The Quasi-Weierstraß form for regular matrix pencils

Thomas Berger* **Achim Ilchmann*** **Stephan Trenn†**

first submission: July 2009; revision: October 2009

Dedicated to Heinrich Voß
on the occasion of his 65$^{\text{th}}$ birthday

Abstract

Regular linear matrix pencils $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$, where $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and the associated differential algebraic equation (DAE) $E\dot{x} = Ax$ are studied. The Wong sequences of subspaces are investigate and invoked to decompose the $\mathbb{K}^n$ into $\mathcal{V}^* \oplus \mathcal{W}^*$, where any bases of the linear spaces $\mathcal{V}^*$ and $\mathcal{W}^*$ transform the matrix pencil into the Quasi-Weierstraß form. The Quasi-Weierstraß form of the matrix pencil decouples the original DAE into the underlying ODE and the pure DAE or, in other words, decouples the set of initial values into the set of consistent initial values $\mathcal{V}^*$ and "pure" inconsistent initial values $\mathcal{W}^* \setminus \{0\}$. Furthermore, $\mathcal{V}^*$ and $\mathcal{W}^*$ are spanned by the generalized eigenvectors at the finite and infinite eigenvalues, resp. The Quasi-Weierstraß form is used to show how chains of generalized eigenvectors at finite and infinite eigenvalues of $A - E\partial$ lead to the well-known Weierstraß form. So the latter can be viewed as a generalized Jordan form. Finally, it is shown how eigenvector chains constitute a basis for the solution space of $E\dot{x} = Ax$.

Keywords: Linear matrix pencils, differential algebraic equations, generalized eigenspaces, Weierstraß form, Quasi-Weierstraß form

1 Introduction

We study linear matrix pencils of the form

$$A - E\partial \in \mathbb{K}^{n\times n}[\partial], \qquad n \in \mathbb{N}, \qquad \text{where } \mathbb{K} \text{ is } \mathbb{Q}, \mathbb{R} \text{ or } \mathbb{C},$$

(assumed *regular* in most cases, i.e. $\det(A - E\partial) \neq 0 \in \mathbb{K}[\partial]$) and the associated differential algebraic equation (DAE)

$$E\dot{x} = Ax, \qquad x(0) = x^0 \in \mathbb{K}^n. \tag{1.1}$$

Our main result is the derivation of the spaces $\operatorname{im} V$ and $\operatorname{im} W$ so that the pencil $A - E\partial$ is transformed into the *Quasi-Weierstraß form*:

$$[EV, AW]^{-1}\,(A - E\partial)\,[V, W] = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}\partial,$$

where J is some matrix and N is nilpotent. This form is weaker than the classical Weierstraß form (where J and N have to be in Jordan form), albeit it contains, as we will show, relevant information such as:

*Institute of Mathematics, Ilmenau University of Technology, Weimarer Straße 25, 98693 Ilmenau, DE, {thomas.berger, achim.ilchmann}@tu-ilmenau.de

†Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1308 W. Main Street, Urbana, 61801, USA, trenn@illinois.edu

– It decouples the DAE $E\dot{x} = Ax$ into the differential and the algebraic part or, more precisely, into the classical ODE $\dot{v} = Jv$ and the pure DAE $N\dot{w} = w$.

– It decouples the set of initial values of the DAE into the set of consistent initial values $\operatorname{im} V$ and "pure" inconsistent initial values $\operatorname{im} W \setminus \{0\}$ in the sense that $\mathbb{K}^n = \operatorname{im} V \oplus \operatorname{im} W$.

– It allows for a vector space isomorphism between the set of consistent initial conditions and all solutions of the homogeneous DAE.

– It decouples the pencil $A - E\partial$ with respect to $\operatorname{im} V \oplus \operatorname{im} W$, where the vector spaces $\operatorname{im} V$ and $\operatorname{im} W$ are spanned by the generalized eigenvectors at the finite and infinite eigenvalues, resp.

The Quasi-Weierstraß form is, conceptually and practically, derived with little effort. There is no need to calculate the eigenvalues and (generalized) eigenvectors of the pencil $A - E\partial$. The spaces $\operatorname{im} V$ and $\operatorname{im} W$ are derived by a recursive subspace iteration in finitely many steps. If the pencil is real, rational or symbolic, then all calculation remain real, rational or symbolic, resp.
Moreover, the Quasi-Weierstraß form may be used to derive chains of generalized eigenvectors at finite and infinite eigenvalues of the pencil $A - E\partial$ which then constitute a basis transforming the pencil into the classical Weierstraß form. This derivation allows to view the Weierstraß form as a generalized Jordan form.

Many results of the present note can, more or less implicitly, be found in the literature; we refer to them. However, our contribution may offer a new view which leads to a simple and coherent analysis of matrix pencils and DAEs.

The paper is organized as follows. In Section 2.1, we study the unifying tool of Wong sequences leading to vector spaces $\mathcal{V}^*$ and $\mathcal{W}^*$; the latter constitute a basis transformation to convert the pencil $A - E\partial$ into Quasi-Weierstraß form. In Section 2.2, the relationship between the Drazin inverse and the Quasi-Weierstraß form is shown. In Section 2.3, we present a vector space isomorphism between $\mathcal{V}^*$ and the space of all solutions of $E\dot{x} = Ax$. This is then also used to derive, in terms of the matrices in the Quasi-Weierstraß form, a Variation-of-Constants variant for inhomogeneous DAEs $E\dot{x} = Ax + f$.
In Section 3.1, we consider chains of generalized eigenvectors at finite and infinite eigenvalues of the pencil $A - E\partial$. The main result is that $\mathcal{V}^*$ can be decomposed into the direct sum of $\mathcal{G}_{\lambda_i}$'s, the vector spaces spanned by generalized eigenvectors at finite eigenvalue λ_i. As an immediate consequence, $A - E\partial$ can be represented in terms of generalized eigenvectors and this is a generalized Jordan form for $A - E\partial$, the Weierstraß form. Finally we show that the chains at finite eigenvalues constitute basis functions spanning the solution space of $E\dot{x} = Ax$.

Nomenclature

$\mathbb{N}$	$:= \{0, 1, \ldots\}$ the set of natural numbers
$\mathbb{K}$	rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$
A^+	Moore-Penrose pseudo inverse $A^+ = (\bar{A}^\top A)^{-1}\bar{A}^\top$ of $A \in \mathbb{K}^{m\times n}$ with $\operatorname{rk} A = n$
I_n	$:= \operatorname{diag}\{1, \ldots, 1\} \in \mathbb{K}^{n\times n}$, or I if the dimension is clear from the context
$\operatorname{spec}(A - E\partial)$	$:= \{ \lambda \in \mathbb{C} \mid \det(A - \lambda E) = 0 \}$ for $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$
$A\mathcal{M}$	$:= \{ Ax \mid x \in \mathcal{M} \}$ the image of a set $\mathcal{M} \subseteq \mathbb{K}^n$ under $A \in \mathbb{K}^{n\times n}$
$A^{-1}\mathcal{M}$	$:= \{ x \in \mathbb{K}^n \mid Ax \in \mathcal{M} \}$ the pre-image of a set $\mathcal{M} \subseteq \mathbb{K}^n$ under $A \in \mathbb{K}^{n\times n}$

2 The Quasi-Weierstraß form

In this section we derive, via Wong sequences, the Quasi-Weierstraß form for regular matrix pencils $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$. This will be applied to characterize the solution space of the DAE $E\dot{x} = Ax$ and to derive a Variation-of-Constants formula for inhomogeneous DAEs $E\dot{x} = Ax + f$. All results hold for $\mathbb{K}$ either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, unless stated otherwise.

