ there exist two functions $F \in H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ and $G \in H_2(\mathbb{C}_{\mathrm{Re}<0}, V)$ such that $\mathcal{B}_2^+\{F\} + \mathcal{B}_2^-\{G\} = \tilde{H}$.

By Theorem 10 there exists $h \in L_2(i\mathbb{R}, V)$ such that $\overline{\mathcal{F}}\{h\} = \tilde{H}$ and there exist $f \in L_2(\mathbb{R}_{\geq 0}, V)$ and $g \in L_2(\mathbb{R}_{\leq 0}, V)$, such that $h = f + g$. Let $F := \overline{\mathcal{L}}\{f\} \in H_2(V)$ and $G := \overline{\mathcal{L}}\{g\} \in H_2^-(V)$ and observe again that $\mathcal{B}_2^+\{F\} = \overline{\mathcal{F}}\{f\}$ and $\mathcal{B}_2^-\{G\} = \overline{\mathcal{F}}\{g\}$. Now linearity of $\overline{\mathcal{F}}$ yields

$$\tilde{H} = \overline{\mathcal{F}}\{h\} = \overline{\mathcal{F}}\{f+g\} = \overline{\mathcal{F}}\{f\} + \overline{\mathcal{F}}\{g\} = \mathcal{B}_2^+\{F\} + \mathcal{B}_2^-\{G\}.$$

qed

The proof of Theorem 24 shows that

$$\overline{\mathcal{F}}^+ : L_2(\mathbb{R}_{\geq 0}, V) \to H_2^+(i\mathbb{R}, V) \quad \textit{and} \quad \overline{\mathcal{F}}^- : L_2(\mathbb{R}_{\leq 0}, V) \to H_2^+(i\mathbb{R}, V)$$

with

$$\overline{\mathcal{F}}^+ := \overline{\mathcal{F}}\big|_{L_2(\mathbb{R}_{\geq 0}, V)} \quad \textit{and} \quad \overline{\mathcal{F}}^- := \overline{\mathcal{F}}\big|_{L_2(\mathbb{R}_{\leq 0}, V)}$$

are isometric isomorphisms (see Figure 7).

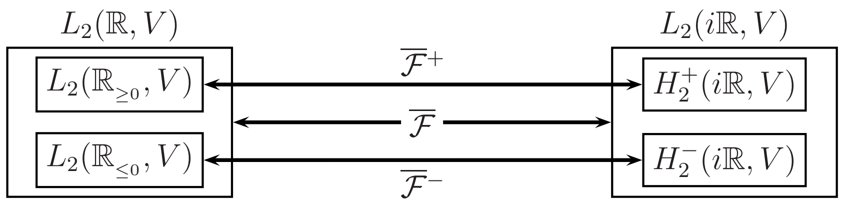

Figure 7: The extended Fourier transform

One may ask, why Hardy spaces and Laplace transform are considered at all. The reason for this is, that functions in the Hardy space $H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ can be characterised very easily (holomorphic and "bounded"), whilst functions in $H_2^+(i\mathbb{R}, V)$ do not have a simple direct characterisation. Furthermore, functions in $H_2(\mathbb{C}_{\mathrm{Re}>0}, V)$ can be often expressed by a simple formula, whilst functions in $H_2^+(i\mathbb{R}, V)$ as L_2-functions do often not have this property.

5 The gain of linear systems

In this section linear systems with m real input signals $u = (u_1, \dots, u_m)^T \in L_2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and n real output signals $y = (y_1, \dots, y_n)^T \in L_2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ are considered. In the frequency domain, the signals fulfil $U = \overline{\mathcal{L}}\{u\} \in H_2(\mathbb{C}_{\text{Re}>0}, \mathbb{C}^m)$ and $Y = \overline{\mathcal{L}}\{y\} \in H_2(\mathbb{C}_{\text{Re}>0}, \mathbb{C}^m)$. The transfer matrix G is an $n \times m$ matrix and a natural assumption is that G is bounded and smooth, i.e. $G \in H_\infty(\mathbb{C}_{\text{Re}>0}, \mathbb{C}^{n\times m})$. It will be assumed that $\mathbb{C}^n$ is equipped with the standard inner product and therefore the induced norm is the 2-norm (Euclidian norm). The matrix space $\mathbb{C}^{n\times m}$ will be equipped with the induced 2-norm, which can be calculated as

$$\|M\|_{n\times m} := \sigma_{\max}[M] \quad \textit{for } M \in \mathbb{C}^{n\times m},$$

where $\sigma_{\max}[\cdot]$ denotes the maximum singular value.

Note that then

$$\|Mx\|_n \leq \|M\|_{n\times m}\|x\|_m \quad \forall\, M \in \mathbb{C}^{n\times m},\ \forall\, x \in \mathbb{C}^m.$$

Definition 26
For $n, m \in \mathbb{N}$ and $1 \leq p \leq \infty$ define

$$H_p^n := H_p(\mathbb{C}_{\text{Re}>0}, \mathbb{C}^n) \quad \text{and} \quad H_p^{n\times m} := H_p(\mathbb{C}_{\text{Re}>0}, \mathbb{C}^{n\times m})$$

and

$$L_p^n := L_p(i\mathbb{R}, \mathbb{C}^n) \quad \text{and} \quad L_p^{n\times m} := L_p(i\mathbb{R}, \mathbb{C}^{n\times m}).$$

The next proposition highlights the fact that bounded (and smooth) transfer matrices can be considered as well defined bounded operators on the Lebesgue spaces L_p (on the Hardy spaces H_p).

Proposition 27
Let $1 \leq p \leq \infty$, $n, m \in \mathbb{N}$ and $G \in H_\infty^{n\times m}$, then the *multiplication operator*

$$\boxed{\mathcal{M}_G^{H_p} : H_p^m \to H_p^n, \quad F \mapsto \mathcal{M}_G^{H_p}\{F\} = \big(s \mapsto G(s)F(s)\big)}$$

is well defined and fulfils

$$\|G\|_{H_\infty^{n\times m}} \geq \left\|\mathcal{M}_G^{H_p}\right\|_{\text{op}} := \sup_{F \in H_p^m \setminus\{0\}} \frac{\|GF\|_{H_p^n}}{\|F\|_{H_p^m}}.$$

If $\tilde{G} \in L_\infty^{n\times m}$, then the *multiplication operator*

$$\boxed{\mathcal{M}_{\tilde{G}}^{L_p} : L_p^m \to L_p^n, \quad \tilde{F} \mapsto \mathcal{M}_{\tilde{G}}^{L_p}\{\tilde{F}\} = \big(i\omega \mapsto \tilde{G}(i\omega)\tilde{F}(i\omega)\big)}$$

is well defined and fulfils

$$\|\tilde{G}\|_{L_\infty^{n\times m}} \geq \left\|\mathcal{M}_{\tilde{G}}^{L_p}\right\|_{\text{op}} := \sup_{\tilde{F} \in L_p^m \setminus\{0\}} \frac{\|\tilde{G}\tilde{F}\|_{L_p^n}}{\|\tilde{F}\|_{L_p^m}}.$$

Proof. Let $1 \leq p < \infty$, $G \in H_\infty^{n\times m}$ and $F \in H_p^m$, then

$$\begin{aligned}
&\sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \|G(x+i\omega)F(x+i\omega)\|_n^p \mathrm{d}\omega \\
&\leq \sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \|G(x+i\omega)\|_{n\times m}^p \|F(x+i\omega)\|_m^p \mathrm{d}\omega \\
&\leq \sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \|G\|_{H_\infty^{n\times m}}^p \|F(x+i\omega)\|_m^p \mathrm{d}\omega \\
&= \|G\|_{H_\infty^{n\times m}}^p \|F\|_{H_p^n}^p < \infty.
\end{aligned}$$

If $p = \infty$, then

$$\begin{aligned}
\sup_{s\in\mathbb{C}_{\mathrm{Re}>0}} \|G(s)F(s)\|_n &\leq \sup_{s\in\mathbb{C}_{\mathrm{Re}>0}} \|G(s)\|_{n\times m}\|F(s)\|_m \\
&\leq \sup_{s\in\mathbb{C}_{\mathrm{Re}>0}} \|G\|_{H_\infty^{n\times m}}\|F(s)\|_m = \|G\|_{H_\infty^{n\times m}}\|F\|_{H_\infty^m} \\
&< \infty.
\end{aligned}$$

Since GF is holomorphic this shows that $GF \in H_p^n$ for $1 \leq p \leq \infty$ and

$$\|GF\|_{H_p^n} \leq \|G\|_{H_\infty^{n\times m}}\|F\|_{H_p^n}$$

and therefore obviously

$$\|G\|_{H_\infty^{n\times m}} \geq \left\|\mathcal{M}_G^{H_p}\right\|_{\mathrm{op}}.$$

The case $\tilde{G} \in L_p^{n\times m}$ follows analogously since $\tilde{G}\tilde{F}$ is measurable. qed

Consider the transfer matrix from Example 2,

$$G(s) = \begin{bmatrix} \frac{m_2 s^2+\gamma_2 s+d+g}{q(s)} & \frac{-d}{q(s)} \\ \frac{-d}{q(s)} & \frac{m_1 s^2+\gamma_1 s+d+g}{q(s)} \end{bmatrix} \quad \textit{for all } s \in \mathbb{C} \textit{ with } q(s) \neq 0,$$

where, for $s \in \mathbb{C}$,

$$\begin{aligned}
q(s) &= m_1 m_2 s^4 + (m_1\gamma_2 + m_2\gamma_1)s^3 + \big((m_1+m_2)(d+g) + \gamma_1\gamma_2\big)s^2 \\
&\quad + (\gamma_1 d + \gamma_2 d)(g+d)s + g^2 + 2dg.
\end{aligned}$$

It is easy to see (Hurwitz-criterion) that the polynomial q *has all zeros in the open left half plane* $\mathbb{C}_{\mathrm{Re}<0}$. *Therefore* G *is holomorphic and bounded on* $\mathbb{C}_{\mathrm{Re}>0}$, *i.e.* $G|_{\mathbb{C}_{\mathrm{Re}>0}} \in H_\infty^{2\times 2}$. *Since* G *is continuous on* $\mathbb{C}_{\mathrm{Re}} \geq 0$, *Proposition 17 yields*

$$\left\|G|_{\mathbb{C}_{\mathrm{Re}>0}}\right\|_{H_\infty^{2\times 2}} = \sup_{i\omega\in i\mathbb{R}} \left\|G(i\omega)\right\|_{2\times 2}.$$

For the parameters $m_1 = 1$, $m_2 = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$, $d = 5$ *and* $g = 9.81$ *the function* $\omega \mapsto \|G(i\omega)\|_{2\times 2}$ *is plotted in Figure 8.*

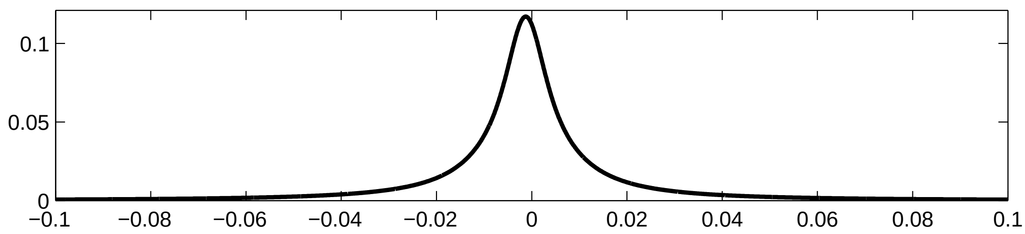

Figure 8: The norm $\|G(i\omega)\|_{2\times 2}$ plotted against $\omega \in \mathbb{R}$.