2.1 The Wong sequences and the Quasi-Weierstraß form

The following sequences of nested subspaces have been introduced in [25]. Apart from very few exceptions, the fundamental role of the Wong sequences has not been realized – neither in the Linear Algebra nor in the DAE community. The Wong sequences are the key to the Quasi-Weierstraß form. In [18] the sequence (2.1) is introduced as a "fundamental geometric tool" in studying some properties of DAEs, but its potential is not fully exploited. In [17] the sequence (2.1), but not (2.2), is considered for constructing projection operators to derive a decoupling of 1.1; in the notation of [17, p. 125] it is $\mathcal{V}_i = \pi_x \mathcal{G}_{i-1} = M_{i-1}$. Related results on the spaces $\mathcal{V}_i$ are considered for abstract infinite dimensional linear operators in [22, Sec. 3]. Some geometric results are derived in [6] via the Wong-sequences. A differential geometry approach is pursued in [23] for deriving a more sophisticated Weierstraß form; in the notation on page 83 it is $\mathcal{V}_i = M^i$; however, the complementary subspaces (2.2) are different.

Definition 2.1 (Wong sequences [25])**.** Let $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$. Then the sequences of subspaces

$$\boxed{\mathcal{V}_0 := \mathbb{K}^n\,, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i) \quad \forall\, i \in \mathbb{N}} \tag{2.1}$$

$$\boxed{\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i) \quad \forall\, i \in \mathbb{N}} \tag{2.2}$$

are called *Wong sequences.* ◇

It is easy to see that the Wong sequences are nested, terminate and satisfy

$$\left.\begin{array}{l} \exists\, k^* \in \mathbb{N}\ \forall\, j \in \mathbb{N} \ :\ \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} =: \mathcal{V}^* = A^{-1}(E\mathcal{V}^*) \supseteq \ker A, \\ \exists\, \ell^* \in \mathbb{N}\ \forall\, j \in \mathbb{N} \ :\ \mathcal{W}_0 \subseteq \ker E = \mathcal{W}_1 \subsetneq \cdots \subsetneq \mathcal{W}_{\ell^*} = \mathcal{W}_{\ell^*+j} =: \mathcal{W}^* = E^{-1}(A\mathcal{W}^*)\,, \end{array}\right\} \tag{2.3}$$

$$A\mathcal{V}^* \subseteq E\mathcal{V}^* \qquad \text{and} \qquad E\mathcal{W}^* \subseteq A\mathcal{W}^*\,. \tag{2.4}$$

Before we derive some elementary properties of the Wong sequences, a motivation in terms of the DAE (1.1) may be warranted.

Remark 2.2 (Motivation of the Wong sequences)**.** Suppose $x(\cdot)\colon \mathbb{R} \to \mathbb{K}^n$ is a classical (i.e. differentiable) solution of (1.1). Invoking the notation as in (2.1), (2.4) and the simple property that the linear spaces $\mathcal{V}_i$ are closed and thus invariant under differentiation, the following implications hold for all $t \in \mathbb{R}$:

$$\begin{array}{rcl} x(t) \in \mathbb{K}^n = \mathcal{V}_0 & \Longrightarrow & \dot{x}(t) \in \mathcal{V}_0 \stackrel{(1.1)}{\Longrightarrow} x(t) \in A^{-1}(E\mathcal{V}_0) = \mathcal{V}_1 \\ & \Longrightarrow & \dot{x}(t) \in \mathcal{V}_1 \stackrel{(1.1)}{\Longrightarrow} x(t) \in A^{-1}(E\mathcal{V}_1) = \mathcal{V}_2 \\ & \Longrightarrow & \text{etc.} \end{array}$$

Therefore, after finitely many iterations it is established that the solution $x(\cdot)$ must evolve in $\mathcal{V}^*$, i.e. $x(t) \in \mathcal{V}^*$ for all $t \in \mathbb{R}$.

The sequence of $\mathcal{W}_i's$ in (2.2) consists of *complementary* subspaces in the sense of (2.5). Although it may be that $\mathcal{V}_i \cap \mathcal{W}_i \supsetneq \{0\}$ as long as $i \in \{1, \ldots, k^* - 1\}$ (see Example 2.5), we can show (see Proposition 2.4 (ii)) that finally $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{K}^n$ and thus any non-trivial $x(\cdot)$ does not intersect with $\mathcal{W}^*$. $\diamond$

In the following Lemma 2.3 some elementary properties of the Wong sequences are derived, they are essential for proving basic properties of the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$ in Proposition 2.4. These results are inspired by the observation of Campbell [7, p. 37] who proves, for $\mathbb{K} = \mathbb{C}$, that the space of consistent initial values is given by $\operatorname{im}\big((A - \lambda E)^{-1}E\big)^{\nu}$ for any $\lambda \in \mathbb{C}\setminus \operatorname{spec}(A - E\partial)$ and $\nu \in \mathbb{N}$ the index of the matrix $(A - \lambda E)^{-1}E$, [7, p. 7]. However, Campbell did not consider the Wong sequences explicitly.

Lemma 2.3 (Properties of $\mathcal{V}_i$ and $\mathcal{W}_i$). *If $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ is regular, then the Wong sequences* (2.1) *and* (2.2) *satisfy*

$$\forall \lambda \in \mathbb{K}\setminus \operatorname{spec}(A - E\partial) \quad \forall i \in \mathbb{N} \ : \quad \mathcal{V}_i = \operatorname{im}\big((A - \lambda E)^{-1}E\big)^i, \quad \mathcal{W}_i = \ker\big((A - \lambda E)^{-1}E\big)^i.$$

In particular,

$$\forall i \in \mathbb{N} \ : \ \dim \mathcal{V}_i + \dim \mathcal{W}_i = n. \tag{2.5}$$

Proof: Since $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ is regular, let

$$\widehat{E} := (A - \lambda E)^{-1}E, \qquad \text{for arbitrary but fixed } \lambda \in \mathbb{K}\setminus \operatorname{spec}(A - E\partial). \tag{2.6}$$

Step 1: We prove by induction: $\mathcal{V}_i = \operatorname{im}\widehat{E}^i$ for all $i \in \mathbb{N}$.
Clearly, $\mathcal{V}_0 = \mathbb{K}^n = \operatorname{im}\widehat{E}^0$. Suppose that $\operatorname{im}\widehat{E}^i = \mathcal{V}_i$ holds for some $i \in \mathbb{N}$.
Step 1a: We show: $\mathcal{V}_{i+1} \supseteq \operatorname{im}\widehat{E}^{i+1}$.
Let $x \in \operatorname{im}\widehat{E}^{i+1} \subseteq \operatorname{im}\widehat{E}^i$. Then there exists $y \in \operatorname{im}\widehat{E}^i$ such that $x = (A - \lambda E)^{-1}Ey$. Therefore, $(A - \lambda E)x = Ey = E(y + \lambda x - \lambda x)$ and so, for $\widehat{y} := y + \lambda x \in \operatorname{im}\widehat{E}^i = \mathcal{V}_i$, we have $Ax = E\widehat{y}$. This implies $x \in \mathcal{V}^{i+1}$.
Step 1b: We show: $\mathcal{V}_{i+1} \subseteq \operatorname{im}\widehat{E}^{i+1}$.
Let $x \in \mathcal{V}_{i+1}$ and choose $y \in \mathcal{V}_i$ such that $Ax = Ey$. Then $(A - \lambda E)x = E(y - \lambda x)$ or, equivalently, $x = (A - \lambda E)^{-1}E(y - \lambda x)$. From $x \in \mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ it follows that $y - \lambda x \in \mathcal{V}_i = \operatorname{im}\widehat{E}^i$ and therefore $x \in \operatorname{im}\widehat{E}^{i+1}$.
Step 2: We prove by induction: $\mathcal{W}_i = \ker\widehat{E}^i$ for all $i \in \mathbb{N}$.
Clearly, $\mathcal{W}_0 = \{0\} = \ker\widehat{E}^0$. Suppose that $\ker\widehat{E}^i = \mathcal{W}_i$ for some $i \in \mathbb{N}$.
First observe that $(I + \lambda\widehat{E})$ restricted to $\ker\widehat{E}^i$ is an operator $(I + \lambda\widehat{E}) : \ker\widehat{E}^i \to \ker\widehat{E}^i$ with inverse $\sum_{j=0}^{i-1}(-\lambda)^j\widehat{E}^j$. Thus the following equivalences hold