The maximum of $\omega \mapsto \|G(i\omega)\|$ *is achieved at* $\omega = 0$ *and has a value of approximately* 0.12. *Proposition 27 only yields that the gain of the example system is not bigger than* 0.12. *However Proposition 27 does not answer the question whether this value is a good approximation of the gain of the considered system.*

As mentioned in the Introduction, the value of the operator norm of the multiplication operator is important in many application, since it is a measure of how strong an input is amplified. Interpreting the H_2*-norm as the energy of the input and output, a value greater one for the norm of the multiplication operator means that the system can "produce" energy, because there exist inputs such that the energy of the output is higher than the energy of the input. Obviously the value of the norm of the multiplication operator cannot easily be computed directly. Whereas Proposition 27 shows that the* H_∞*-norm of the transfer matrix* G *is an upper bound of the gain of a given system, the following theorem shows that under certain assumptions on the transfer matrix* G *these two values do actually coincide.*

Theorem 28
Let $n, m \in \mathbb{N}$ and suppose that $\boxed{G \in H_\infty^{n\times m}}$ satisfies

$$\forall\, \varepsilon > 0\, \exists\, i\omega_0 \in i\mathbb{R}\backslash\{0\} : \left\{ \begin{array}{l} \|\mathcal{B}_\infty^+\{G\}(i\omega_0)\|_{n\times m} \geq \|G\|_{H_\infty^{n\times m}} - \varepsilon \\ \text{and } \boxed{\mathcal{B}_\infty^+\{G\} \text{ is continuous at } i\omega_0}. \end{array} \right\} \tag{11}$$

Then

$$\left\|\mathcal{M}_G^{H_2}\right\|_{\text{op}} = \|G\|_{H_\infty^{n\times m}}.$$

This theorem without assumption (11) *can be found in (Francis, 1987), Theorem 2 in Sub-Section 2.4, but no proof is given and it is mentioned there that "A complete proof of Theorem 4.2 is not readily available in the literature". Although there are proofs now available, e.g. in (Zhou* et al.*, 1996), Remark 4.2 and in (Vinnicombe, 2001), Sub-Section 1.1.2, a proof is given* *in this work. The reason for this is that the cited proofs are not convincing and it is not clear how to fix the gaps in their setup. This is discussed in detail in Remark 32.*

Proof.
Suppose that $\|G\|_{H_\infty^{n\times m}} > 0$, otherwise the claim is obvious.

It will be shown that

$$\forall\,\varepsilon > 0\ \exists\, F_\varepsilon \in H_2^m\ :\ \|F_\varepsilon\|_{H_2^m} = 1 \ \text{ and } \ \|GF_\varepsilon\|_{H_2^n} > \|G\|_{H_\infty^{n\times m}} - \varepsilon. \tag{12}$$

It then followed, together with Proposition 27, that $\|G\|_{H_\infty^{n\times m}}$ were the least upper bound for $\|\mathcal{M}_G^{H_2}\|_{\text{op}}$ and the proof were complete.

Fix $\varepsilon \in (0, \|G\|_{H_\infty})$ and let $\tilde{G} := \mathcal{B}_\infty^+\{G\}$.

STEP 1: Showing existence of $\gamma > 0$, $\omega_0 \in \mathbb{R}\backslash\{0\}$ and $v_1 \in \mathbb{C}^m$ such that

$$\|\tilde{G}(i\omega)v_1\|_n^2 > \|G\|_{H_\infty^{n\times m}}^2 - \varepsilon/2 \quad \forall\,\omega \in (-\gamma + \omega_0, \omega_0 + \gamma). \tag{13}$$

Note that

$$\|\tilde{G}(i\omega)\|_{n\times m} = \sigma_{\max}\big[\tilde{G}(i\omega)\big] \quad \forall\, i\omega \in i\mathbb{R}$$

and let $\tilde{G}(i\omega)$ be presented as a singular value decomposition (SVD, see e.g. Theorem 3.1.1 in (Horn and Johnson, 1991)),

$$\tilde{G}(i\omega) = \sigma_{\max}\big[\tilde{G}(i\omega)\big]u_1(i\omega)v_1(i\omega)^* + \sum_{k=2}^{r(i\omega)} \sigma_k(i\omega)u_k(i\omega)v_k(i\omega)^*, \tag{14}$$

where $u_1(i\omega), \ldots, u_{r(i\omega)}(i\omega) \in \mathbb{C}^n$ and $v_1(i\omega), \ldots, v_{r(i\omega)}(i\omega) \in \mathbb{C}^m$ are orthonormal vectors, $r(i\omega) \in \mathbb{N}$ is the rank of the matrix $\tilde{G}(i\omega)$ and

$$\sigma_{\max}\big[\tilde{G}(i\omega)\big] \geq \sigma_2(i\omega) \geq \ldots \geq \sigma_{r(i\omega)}(i\omega) > 0$$

are the singular values. Then

$$\|\tilde{G}(i\omega)v_1(i\omega)\|_n = \sigma_{\max}\big[\tilde{G}(i\omega)\big] \quad \big(= \|\tilde{G}(i\omega)\|_{n\times m}\big) \quad \forall\, i\omega \in i\mathbb{R}. \tag{15}$$

Note that $\|G\|_{H_\infty} = \|\tilde{G}\|_{L_\infty(i\mathbb{R})}$ by Proposition 17. Invoking Property (11), choose $i\omega_0 \in i\mathbb{R}\backslash\{0\}$ such that $\tilde{G}$ is continuous at $i\omega_0$ and

$$\|\tilde{G}(i\omega_0)\|_{n\times m}^2 > \|G\|_{H_\infty}^2 - \varepsilon/8. \tag{16}$$

Let $v_1 := v_1(i\omega_0)$, then, by continuity of $\tilde{G}$ at $i\omega_0$,

$$i\omega \mapsto \|\tilde{G}(i\omega)v_1\|_n$$

is continuous at $i\omega_0$. Hence, together with (16), one may choose $\gamma > 0$ such that (13) is fulfilled.

Fix $\omega_0 \neq 0$ and $\gamma > 0$.

In the following $F_\varepsilon \in H_2^m$ is constructed so that it has the most energy in the direction of v_1 and at the frequency ω_0. Therefore, scalar functions are defined which approximates the Dirac impulse on $i\mathbb{R}$ at the point $i\omega_0$. Let $(F_k) \in L_2(i\mathbb{R}, \mathbb{C})^{\mathbb{N}}$ be a scalar function sequence, then

"*(F_k) approximates the Dirac impulse at $i\omega_0$*"
$:\Leftrightarrow$

$$\left[\forall\, \gamma > 0 : \lim_{k\to\infty} \int_{|\omega-\omega_0|>\gamma} \left|F_k(i\omega)\right|^2 \mathrm{d}\omega = 0\right] \textit{ and } \left[\forall\, k \in \mathbb{N} : \|F_k\|_{L_2(i\mathbb{R})} = 1\right].$$

STEP 2: It is shown constructively that, for $k \in \mathbb{N}_{>0}$, there exist scalar functions

$$S_k \in H_2^1 : \ \|S_k\|_{H_2^1} = 1 \text{ and } \lim_{k\to\infty} \int_{|\omega-\omega_0|\geq\gamma} |\mathcal{B}_2^+\{S_k\}(i\omega)|^2 \mathrm{d}\omega = 0. \tag{17}$$

Define, for $k \in \mathbb{N}_{>0}$,

$$s_k : \mathbb{R} \to \mathbb{C}, \quad t \mapsto \begin{cases} \xi_k e^{i\omega_0 t}, & \text{for } t \in \left[0, \frac{2k\pi}{|\omega_0|}\right], \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\xi_k := \sqrt{\frac{|\omega_0|}{2k\pi}}$$

is a scaling factor. Then

$$s_k \in L_2(\mathbb{R}_{\geq 0}, \mathbb{C}) \cap L_1(\mathbb{R}_{\geq 0}, \mathbb{C}) \quad \text{with} \quad \|s_k\|_{L_2(\mathbb{R}_{\geq 0})} = 1,$$

and

$$\xi_k \to 0 \quad \text{as } k \to \infty. \tag{18}$$

Now Proposition 23 yields

$$S_k := \mathcal{L}\{s_k\} \in H_2^1 \quad \text{and} \quad \|S_k\|_{H_2^1} = 1$$

and furthermore

$$S_k(s) = \int_{\mathbb{R}} e^{-st} s_k(t) \mathrm{d}t = \xi_k \frac{e^{\frac{2k\pi}{|\omega_0|}(i\omega_0 - s)} - 1}{i\omega_0 - s}, \quad \text{for } s \in \mathbb{C}_{\mathrm{Re}>0}.$$

Note that S_k has a continuous extension $\tilde{S}_k = \mathcal{B}_2^+\{S_k\} : i\mathbb{R} \to \mathbb{C}$ with

$$\tilde{S}_k(i\omega) = \begin{cases} \sqrt{\frac{2k\pi}{|\omega_0|}}\ e^{-\frac{k\pi}{|\omega_0|} i\omega}\ \frac{\sin\left(\frac{k\pi}{|\omega_0|}(\omega-\omega_0)\right)}{\frac{k\pi}{|\omega_0|}(\omega-\omega_0)}, & \omega \neq \omega_0, \\ \sqrt{\frac{2k\pi}{|\omega_0|}}, & \omega = \omega_0. \end{cases}$$

A simple calculation gives

$$\int_{|\omega-\omega_0|\geq\gamma} |\tilde{S}_k(i\omega)|^2 \mathrm{d}\omega \leq \int_{|\omega-\omega_0|\geq\gamma} \frac{4(\xi_k)^2}{|\omega-\omega_0|^2} \mathrm{d}\omega = \frac{8(\xi_k)^2}{\gamma} \quad \forall\, \gamma > 0,$$

and therefore (17) follows from (18).

STEP 3: Showing existence $F_\varepsilon \in H_2^m$ with $\|F_\varepsilon\|_{H_2^m} = 1$ and

$$\|GF_\varepsilon\|_{H_2^n} \geq \|G\|_{H_\infty^{n\times m}} - \varepsilon.$$

By (18) it is possible to choose $k_0 \in \mathbb{N}$ sufficiently large so that

$$\frac{8(\xi_{k_0})^2}{\gamma} < \frac{\varepsilon}{2\|G\|_{H_\infty}^2}. \tag{19}$$

The desired function F_ε is then

$$\boxed{F_\varepsilon : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}^m, \quad s \mapsto S_{k_0}(s) v_1}$$

and it satisfies

$$F_\varepsilon \in H_2^m \quad \text{and} \quad \|F_\varepsilon\|_{H_2^m} = \|S_{k_0}\|_{H_2^1} \|v_1\|_m = 1.$$

Note that $\tilde{F}_\varepsilon := \mathcal{B}_2^+\{F_\varepsilon\} = \tilde{S}_{k_0} v_1$.