$$\begin{aligned}
x \in \mathcal{W}_{i+1} \quad &\Longleftrightarrow \quad \exists\, y \in \mathcal{W}_i :\ Ex = Ay = (A - \lambda E)y + \lambda Ey \\
&\Longleftrightarrow \quad \exists\, y \in \mathcal{W}_i = \ker\widehat{E}^i :\ \widehat{E}x = (I + \lambda\widehat{E})y =: \widehat{y} \\
&\Longleftrightarrow \quad \exists\, \widehat{y} \in \ker\widehat{E}^i :\ \widehat{E}x = \widehat{y} \\
&\Longleftrightarrow \quad x \in \ker\widehat{E}^{i+1}.
\end{aligned}$$

$\square$

Next we prove important properties of the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$, some of which can be found in [25], but the present presentation is more straightforward.

Proposition 2.4 (Properties of $\mathcal{V}^*$ and $\mathcal{W}^*$)**.** *If $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ is regular, then $\mathcal{V}^*$ and $\mathcal{W}^*$ as in* (2.3) *satisfy:*

(i) $k^* = l^*$, *where k^*, l^* are given in* (2.3),

(ii) $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{K}^n$,

(iii) $\boxed{\ker E \cap \mathcal{V}^* = \{0\}}$ *and* $\boxed{\ker A \cap \mathcal{W}^* = \{0\}}$ *and* $\boxed{\ker E \cap \ker A = \{0\}}$.

Proof: (i): This is a consequence of (2.5).
(ii): In view of (2.5), it suffices to show that $\mathcal{V}^* \cap \mathcal{W}^* = \{0\}$.
Using the notation as in (2.6), we may conclude: If $x \in \mathcal{V}^* \cap \mathcal{W}^* = \operatorname{im} \widehat{E}^{k^*} \cap \ker \widehat{E}^{k^*}$, then there exists $y \in \mathbb{K}^n$ such that $x = \widehat{E}^{k^*} y$ and so $0 = \widehat{E}^{k^*} x = \left(\widehat{E}^{k^*}\right)^2 y = \widehat{E}^{2k^*} y$, whence, in view of $y \in \ker \widehat{E}^{2k^*} = \ker \widehat{E}^{k^*}$, $0 = \widehat{E}^{k^*} y = x$.
(iii): This is a direct consequence from (2.3) and (ii). $\square$

Example 2.5 (Regular pencil)**.** Consider the linear pencil $A - E\partial \in \mathbb{K}^{4\times 4}[\partial]$ given by

$$A := \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & 2 & -1 \\ 1 & 2 & 3 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \qquad E := \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 2 & 0 & -1 \\ -3 & -1 & 1 & 2 \\ -2 & -2 & 0 & 2 \end{bmatrix}.$$

Since $\det(A - E\partial) = 36\,\partial\,(\partial - 1)$, the pencil is regular and not equivalent to an ordinary differential equation. A straightforward calculation gives

$$\mathcal{V}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathcal{W}_1 = \ker E = \operatorname{im} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

and

$$\mathcal{V}_2 = \operatorname{im} V, \quad \text{where } V := \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \mathcal{W}_2 = \operatorname{im} W, \quad \text{where } W := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Both sequences terminate after these two iterations and therefore $\mathcal{V}^* = \mathcal{V}_2$, $\mathcal{W}^* = \mathcal{W}_2$ and $k^* = \ell^* = 2$. The statements of Proposition 2.4 and (2.4) are readily verified. Finally, we stress, in view of (2.5), that for this example

$$\mathcal{V}_1 \cap \mathcal{W}_1 = \mathcal{W}_1 \supsetneqq \{0\}.$$

$\diamond$

We are now in a position to state the main result of this note: The Wong sequences $\mathcal{V}_i$ and $\mathcal{W}_i$, converging in finitely many steps to the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$, constitute a transformation of the original pencil $A - E\partial$ into two decoupled pencils. Something which could be interpreted as a Quasi-Weierstraß form is implicitly hidden in the proof of [25, Cor. 3.3].

Theorem 2.6 (The Quasi-Weierstraß form)**.** *Consider a regular matrix pencil $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ and corresponding spaces $\mathcal{V}^*$ and $\mathcal{W}^*$ as in* (2.3)*. Let*

$$n_1 := \dim \mathcal{V}^*,\ V \in \mathbb{K}^{n\times n_1} \,:\, \operatorname{im} V = \mathcal{V}^* \quad \text{and} \quad n_2 := n - n_1 = \dim \mathcal{W}^*,\ W \in \mathbb{K}^{n\times n_2} \,:\, \operatorname{im} W = \mathcal{W}^*.$$

Then $[V, W]$ and $[EV, AW]$ are invertible and transform $A - E\partial$ into the Quasi-Weierstraß form

$$\boxed{[EV, AW]^{-1} (A - E\partial)\ [V, W] = \left(\begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} - \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \partial \right),} \tag{2.7}$$

for some $J \in \mathbb{K}^{n_1 \times n_1}$, $N \in \mathbb{K}^{n_2 \times n_2}$ so that $N^{k^*} = 0$ for k^* as given in (2.3)*.*

Before we prove Theorem 2.6, some comments may be warranted.

Remark 2.7 (The Quasi-Weierstraß form)**.** Let $A - E\partial \in \mathbb{K}^{n \times n}[\partial]$ be a regular matrix pencil and use the notation from Theorem 2.6.