Now

$$\begin{aligned}
\|GF_\varepsilon\|_{H_2^n}^2 &\overset{\text{Prop.17}}{=} \int_{\mathbb{R}} \|\tilde{G}(i\omega)\tilde{F}_\varepsilon(i\omega)\|_n^2 \mathrm{d}\omega \\
&\geq \int_{|\omega-\omega_0|<\gamma} \|\tilde{G}(i\omega)v_1\|_n^2 |\tilde{S}_{k_0}(i\omega)|^2 \mathrm{d}\omega \\
&\overset{(13)}{>} \int_{|\omega-\omega_0|<\gamma} (\|G\|_{H_\infty^{n\times m}}^2 - \varepsilon/2) |\tilde{S}_k(i\omega)|^2 \mathrm{d}\omega \\
&= (\|G\|_{H_\infty}^2 - \varepsilon/2) \left(\|S_k\|_{H_2^1} - \int_{|\omega-\omega_0|\geq\gamma} |\tilde{S}_k(i\omega)|^2 \mathrm{d}\omega \right) \\
&\overset{(17)}{>} (\|G\|_{H_\infty}^2 - \varepsilon/2) \left(1 - \frac{8\xi_k}{\gamma} \right) \\
&\overset{(19)}{>} \|G\|_{H_\infty}^2 - \varepsilon
\end{aligned}$$

This completes the proof of the theorem. qed

Remark 29
It is not clear whether there actually exists $G \in H_\infty^{n\times m}$ such that Property (11) is not fulfilled, or whether

$$\left\{ \; G \in H_\infty^{n\times m} \;\mid G \text{ fulfils (11) } \right\}$$

is at least dense in $H_\infty^{n\times m}$.

It is easy to see, that for example the class of real rational functions bounded on $\mathbb{C}_{\text{Re}>0}$ (see Definition 33) fulfil Assumption (11). Transfer functions $G \in H_\infty^{n\times m}$ which are elements of the *Callier-Desoer class* $\mathcal{M}\hat{\mathcal{B}}(0)$ defined in (Curtain and Zwart, 1995), Section 7.2 are, by (Curtain and Zwart, 1995), Property 7.1.7, continuous on the imaginary axis and fulfil therefore Assumption (11).

It is stressed that there exists $\tilde{G} \in L_\infty(i\mathbb{R}, \mathbb{C})$ such that Property (11) is not fulfilled for any representation of $\tilde{G}$: consider for example

$$\chi_Q : i\mathbb{R} \to \{0, 1\}$$

where $Q \subset i\mathbb{R}$ is defined as (assume that the rational numbers $\mathbb{Q}$ are the numbers $\{q_1, q_2, \ldots\}$)

$$Q := i\mathbb{R} \Big\backslash \bigcup_{q_k \in \mathbb{Q}} \mathbb{B}_{2^{-k}}(iq_k).$$

Then it is easy to see that $\lambda(Q) \leq 1$ and therefore

$$\underset{i\omega \in i\mathbb{R}}{\text{ess-sup}} |\chi_Q(i\omega)| = 1.$$

Let $\varepsilon \in (0, 1)$ and $i\omega_0 \in i\mathbb{R}$ with $|\chi_Q(i\omega_0)| > \|\chi_Q\|_{L_\infty(i\mathbb{R})} - \varepsilon$ (in fact this implies $\chi_Q(i\omega_0) = 1$). Therefore the first property of (11) holds. Choose a sequence of rational numbers $(q_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $iq_n \to i\omega_0$ as $n \to \infty$. Observe that $\chi_Q(iq) = 0$ for all rational numbers $q \in \mathbb{Q}$, therefore χ_Q cannot be continuous at $i\omega_0$, and hence the second property in (11) does not hold. Since $i\mathbb{R}\backslash Q$ is open and dense in $i\mathbb{R}$ this is also true if χ_Q is changed on a Lebesgue zero-set.

The following theorem can be found in (Francis, 1987), Theorem 2 in Sub-Section 2.4, and in (Zhou et al.*, 1996), Theorem 4.4. Like Theorem 28 in (Francis, 1987) a proof is not given (it is stated there, that a proof is not available yet) and the proof given in (Zhou* et al.*, 1996) is not convincing (see Remark 32, which mainly deals with Theorem 28, but also apply here).*

Theorem 30
Let $n, m \in \mathbb{N}$ and $\boxed{\tilde{G} \in L_\infty^{n\times m}}$, then

$$\left\| \mathcal{M}_{\tilde{G}}^{L_2} \right\|_{\text{op}} = \|\tilde{G}\|_{L_\infty^{n\times m}}.$$

The result of Theorem 30 compared to Theorem 28 might be surprising, since extra assumptions are not required. However, most problems in the proof of Theorem 28 occur because of the "bad" properties of the function

space L_∞. The essential difference between both theorems is the space of "input functions". In Theorem 30, all functions $\tilde{F} \in L_2^m$ are allowed and therefore the "bad" properties of L_2 functions can compensate those of the L_∞ function.
The main idea of the proof to Theorem 30 is again to construct an input function $\tilde{F}_\varepsilon$ which has the most energy in the direction and at the frequencies where $\tilde{G}$ takes its largest values. This is done by a kind of approximation to the Dirac-impulse which depends on the transfer matrix $\tilde{G}$. In contrast to this, the approximation of the Dirac-impulse in the proof of Theorem 28 only depends on the frequency ω_0 where the Dirac-impulse should be located.
Since $\tilde{G}$ is not necessarily continuous at $i\omega_0$ it is not sufficient to consider only one fixed direction $v(i\omega_0) \in \mathbb{C}^m$ as in the proof of Theorem 28. One is interested in the value $\|\tilde{G}(i\omega)\|_{n\times m} = \|\tilde{G}(i\omega)v_1(i\omega)\|_n$, where $v_1(i\omega)$ is the first singular vector of $\tilde{G}(i\omega)$ (see SVD, (14)). Therefore, if the mapping $i\omega \mapsto v_1(i\omega)$ should be used in the construction of $\tilde{F}_\varepsilon$, then it must be at least Lebesgue measurable. That the first singular value singular value can be chosen measurably shows the following lemma.

Lemma 31
There exists a Lebesgue measurable mapping

$$v_1 : \mathbb{C}^{n\times m} \to \mathbb{S}^{m-1}, \quad A \mapsto v_1(A)$$

such that

$$\|Av_1(A)\| = \|A\| \quad \forall\, A \in \mathbb{C}^{n\times m}.$$

Proof. The mapping

$$\mathcal{S} : \mathbb{C}^{n\times m} \times \mathbb{S}^{m-1} \to \mathbb{R}, \quad (A, v) \mapsto \|Av\|_n - \|A\|_{n\times m},$$

is continuous. Then

$$\begin{aligned} \mathcal{S}^{-1}(\{0\}) &= \left\{ \ (A, v) \in \mathbb{C}^{n\times m} \times \mathbb{S}^{m-1} \ \mid \|Av\|_n = \|A\|_{n\times m} \ \right\} \\ &\in \mathfrak{B}(\mathbb{C}^{n\times m}) \otimes \mathfrak{B}(\mathbb{S}^{m-1}) \\ &\subseteq \mathfrak{L}(\mathbb{C}^{n\times m}) \otimes \mathfrak{B}(\mathbb{S}^{m-1}), \end{aligned}$$

where $\mathfrak{B}(\mathbb{C}^{n\times m})$ and $\mathfrak{B}(\mathbb{S}^{m-1})$ are the Borel σ-fields of $\mathbb{C}^{n\times m}$ and $\mathbb{S}^{m-1}$, resp., and $\mathfrak{L}(\mathbb{C}^{n\times m})$ is the Lebesgue σ-field of $\mathbb{C}^{n\times m}$.

The graph of the set-valued map (or multifunction)

$$V_1 : \mathbb{C}^{n\times m} \to \mathfrak{P}(\mathbb{S}^{m-1})\backslash\{\emptyset\}, \quad A \mapsto \left\{ \ v \in \mathbb{S}^{m-1} \ \mid \|Av\|_m = \|A\|_{n\times m} \ \right\}$$

is then $\mathcal{S}^{-1}(\{0\})$. By completeness of $\mathfrak{L}(\mathbb{C}^{n\times m})$ under the Lebesgue measure Theorem III.30 together with Theorem III.6 in (Castaing and Valadier, 1977) yields that for V_1 there exists a measurable selection

$$v_1 : \mathbb{C}^{n\times m} \to \mathbb{S}^{m-1},$$

i.e. v_1 is Lebesgue measurable and $v_1(A) \in V_1(A)$ for all $A \in \mathbb{C}^{n\times m}$. qed

Proof of Theorem 30 Set, for $\varepsilon > 0$,

$$\begin{aligned} S_\varepsilon &:= \left\{ i\omega \in i\mathbb{R} \;\middle|\; \|\tilde{G}(i\omega)\|_{n\times m} > \|\tilde{G}\|_{L_\infty(i\mathbb{R})} - \varepsilon \right\} \\ &= (i\omega \mapsto \|\tilde{G}(i\omega)\|_{n\times m})^{-1}\big((\|\tilde{G}\|_{L_\infty(i\mathbb{R})} - \varepsilon, \infty)\big), \end{aligned}$$

which is a Lebesgue measurable set with positive measure. If the measure of S_ε is infinite then consider, instead of S_ε, the set $S_\varepsilon \cap \mathbb{B}_R(0)$ for sufficiently large $R > 0$. Let $\delta \in (0, \infty)$ be the measure of S_ε.

Define

$$\boxed{\tilde{F}_\varepsilon : i\mathbb{R} \to \mathbb{C}^m, \quad i\omega \mapsto \sqrt{\frac{2\pi}{\delta}}\chi_{S_\varepsilon} v_1\big(\tilde{G}(i\omega)\big)},$$

where

$$v_1 : \mathbb{C}n \times m \to \mathbb{C}^m$$

is defined as in Lemma 31. Since $\tilde{G}$ is Lebesgue measurable, it follows from Lemma 31 that $\tilde{F}_\varepsilon$ is Lebesgue measurable. Furthermore,

$$\tilde{F}_\varepsilon \in L_2^m \quad \text{with} \quad \|\tilde{F}_\varepsilon\|_{L_2^m} = 1.$$

Now

$$\begin{aligned} \|\tilde{G}\tilde{F}_\varepsilon\|_{L_2(i\mathbb{R})} &= \frac{1}{2\pi}\int_{\mathbb{R}} \|\tilde{G}(i\omega)\tilde{F}_\varepsilon(i\omega)\|_n^2 \mathrm{d}\omega \\ &= \frac{1}{\delta}\int_{S_\varepsilon} \|\tilde{G}(i\omega)v_1\big(\tilde{G}(i\omega)\big)\|_n^2 \mathrm{d}\omega \\ &= \frac{1}{\delta}\int_{S_\varepsilon} \|\tilde{G}(i\omega)\|_{n\times m} \mathrm{d}\omega \\ &> \|\tilde{G}\|_{L_\infty^{n\times m}} - \varepsilon. \end{aligned}$$

In Proposition 27 it was already shown that $\|\tilde{G}\|_{L_\infty^{n\times m}}$ is an upper bound for $\left\|\mathcal{M}_{\tilde{G}}^{L_2}\right\|_{\mathrm{op}}$ and therefore the proof is complete. qed