(i) It is immediate, and will be used in later analysis, that (2.7) is equivalent to

$$AV = EVJ \qquad \text{and} \qquad EW = AWN \tag{2.8}$$

and to

$$E = [EV, AW] \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} [V, W]^{-1} \qquad \text{and} \qquad A = [EV, AW] \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} [V, W]^{-1}. \tag{2.9}$$

(ii) If (2.8) is solvable and if $[EV, AW]$ is invertible, then it is straightforward to see that J and N in (2.8), or equivalently in (2.7), are uniquely given by

$$J := (EV)^+ AV \quad \text{and} \quad N := (AW)^+ EW, \quad \text{resp.} \tag{2.10}$$

(iii) The spaces $\mathcal{V}^*$ and $\mathcal{W}^*$ determine uniquely – up to similarity – the solutions J and N of (2.7), resp. More precisely, let

$$\widehat{V} \in \mathbb{K}^{n \times n_1} \ : \ \operatorname{im} \widehat{V} = \mathcal{V}^* \quad \text{and} \quad \widehat{W} \in \mathbb{K}^{n \times n_2} \ : \ \operatorname{im} \widehat{W} = \mathcal{W}^*.$$

Then

$$\exists\, S \in \mathbb{K}^{n_1 \times n_1} \ \text{invertible} \ : \ VS = \widehat{V} \quad \text{and} \quad \exists\, T \in \mathbb{K}^{n_2 \times n_2} \ \text{invertible} \ : \ WT = \widehat{W},$$

and a simple calculation yields that J and N are similar to

$$(E\widehat{V})^+ A\widehat{V} = S^{-1} J S \qquad \text{and} \qquad (A\widehat{W})^+ E\widehat{W} = T^{-1} N T, \qquad \text{resp.}$$

(iv) If $\det E \neq 0$, then $\mathcal{V}^* = \mathcal{V}_i = \mathbb{K}^n$ and $\mathcal{W}^* = \mathcal{W}_i = \{0\}$ for all $i \in \mathbb{N}$. Therefore

$$E^{-1}\ (A - E\partial) \ = \ (E^{-1}A - I\partial)$$

is in Quasi-Weierstraß form.

(v) Let $\mathbb{K} = \mathbb{C}$. In view of (iii), the matrices V and W may always be chosen so that J and N in (2.7) are in Jordan form, in this case (2.7) is said to be in Weierstraß form.

(vi) For $\mathbb{K} = \mathbb{C}$, there are various numerical methods available to calculate the Weierstraß form, see e.g. [2]. However, since the Quasi-Weierstraß form does not invoke any eigenvalues and eigenvectors (here only the decoupling (2.7) and J and N without any special structure is important) the above mentioned algorithms are "too expensive". To calculate the subspaces (2.1) and (2.2) of the Wong sequences, one may use methods to obtain orthogonal basis for deflating subspaces;

see for example [4] and [9].
Furthermore, the Quasi-Weierstraß form - in contrast to the Weierstraß form - allows to consider matrix pencil over rational or even symbolic rings and the algorithm is still applicable. In fact, we will show in Proposition 2.10 that the number of subspace iterations equals the index of the matrix pencil; hence in many practical situations only one or two iterations must be carried out. $\diamond$

Proof of Theorem 2.6: Invertibility of $[V, W]$ follows from Proposition 2.4 (ii). Suppose

$$[EV, AW]\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \quad \text{for some} \quad \alpha \in \mathbb{K}^{n_1},\ \beta \in \mathbb{K}^{n_2}\,.$$

Then $V\alpha \in \mathcal{V}^* \cap \ker E$ and $W\beta \in \mathcal{W}^* \cap \ker A$, and thus Proposition 2.4 (iii) gives $V\alpha = 0$ and $W\beta = 0$. Since V and W have full column rank, we conclude $\alpha = 0$ and $\beta = 0$, and therefore $[EV, AW]$ is invertible and the inverse in (2.7) is well defined.
The subset inequalities (2.4) imply that (2.8) is solvable and (2.7) holds.
It remains to prove that N is nilpotent. To this end we show

$$\forall\, i \in \{0, \dots, k^*\} \ :\ \operatorname{im} WN^i \subseteq \mathcal{W}_{k^*-i}\,. \tag{2.11}$$

The statement is clear for $i = 0$. Suppose, for some $i \in \{0, \dots, k^* - 1\}$, we have

$$\operatorname{im} WN^i \subseteq \mathcal{W}_{k^*-i}\,. \tag{2.12}$$

Then

$$\operatorname{im} AWN^{i+1} \overset{(2.8)}{=} \operatorname{im} EWN^i \overset{(2.12)}{\subseteq} E\mathcal{W}_{k^*-i} \overset{(2.2)}{\subseteq} A\mathcal{W}_{k^*-i-1}$$

and, by invoking Proposition 2.4 (iii),

$$\operatorname{im} WN^{i+1} \subseteq \mathcal{W}_{k^*-i-1}.$$

This proves (2.11).
Finally, (2.11) for $i = k^*$ together with the fact that W has full column rank and $\mathcal{W}_0 = \{0\}$, implies that $N^{k^*} = 0$. $\square$

Example 2.8 (Example 2.5 revisited)**.** For V and W as defined in Example 2.5 we have

$$[V, W] = \begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ -1 & -1 & 1 & -1\\ 0 & 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad [EV, AW] = \begin{bmatrix} 4 & 1 & 4 & -1\\ 0 & 3 & 2 & -2\\ -4 & -1 & 4 & -1\\ -2 & -2 & 0 & 3 \end{bmatrix}$$

and the corresponding transformation of (2.7) shows that a Quasi-Weierstraß form of this example is given by

$$\begin{bmatrix} J & 0\\ 0 & I_2\end{bmatrix} - \begin{bmatrix} I_2 & 0\\ 0 & N\end{bmatrix}\partial \qquad \text{where} \quad J := \frac{1}{3}\begin{bmatrix} 2 & 1\\ -2 & 1\end{bmatrix}, \quad N := \frac{2}{3}\begin{bmatrix} -1 & 1\\ -1 & 1\end{bmatrix}.$$

$\diamond$

It follows from Remark 2.7 (iii) that the following definition of the index of a regular pencil is well defined since it does not depend on the special choice of N in the Quasi-Weierstraß form.

Definition 2.9 (Index of $A - E\partial$)**.** Let $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ be regular matrix pencil and consider the Quasi-Weierstraß form (2.7). Then

$$\nu^* \;:=\; \begin{cases} \min\{\nu \in \mathbb{N} \,|\, N^\nu = 0\}, & \text{if } N \text{ exists} \\ 0, & \text{otherwise} \end{cases}$$

is called the *index* of $A - E\partial$. ◇

The classical definition of the index of a regular matrix pencil (see e.g. [10, Def. 2.9]) is via the Weierstraß form. However, invoking Remark 2.7 (v), we see that ν^* in Definition 2.9 is the same number.

Proposition 2.10 (Index of $A - E\partial$)**.** *If $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ is regular, then the Wong sequence in* (2.2) *and W and N as in Theorem 2.6 satisfy*

$$\forall\, i \in \mathbb{N} \;:\; \mathcal{W}_i = W \ker N^i\,. \tag{2.13}$$

This implies that $\nu^* = k^*$; i.e. the index ν^* coincides with k^* determined by the Wong sequences in (2.3)*.*

Proof: We use the notation as in Theorem 2.6 and also the following simple formula

$$\forall\, i \in \mathbb{N} \;:\; \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \{0_{n_1}\} \\ \ker N^i \end{pmatrix} = \begin{pmatrix} \{0_{n_1}\} \\ \ker N^{i+1} \end{pmatrix}. \tag{2.14}$$

Next, we conclude, for $\mathcal{W}_0 := \{0\}$,

$$\begin{aligned} \forall\, i \in \mathbb{N}\setminus\{0\} \;:\; \widehat{\mathcal{W}_i} \;&:=\; [V,W]^{-1}\mathcal{W}_i \\ &\overset{(2.3)}{=}\; [V,W]^{-1}E^{-1}A\mathcal{W}_{i-1} \overset{(2.9)}{=} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} [EV,AW]^{-1}A[V,W]\widehat{\mathcal{W}}_{i-1} \\ &=\; \underbrace{\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right) \cdots \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)}_{i\text{-times}} \widehat{\mathcal{W}_0} \\ &\overset{(2.14)}{=}\; \underbrace{\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right) \cdots \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)}_{(i-1)\text{-times}} \begin{pmatrix} \{0_{n_1}\} \\ \ker N \end{pmatrix} \overset{(2.14)}{=} \begin{pmatrix} \{0_{n_1}\} \\ \ker N^i \end{pmatrix} \end{aligned}$$

and hence (2.13). □

Example 2.11 (Example 2.5, 2.8 revisited)**.** For W and N as defined in Example 2.5 and 2.8, resp., we see that $N^2 = 0$ and confirm the statement of Proposition 2.10:

$$W\ \ker N \;=\; W\ \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;=\; \operatorname{im} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \;=\; \mathcal{W}_1.$$

◇

An immediate consequence of the Quasi-Weierstraß form (2.7) is

$$\det(A - E\partial) \;=\; \det([EV,AW])\,\det(J - I\partial)\,\det(I - N\partial)\,\det([V,W]^{-1})\,,$$

and since any nilpotent matrix N satisfies $\det(I - N\partial) = 1$, we arrive at the following corollary.