Remark 32

Theorem 4.4 (with Remark 4.2) in (Zhou *et al.*, 1996) and Sub-Section 1.1.2 in (Vinnicombe, 2001) are both stating the result of Theorem 28:

$$\left\|\mathcal{M}_G^{H_2}\right\|_{\mathrm{op}} = \|G\|_{H_\infty^{n\times m}}.$$

A careful inspection of the proofs given there shows that they do not prove the general case $G \in H_\infty^{n\times m}$, but only the special case that G is a real rational function (see Definition 33 and Corollary 35). In addition, their proofs have big gaps and it is not obvious how to close them. The following problems occur in both proofs

(i) *Choosing* $\omega_0 \in \mathbb{R}$ *such that* $\sigma_{\max}\big[\tilde{G}(i\omega_0)\big] = \|G\|_{H_\infty^{n\times m}}$.

In (Vinnicombe, 2001), p. 6, it is mentioned that "there will exist an ω_0 such that $\overline{\sigma}\big(P(j\omega_0)\big) = \gamma$ $\Big[$*i.e.* $\sigma_{\max}\big[\tilde{G}(i\omega_0)\big] = \|G\|_{H_\infty^{n\times m}}\Big]$" and (Zhou *et al.*, 1996), p. 101, start the proof with "choose a frequency ω_0 where $\overline{\sigma}\big[G(i\omega)\big]$ $\Big[= \sigma_{\max}\big[\tilde{G}(i\omega_0)\big]\Big]$ is maximum". For a transfer function $G \in H_\infty^{n\times m}$, Proposition 17 only gives $\tilde{G} \in L_\infty(i\mathbb{R}, \mathbb{C}^{n\times m})$. The elements of the Lebesgue-space L_∞ are equivalence classes and it is always only a representative which is considered. Therefore it cannot be assumed that the essential supremum is achieved at some point $i\omega_0 \in i\mathbb{R}$. But even if one only considers piecewise continuous functions, it can not be guaranteed, that one can choose ω_0 such that $\sigma_{\max}\big[\tilde{G}(i\omega_0)\big] = \|G\|_{H_\infty}$. Consider for example

$$g : \mathbb{R} \to \mathbb{R}, \quad \begin{cases} t & \text{for } t \in (0,1), \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Even for continuous transfer matrices it is possible that the maximum is only achieved at $\omega_0 = \infty$ which leads to technical difficulties.

(ii) *Approximating the Dirac impulse for "picking" the value* $\|\tilde{G}(i\omega_0)\|_{n\times m}$.

Consider a sequence of functions $(f_k) \in L_2(\mathbb{R}, \mathbb{C})^{\mathbb{N}}$ with $\|f_k\|_{L_2(\mathbb{R})} = 1$ and an analogous property as (17). It is easy to see, that for a continuous (at least at t_0) function $g : \mathbb{R} \to \mathbb{C}$,

$$\int_{\mathbb{R}} |g(t) f_k(t)|^2 \mathrm{d}t \to |g(t_0)|^2 \quad \text{as } k \to \infty.$$

If g is not continuous, then it is not necessarily possible to "pick" the value $g(t_0)$ with approximations of the Dirac-impulse (in particular if the same approximation sequence (f_k) is used for a whole class of functions g). Consider for example again the function g as defined in (20) and a sequence of functions $(f_k) \in L_2(\mathbb{R}, \mathbb{C})^{\mathbb{N}}$ which approximate the Dirac-impulse at $t_0 = 1$. If the limit $\lim_{k\to\infty} \int_{\mathbb{R}} |g(t) f_k(t)|^2 \mathrm{d}\omega$ exists at all, the value of it depends on the concrete approximation sequence $(f_k)_{k\in\mathbb{N}}$.

(iii) *Assuming that* $\|\tilde{G}(i\omega)\|_{n\times m} = \|\tilde{G}(-i\omega)\|_{n\times m}$.

In both proofs a sequence of functions which approximates the Dirac-impulse is constructed. In (Vinnicombe, 2001) these are basically

$$F_k = \mathcal{L}\{f_k\}$$

with

$$f_k : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \chi_{[0,2k\pi/\omega_0]}(t) \sin(\omega_0 t) = \chi_{[0,2k\pi/\omega_0]}(t) \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$

and in (Zhou *et al.*, 1996)

$$F_k = c_k \chi_{(-\varepsilon_k+\omega_0,\omega_0+\varepsilon_k)\cup(-\varepsilon_k-\omega_0,-\omega_0+\varepsilon_k)}$$

where $\varepsilon_k \to 0$ as $k \to \infty$ and $c_k = \sqrt{\pi/2\varepsilon_k}$. These sequences do not approximate a single Dirac-impulse at ω_0 but the sum of two Dirac-impulses at $-\omega_0$ and ω_0. If $\|\tilde{G}(i\omega_0)\|_{n\times m} \neq \|\tilde{G}(-i\omega_0)\|_{n\times m}$, which is the general case, this would not lead to the desired result. (Zhou *et al.*, 1996) restrict themselves to the case that $\|\tilde{G}(i\omega)\|_{n\times m} = \|\tilde{G}(-i\omega)\|_{n\times m}$ and it is not clear how the general case can be treated.

The underlying problem is that both (Vinnicombe, 2001) and (Zhou *et al.*, 1996) try to construct a sequence of *real valued* functions in such a way that the Laplace transform is the desired sequence of functions, i.e. approximating the Dirac-impulse. But it is easy to see, that the Laplace transform F of a real valued function f satisfies

$$F(\overline{s}) = \overline{F(s)},$$

(the converse is true, too) and therefore it is not possible to construct a sequence of real valued functions with its Laplace transforms approximating a single Dirac-impulse at $\omega_0 \neq 0$.