Corollary 2.12. *Suppose $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ is a regular matrix pencil. Then, using the notation of Theorem 2.6, we have:*

(i) $\det(A - E\partial) \;=\; c\,\det(J - I_{n_1}\partial) \quad$ *for* $\quad c := \det([EV, AW])\det([V, W]^{-1}) \neq 0$,

(ii) $\operatorname{spec}(A - E\partial) \;=\; \operatorname{spec}(J - I_{n_1}\partial)$,

(iii) $\dim \mathcal{V}^* \;=\; \deg\big(\det(A - E\partial)\big)$.

In the remainder of this subsection we characterize $\mathcal{V}^*$ in geometric terms as a largest subspace. [6] already stated that $\mathcal{V}^*$ is the largest subspace such that $A\mathcal{V}^* \subseteq E\mathcal{V}^*$.

Proposition 2.13 ($\mathcal{V}^*$ largest subspaces)**.** *Let $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ be a regular matrix pencil. Then $\mathcal{V}^*$ determined by the Wong sequences* (2.3) *is the largest subspace of $\mathbb{K}^n$ such that $A\mathcal{V}^* \subseteq E\mathcal{V}^*$.*

Proof: We have to show that any subspace $\mathcal{U} \subseteq \mathbb{K}^n$ so that $A\mathcal{U} \subseteq E\mathcal{U}$ satisfies $\mathcal{U} \subseteq \mathcal{V}^*$. Let $u_0 \in \mathcal{U}$. Then

$$\exists\, u_1, \ldots, u_{k^*} \in \mathcal{U}\ \forall\, i = 1, \ldots, k^* \;:\; Au_{i-1} = Eu_i\,.$$

By Theorem 2.6,

$$\exists\, \alpha_0, \ldots, \alpha_{k^*} \in \mathbb{K}^{n_1} \quad \exists\, \beta_0, \ldots, \beta_{k^*} \in \mathbb{K}^{n_2} \quad \forall\, i = 0, \ldots, k^* \;:\; u_i = [V, W]\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}$$

and hence

$$\forall\, i = 1, \ldots, k^* \;:\; A[V, W]\begin{pmatrix}\alpha_{i-1}\\ \beta_{i-1}\end{pmatrix} \;=\; E[V, W]\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}$$

or, equivalently,

$$\forall\, i = 1, \ldots, k^* \;:\; [EV, AW]\begin{pmatrix}-\alpha_i\\ \beta_{i-1}\end{pmatrix} = [AV, EW]\begin{pmatrix}-\alpha_{i-1}\\ \beta_i\end{pmatrix} \overset{(2.8)}{=} [EV, AW]\begin{bmatrix}J & 0\\ 0 & N\end{bmatrix}\begin{pmatrix}-\alpha_{i-1}\\ \beta_i\end{pmatrix}\,;$$

since $[EV, AW]$ is invertible, we arrive at

$$\forall\, i = 1, \ldots, k^* \;:\; \beta_{i-1} = N\beta_i$$

and therefore

$$\beta_0 = N\beta_1 = \ldots = N^{k^*}\beta_{k^*} = 0.$$

This yields $u_0 = V\alpha_0 \in \operatorname{im} V = \mathcal{V}^*$ and proves $\mathcal{U} \subseteq \mathcal{V}^*$. □

2.2 The Drazin inverse

It is well known [7, 10] that the solution of the inhomogeneous differential algebraic equation $E\dot{x} = Ax + f$ can be expressed – provided E and A commute – in terms of the *Drazin inverses* E^D and A^D of E and A, resp. We will show that the Drazin inverses may be determined in terms of V and W of the Quasi-Weierstraß form if $EA = AE$.

First, we recall the well known definition of the Drazin inverse, see e.g. [20, p. 114].

Definition 2.14 (Drazin inverse)**.** For $M \in \mathbb{K}^{n\times n}$, the matrix $M^D \in \mathbb{K}^{n\times n}$ is called *Drazin inverse of* M if, and only if,

$$M^D M = M M^D \quad \wedge \quad M^D M M^D = M^D \quad \wedge \quad \exists\, \nu \in \mathbb{N} : M^D M^{\nu+1} = M^\nu\,. \tag{2.15}$$

◇

Definition 2.14 is, on first sight, more general than [10, Def. 2.17]. However, existence of a Drazin inverse according to Definition 2.14 follows for every $M \in \mathbb{K}^{n\times n}$ as in the proof of [10, Thm. 2.19]. To show uniqueness of M^D for $M \in \mathbb{K}^{n\times n}$, consider two Drazin inverses M_1^D and M_2^D of $M \in \mathbb{K}^{n\times n}$ with ν_1, ν_2 satisfying the third condition in Definition 2.14, resp. Then, the same idea as in the proof of [10, Thm. 2.19] yields $M_1^D = M_2^D$.
Finally, existence and uniqueness of the Drazin inverse show that Definition 2.14 and [10, Def. 2.17] coincide.

Proposition 2.15 (Drazin inverses of E and A)**.** *Consider a regular matrix pencil $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ and the Quasi-Weierstraß form* (2.7)*. If $EA = AE$, then the Drazin inverses of E and A are given by*

$$E^D = [V, W]\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}[EV, AW]^{-1} \quad \textit{and} \quad A^D = [V, W]\begin{bmatrix} J^D & 0 \\ 0 & I_{n_2} \end{bmatrix}[EV, AW]^{-1}, \ \textit{resp.} \tag{2.16}$$

The proof of Proposition 2.15 is based on the following observation for the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$ if E and A commute.

Lemma 2.16. *Consider a regular matrix pencil $A-E\partial \in \mathbb{K}^{n\times n}[\partial]$ and let $\mathcal{V}^*$ and $\mathcal{W}^*$ be given by* (2.3)*. Then*

$$AE = EA \quad \Longrightarrow \quad E\mathcal{V}^* = \mathcal{V}^* \quad \wedge \quad A\mathcal{W}^* = \mathcal{W}^*.$$

Proof: We use the notation as in Theorem 2.6. To prove $E\mathcal{V}^* = \mathcal{V}^*$, note that in view of the full rank of EV it suffices to show that

$$\forall\, i \in \mathbb{N} \ : \ E\mathcal{V}_i \subseteq \mathcal{V}_i\,. \tag{2.17}$$

The claim holds for $i = 0$. Suppose it holds for some $i \in \mathbb{N}$. Then the following implications are valid

$$\begin{aligned} x \in E\mathcal{V}_{i+1} &\overset{(2.1)}{\Rightarrow} \ \exists\, y \in \mathcal{V}_{i+1} : \ x = Ey \quad \wedge \quad \exists\, z \in \mathcal{V}_i : \ Ay = Ez \\ &\Rightarrow \ Ax = AEy = EAy = E(Ez), \quad \text{where } Ez \in \mathcal{V}_i \text{ by induction hypothesis} \\ &\overset{(2.1)}{\Rightarrow} \ x \in \mathcal{V}_{i+1} \end{aligned}$$

Therefore, $E\mathcal{V}^* \subseteq \mathcal{V}^*$. The equality $A\mathcal{W}^* = \mathcal{W}^*$ follows analogously and is omitted. □

Proof of Proposition 2.15: We use the notation as in Theorem 2.6. To show the properties of the Drazin inverse, first note that Lemma 2.16 yields

$$\exists \text{ invertible matrices } C_1 \in \mathbb{R}^{n_1\times n_1},\, C_2 \in \mathbb{R}^{n_2\times n_2} \ : \ [EV, AW] = [V, W]\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \tag{2.18}$$

and therefore, in view of $EA = AE$, the matrices J and C_1 as well as N and C_2 commute:

$$\begin{aligned} \begin{bmatrix} C_1 J & 0 \\ 0 & NC_2 \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \overset{(2.18)}{=} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}[V, W]^{-1}[EV, AW]\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \\ &\overset{(2.9)}{=} [EV, AW]^{-1}EA[V, W] = [EV, AW]^{-1}AE[V, W] \\ &\overset{(2.9)}{=} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}[V, W]^{-1}[EV, AW]\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \\ &\overset{(2.18)}{=} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} JC_1 & 0 \\ 0 & C_2 N \end{bmatrix}. \end{aligned} \tag{2.19}$$

Furthermore, by (2.9), for all $i \geq 1$,

$$E^i = [EV, AW] \begin{bmatrix} C_1^{i-1} & 0 \\ 0 & C_2^{i-1} N^i \end{bmatrix} [V, W]^{-1} \quad \wedge \quad A^i = [EV, AW] \begin{bmatrix} J^{i-1} C_1^{i-1} & 0 \\ 0 & C_2^{i-1} \end{bmatrix} [V, W]^{-1}. \quad (2.20)$$

Step 1: We show that E^D is the Drazin inverse of E.
Invoking (2.9) and (2.18), it is easy to see that $E^D E = E E^D$ and $E^D E E^D = E^D$. Finally, for k^* as in (2.3) and invoking $N^{k^*} = 0$,

$$E^D E^{k^*+1} \overset{(2.20)}{=} [V, W] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_1^{k^*} & 0 \\ 0 & 0 \end{bmatrix} [V, W]^{-1} \overset{(2.18)}{=} [EV, AW] \begin{bmatrix} C_1^{k^*-1} & 0 \\ 0 & 0 \end{bmatrix} [V, W]^{-1} \overset{(2.20)}{=} E^{k^*}.$$

Step 2: We show that A^D is the Drazin inverse of A.
First note that Definition 2.14 yields $C_1^{-1} J^D C_1 = (C_1^{-1} J C_1)^D \overset{(2.19)}{=} J^D$ and applying (2.19) again gives

$$C_1^{-1} J^D J C_1 = C_1^{-1} J J^D C_1 = J C_1^{-1} J^D C_1 = J J^D \quad (2.21)$$

and so

$$\begin{aligned} A^D A \underset{(2.18)}{\overset{(2.9)}{=}} [EV, AW] \begin{bmatrix} C_1^{-1} J J^D C_1 & 0 \\ 0 & I \end{bmatrix} [EV, AW]^{-1} &= [EV, AW] \begin{bmatrix} C_1^{-1} J^D J C_1 & 0 \\ 0 & I \end{bmatrix} [EV, AW]^{-1} \\ &\overset{(2.21)}{=} [EV, AW] \begin{bmatrix} J^D J & 0 \\ 0 & I \end{bmatrix} [EV, AW]^{-1} \underset{(2.18)}{\overset{(2.9)}{=}} A A^D. \end{aligned}$$

It is easy to see that $A^D A A^D = A^D$ and, for $\nu \in \mathbb{N}$ such that $J^D J^{\nu+1} = J^\nu$, we arrive at

$$\begin{aligned} A^D A^{\nu+1} &\overset{(2.20)}{=} [V, W] \begin{bmatrix} J^D J^\nu C_1^\nu J^D & 0 \\ 0 & C_2^\nu \end{bmatrix} [V, W]^{-1} \\ &\overset{(2.19)}{=} [V, W] \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} J^{\nu-1} C_1^{\nu-1} & 0 \\ 0 & C_2^{\nu-1} \end{bmatrix} [V, W]^{-1} \underset{(2.18)}{\overset{(2.20)}{=}} A^\nu. \end{aligned}$$

This completes the proof of the proposition. □

2.3 Differential algebraic equations

In this section we consider, for the matrix pencil $A - E\partial \in \mathbb{K}^{n \times n}[\partial]$, the initial value problem

$$E\dot{x} = Ax, \qquad x(0) = x^0, \quad (2.22)$$

where $x^0 \in \mathbb{K}^n$.
A *solution* of the initial value problem (2.22) is a differentiable function $x(\cdot) : I \to \mathbb{K}^n$ which solves (2.22) for all $t \in I$, $I \subseteq \mathbb{R}$ an interval, and $x(0) = x^0$; the solution is called *global* if, and only if, $I = \mathbb{R}$.

The main result of this subsection is the vector space isomorphism between the *global behaviour* of (2.22), i.e.

$$\ker(A - E\tfrac{\mathrm{d}}{\mathrm{d}t}) \ := \ \left\{ x : \mathbb{R} \to \mathbb{K}^n \;\middle|\; x(\cdot) \text{ is differentiable and solves } (A - E\tfrac{\mathrm{d}}{\mathrm{d}t})x(t) = 0 \text{ for all } t \in \mathbb{R} \right\},$$

and the set of *consistent initial values*, i.e. the set of all $x^0 \in \mathbb{K}^n$ such that (2.22) has a global solution. We apply the Quasi-Weierstraß form (2.7) to formulate and prove this result; a similar result is in [11, Th. 1].

Theorem 2.17 (Vector space isomorphism)**.** *Suppose that $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ is a regular matrix pencil and use the notation from Theorem 2.6. Let $V_W^+ := [I,0][V,W]^{-1}$, then the linear map*

$$\varphi\colon \mathcal{V}^* \to \ker\left(A - E\tfrac{\mathrm{d}}{\mathrm{d}t}\right), \quad x^0 \mapsto \left(t \mapsto Ve^{Jt}V_W^+x^0\right)$$

is a vector space isomorphism.