Note that in the proof to Theorem 28 the approximation of the Dirac impulse has basically the form

$$f_k : \mathbb{R} \to \mathbb{C}, \quad t \mapsto \chi_{[0,2k\pi/\omega_0]}(t) e^{i\omega_0 t}$$

which is not real valued at all. But under the assumption $\|\tilde{G}(i\omega)\|_{n\times m} = \|\tilde{G}(-i\omega)\|_{n\times m}$ essentially the same proof with the real valued f_k as suggested in (Vinnicombe, 2001) will show the assertion of Theorem 28.

In addition to the problems described above, in (Vinnicombe, 2001) it is not clear why the inequality

$$\|y_k - Pu_k\|_2 < \beta$$

is true for some $\beta > 0$ and for all $k \in \mathbb{N}$.

In (Zhou *et al.*, 1996), Remark 4.2, it is assumed that the function $f : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}^q$ defined in the proof of Theorem 4.4 is in $H_2(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C}^q)$. It is not clear whether a scalar function $\hat{f} \in H_2(\mathbb{C}_{\mathrm{Re}>0}, \mathbb{C}^q)$ exists with

$$|\hat{f}| = c\chi_{(-\varepsilon-\omega_0,-\omega_0+\varepsilon)\cup(-\varepsilon+\omega_0,\omega_0+\varepsilon)},$$

which is claimed and necessary for the assertion of Remark 4.2.

This work ends with the special case that the transfer matrix G is a real rational transfer matrix, which is an important case, because as mentioned in the Introduction, every linear system described by linear differential equations (1) *has a real rational transfer matrix, see also Example 2.*

Definition 33
The real rational subspace of $H_\infty^{n\times m}$ is defined as

$$\mathcal{R}H_\infty^{n\times m} := \left\{ G \in H_\infty^{n\times m} \;\middle|\; \begin{array}{l} \big[G(s)\big]_{kl} = \frac{p_{kl}(s)}{q_{kl}(s)} \text{ for all } s \in \mathbb{C}_{\mathrm{Re}>0}, \\ p_{kl}, q_{kl} \text{ polynomials with real coefficients,} \\ k = 1,\ldots,n,\ l = 1,\ldots,m \end{array} \right\}.$$

Analogously the real rational subspace of $L_\infty(i\mathbb{R}, \mathbb{C}^{n\times m})$ is defined as

$$\mathcal{R}L_\infty^{n\times m} := \left\{ \tilde{G} \in L_\infty^{n\times m} \;\middle|\; \begin{array}{l} \big[\tilde{G}(i\omega)\big]_{kl} = \frac{p_{kl}(i\omega)}{q_{kl}(i\omega)} \text{ for all } i\omega \in i\mathbb{R}, \\ p_{kl}, q_{kl} \text{ polynomials with real coefficients,} \\ k = 1,\ldots,n,\ l = 1,\ldots,m \end{array} \right\}.$$

Remark 34
It is easy to see that $\mathcal{R}H_\infty^{n\times m}$ are exactly those real rational transfer matrices which entries do not have any poles in $\mathbb{C}_{\mathrm{Re}\geq 0}$ and are proper, i.e. the degree of denominator is not smaller than that of the numerator.
The real rational transfer matrices in $\mathcal{R}L_\infty^{n\times m}$ are those which do not have any poles on the imaginary axis $i\mathbb{R}$.

The following Corollary formulates the results which are covered by the proofs given in (Vinnicombe, 2001) and (Zhou et al.*, 1996). The proof of (Vinnicombe, 2001) is convincing if $G \in \mathcal{R}H_\infty^{n\times m}$, and the case $\tilde{G} \in \mathcal{R}L_\infty^{n\times m}$ is proven in (Zhou* et al.*, 1996). It is worth noting that the "worst case" input functions can be chosen to be real valued. The problems mentioned in Remark 32 do not apply for these two cases, because real rational transfer functions without poles on $i\mathbb{R}$ are continuous on $i\mathbb{R}$ and fulfil $\|\tilde{G}(i\omega)\|_{n\times m} = \|\tilde{G}(-i\omega)\|_{n\times m}$ for all $i\omega \in i\mathbb{R}$.*

Corollary 35

(i) Let $n, m \in \mathbb{N}$ and $\boxed{G \in \mathcal{R}H_\infty^{n\times m}}$. Then

$$\left\|\mathcal{M}_G^{H_2}\right\|_{\mathrm{op}} = \|G\|_{H_\infty^{n\times m}}$$

and for every $\varepsilon > 0$ there exists a real (vector-)valued function $f_\varepsilon \in L_2(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ with unit norm such that its Laplace transform $F_\varepsilon \in H_2^m$ satisfies

$$\big| \|GF_\varepsilon\|_{H_2^n} - \|G\|_{H_\infty^{n\times m}} \big| < \varepsilon.$$

(ii) Let $n, m \in \mathbb{N}$ and $\boxed{\tilde{G} \in \mathcal{R}L_\infty^{n\times m}}$. Then

$$\left\|\mathcal{M}_{\tilde{G}}^{L_2}\right\|_{\mathrm{op}} = \|\tilde{G}\|_{L_\infty^{n\times m}}$$

and for every $\varepsilon > 0$ there exists a real (vector-)valued function $f_\varepsilon \in L_2(\mathbb{R}, \mathbb{R}^m)$ with unit norm such that its Fourier transform $\tilde{F}_\varepsilon \in L_2^m$ satisfies

$$\big| \|\tilde{G}\tilde{F}_\varepsilon\|_{L_2^n} - \|\tilde{G}\|_{H_\infty^{n\times m}} \big| < \varepsilon.$$

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