Proof: Note that $V_W^+x^0 = V^+x^0$ for any $x^0 \in \mathcal{V}^*$ and

$$\eta^0 := V^+x^0 \quad \text{is the unique solution of} \quad V\eta^0 = x^0,$$

and therefore,

$$x(t) = Ve^{Jt}V^+x^0 = Ve^{Jt}\eta^0 \qquad \forall\, t \in \mathbb{R}.$$

Step 1: It follows from

$$Ax(t) = AVe^{Jt}\eta^0 \overset{(2.8)}{=} EVJe^{Jt}\eta^0 = E\dot{x}(t) \qquad \forall\, t \in \mathbb{R}$$

that $x(\cdot) \in \ker(A - E\frac{\mathrm{d}}{\mathrm{d}t})$.
Step 2: It is immediate from Remark 2.7 (iii) that φ is well defined, that means it does not depend on the special choice of V.
Step 3: We show that $\varphi(\cdot)$ is surjective. It follows from Theorem 2.6 that

$$x(\cdot) \in \ker(A - E\tfrac{\mathrm{d}}{\mathrm{d}t}) \qquad \Longleftrightarrow \qquad [V,W]^{-1}x(\cdot) = \begin{pmatrix} z_1(\cdot) \\ z_2(\cdot) \end{pmatrix} \quad \text{solves} \quad \dot{z}_1 = Jz_1,\ N\dot{z}_2 = z_2\,.$$

Therefore, $z_2(\cdot) \equiv 0$ and $x(\cdot) = Vz_1(\cdot)$, and we arrive at $\varphi(Vz_1(0))(\cdot) = x(\cdot)$.
Step 4: We show that $\varphi(\cdot)$ is injective. Let $x^1, x^2 \in \mathcal{V}^*$ such that $\varphi(x^1) = \varphi(x^2)$. Choose unique $\eta^i = V^+x^i$ such that $V\eta^i = x^i$, for $i = 1,2$ resp. Then

$$x^1 = VV^+V\eta^1 = VV^+x^1 = \varphi(x^1)(0) = \varphi(x^2)(0) = VV^+V\eta^2 = x^2.$$

This completes the proof. $\square$

Remark 2.18. As mentioned in the proof of Theorem 2.17, V^+ and V_W^+ act identically on $\mathcal{V}^*$. Thus it might seem artificial to define φ in terms of V_W^+ instead of the standard pseudo inverse V^+. However, if x^0 is not a consistent initial value, then the formula for φ with V_W^+ yields the unique solution for the *inconsistent initial value problem.* For more details see [21, Sec. 4.2.2]. $\diamond$

The following (well known) statements of Corollary 2.19 are an immediate consequence of Theorem 2.17.

Corollary 2.19. *For any regular matrix pencil $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ we have:*

(i) $\mathcal{V}^* = \left\{x^0 \in \mathbb{K}^n \;\middle|\; \text{exists a global solution } x(\cdot) \text{ of } E\dot{x} = Ax,\ x(0) = x^0\right\}$ *with dimension* $\dim\ker(A - E\frac{\mathrm{d}}{\mathrm{d}t}) = \deg\left(\det(A - E\partial)\right)$.

(ii) *Any global solution $x(\cdot)$ of the initial value problem* (2.22) *satisfies for all $t \in \mathbb{R}$: $x(t) \in \mathcal{V}^*$.*

(iii) *Any local solution $x(\cdot)\colon I \to \mathbb{K}^n$ of* (2.22) *on an interval $I \subseteq \mathbb{R}$ can be uniquely extended to a global solution on $\mathbb{R}$; any global solution of the initial value problem* (2.22) *is unique.*

In the remainder of this section we show that the solution formula in Theorem 2.17 can be extended to inhomogeneous DAEs initial value problems

$$E\dot{x} = Ax + f, \qquad x(0) = x^0, \tag{2.23}$$

where $x^0 \in \mathbb{K}^n$, $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ is regular and $f : \mathbb{R} \to \mathbb{K}^n$ sufficiently often differentiable. It is a variant of the Variation-of-Constants formula for ordinary differential equations.

Proposition 2.20 (Solution to the inhomogeneous DAE). *Let $A - E\partial \in \mathbb{K}^{n\times n}[\partial]$ be a regular matrix pencil, use the notation from Theorem 2.6, let $f\colon \mathbb{R} \to \mathbb{K}^n$ be k^*-times continuously differentiable and define*

$$\begin{pmatrix} f_V(\cdot) \\ f_W(\cdot) \end{pmatrix} := [EV, AW]^{-1} f(\cdot), \quad \textit{where } f_V(t) \in \mathbb{K}^{n_1},\ f_W(t) \in \mathbb{K}^{n_2} \ \textit{ for all } \ t \in \mathbb{R}. \tag{2.24}$$

Then (2.23) *has a solution if, and only if,*

$$x^0 + W \sum_{i=0}^{k^*-1} N^i f_W^{(i)}(0) \in \mathcal{V}^*. \tag{2.25}$$

Any solution $x(\cdot)$ of (2.23) *is global, unique and satisfies, for $V_W^+ := [I, 0][V, W]^{-1}$,*

$$\boxed{x(t) = Ve^{Jt}V_W^+ x^0 + \textstyle\int_0^t Ve^{J(t-s)} f_V(s)\,\mathrm{d} - W\sum_{i=0}^{k^*-1} N^i f_W^{(i)}(t), \quad t \in \mathbb{R}\,.} \tag{2.26}$$

Proof: *Step 1:* We show that $x(\cdot)$ as in (2.26) satisfies $E\dot{x}(t) = Ax(t) + f(t)$ for all $t \in \mathbb{R}$:

$$\begin{aligned} E\dot{x}(t) &= EVJe^{Jt}V_W^+x^0 + \textstyle\int_0^t EVJe^{J(t-s)} f_V(s)\,ds + EVf_V(t) - \sum_{i=0}^{k^*-1} EWN^i f_W^{(i+1)}(t) \\ &\underset{N^{k^*}=0}{\overset{(2.8)}{=}} AVe^{Jt}V_W^+x^0 + \textstyle\int_0^t AVe^{J(t-s)} f_V(s)\,ds + EVf_V(t) + AWf_W(t) - \sum_{i=0}^{k^*-1} AWN^i f_W^{(i)}(t) \\ &\overset{(2.24)}{=} Ax(t) + f(t). \end{aligned}$$

Step 2: We show that $x(0) = x^0$ for $x(\cdot)$ as in (2.26) if, and only if, (2.25) holds. Choose $\alpha \in \mathbb{K}^{n_1}$ and $\beta \in \mathbb{K}^{n_2}$ such that $x_0 + W\sum_{i=0}^{k^*-1} N^i f_W^{(i)}(0) = V\alpha + W\beta$. In view of $V_W^+W = 0$ and $V_W^+V = I$,

$$\begin{aligned} x(0) &= VV_W^+x^0 - \sum_{i=0}^{k^*-1} WN^i f_W^{(i)}(0) \\ &= VV_W^+V\alpha + VV_W^+W\beta - VV_W^+W \sum_{i=0}^{k^*-1} N^i f_W^{(i)}(0) - \sum_{i=0}^{k^*-1} WN^i f_W^{(i)}(0) \\ &= V\alpha - \sum_{i=0}^{k^*-1} WN^i f_W^{(i)}(0) \ = \ x^0 - W\beta. \end{aligned}$$

Since W has full column rank, $x(0) = x^0$ if, and only if, $\beta = 0$. This shows (2.25).
Step 3: Finally, we show that any solution $x(\cdot)$ of (2.23) can be written in the form (2.26). Let

$$z(t) := Ve^{Jt}V_W^+x^0 + \int_0^t Ve^{J(t-s)} f_V(s)\,ds - W \sum_{i=0}^{k^*-1} N^i f_W^{(i)}(t), \quad t \in \mathbb{R}.$$

Then $z(\cdot)$ solves, by Step 2, the inhomogeneous DAE

$$\dot{z} = Az + f(t), \qquad z(0) = VV_W^+ x^0 - W \sum_{i=0}^{k^*-1} N^i f_W^{(i)}(0)$$

and $(x-z)(\cdot)$ solves the homogeneous DAE

$$E\tfrac{\mathrm{d}}{\mathrm{d}t}(x-z) = A(x-z), \qquad (x-z)(0) = x^0 - VV_W^+ x^0 + W \sum_{i=0}^{k^*-1} N^i f_W^{(i)}(0). \tag{2.27}$$

Since Corollary 2.19 (iii) gives $(x-z)(0) \in \mathcal{V}^*$, and since $x^0 - VV_W^+ x^0 = [0, W][V, W]^{-1} x^0 \in \mathcal{W}^*$, we conclude from (2.27) that $(x-z)(0) \in \mathcal{V}^* \cap \mathcal{W}^* = \{0\}$. Therefore, a repeated application of Corollary 2.19 yields $x(\cdot) \equiv z(\cdot)$. This concludes the proof. $\square$

Remark 2.21 (Solution formula in terms of Drazin inverse)**.** It is well known, see for example [10, Th. 2.29], that for any regular $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$ such that $EA = AE$, the solution of (2.23) may be expressed in terms of the Drazin inverses of E and A. So alternatively to (2.26) we have

$$x(t) = e^{E^D A t} E^D E x^0 + \int_0^t e^{E^D A(t-s)} f(s)\, ds - \big(I - E^D E\big) \sum_{i=0}^{k^*-1} (EA^D)^i A^D f^{(i)}(t), \quad t \in \mathbb{R}\,, \tag{2.28}$$

where k^* is determined in (2.3) (see also Proposition 2.10) and the Drazin inverses are given in (2.16).

The solution formula (2.26) compares favourably to (2.28): The latter Drazin inverse approach relies on $EA = AE$ which, if not satisfied, requires to transform the system (2.23) to

$$\widehat{E} := (A - \lambda E)^{-1} E,\ \widehat{A} := (A - \lambda E)^{-1} A,\ \widehat{f}(\cdot) := (A - \lambda E)^{-1} f(\cdot) \qquad \text{for any} \ \ \lambda \notin \operatorname{spec}(A - E\partial).$$

Then $\widehat{E}$ and $\widehat{A}$ commute and the solution of (2.23) is identical to the solution of $\widehat{E}\dot{x} = \widehat{A}x + \widehat{f},\ x(0) = x^0$. However, the matrix multiplication with the inverse $(A - \lambda E)^{-1}$ may be numerically questionable and one may loose structural properties of E and A. Moreover, the Drazin inverses $\widehat{E}^D$ and $\widehat{A}^D$ have to be computed. Whereas the former Wong sequence approach for (2.23) might require much less computational effort. Furthermore, the Quasi-Weierstraß form directly reveals the underlying ODE (slow system) and the pure DAE (fast system); this is not the case for the Drazin inverse approach. $\diamond$

3 Chains of generalized eigenvectors

In Section 3.1 we show that the generalized eigenvectors of a regular pencil $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$ constitute a basis which transforms $A - E\partial$ into Weierstraß form. From this point of view, the Weierstraß form is a generalized Jordan form. The chains of eigenvectors and eigenspaces are derived in terms of the matrices E and A; the Quasi-Weierstraß form is only used in the proofs. This again shows the unifying power of the Wong-sequences and allows for a “natural” proof of the Weierstraß form.
In Section 3.2 we show - by using results from Section 3.1 and so by using the Wong-sequences – how the generalized eigenvectors constitute a basis of solutions for the DAE (1.1).

3.1 The Weierstraß form

We recall the well known concept of chains of generalized eigenvectors; see for fairly general operator valued functions [16, (11.2)], for infinite eigenvectors see [3, Def. 2] and also [14, 15].
Note that eigenvalues and eigenvectors of real or rational matrix pencils are in general complex valued, thus in the following we restrict the analysis to the case $\mathbb{K} = \mathbb{C}$.

Definition 3.1 (Chains of generalized eigenvectors)**.** Let $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$ be a matrix pencil. Then $(v_1, \ldots, v_k) \in (\mathbb{C}^n \setminus \{0\})^k$ is called a *chain (of $A - E\partial$ at eigenvalue λ)* if, and only if,

$$\left.\begin{array}{rcccc} \lambda \in \operatorname{spec}(A - E\partial) : & (A-\lambda E)v_1 = 0, & (A-\lambda E)v_2 = Ev_1, & \ldots, & (A-\lambda E)v_k = Ev_{k-1} \\ \lambda = \infty : & Ev_1 = 0, & Ev_2 = Av_1, & \ldots, & Ev_k = Av_{k-1}, \end{array}\right\} \tag{3.1}$$

the i^{th} vector v_i of the chain is called *generalized eigenvector of order i at λ*. ⋄

Note that $(v_1, \ldots, v_k)$ is a chain at $\lambda \in \operatorname{spec}(A - E\partial)$ if, and only if, for all $i \in \{1, \ldots, k\}$,

$$AV_i = EV_i \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \quad \text{where } V_i := [v_1, \ldots, v_i]\,. \tag{3.2}$$

Remark 3.2 (Linear relations)**.** It may be helpful to consider the concept of generalized eigenvectors, in particular for eigenvalues at ∞, from the viewpoint of linear relations, see e.g. [1]:
$\mathcal{R} \subset \mathbb{C}^n \times \mathbb{C}^n$ is called a *linear relation* if, and only if, $\mathcal{R}$ is a linear space; its *inverse relation* is $\mathcal{R}^{-1} := \{(y, x) \in \mathbb{C}^n \times \mathbb{C}^n \,|\, (x, y) \in \mathcal{R}\}$, and the *multiplication* with a relation $\mathcal{S} \subset \mathbb{C}^n \times \mathbb{C}^n$ is $\mathcal{R}\mathcal{S} := \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \,|\, \exists\, z \in \mathbb{C}^n : (x, z) \in \mathcal{S} \wedge (z, y) \in \mathcal{R}\}$. $\lambda \in \mathbb{C}$ is called an *eigenvalue* of a relation $\mathcal{R}$ with eigenvector $x \in \mathbb{C}^n \setminus \{0\}$ if, and only if, $(x, \lambda x) \in \mathcal{R}$; see [19]. Clearly, $\lambda \neq 0$ is an eigenvalue of $\mathcal{R}$ if, and only if, $1/\lambda$ is an eigenvalue of $\mathcal{R}^{-1}$; this justifies to call ∞ an eigenvalue of $\mathcal{R}$ if, and only if, 0 is an eigenvalue of $\mathcal{R}^{-1}$.
In the context of a matrix pencil $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$, the matrices A and E induce the linear relations

$$\mathcal{A} := \{(x, Ax) \,|\, x \in \mathbb{C}^n\} \qquad \text{and} \qquad \mathcal{E} := \{(x, Ex) \,|\, x \in \mathbb{C}^n\}\,, \qquad \text{resp.}$$

and therefore,

$$\begin{array}{rclrcl} \mathcal{E}^{-1} & = & \{(Ex, x) \,|\, x \in \mathbb{C}^n\}\,, & \mathcal{A}^{-1} & = & \{(Ax, x) \,|\, x \in \mathbb{C}^n\}\,, \\ \mathcal{E}^{-1}\mathcal{A} & = & \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \,|\, Ax = Ey\}\,, & \mathcal{A}^{-1}\mathcal{E} & = & \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \,|\, Ex = Ay\}\,. \end{array}$$

It now follows that

$$\begin{array}{rcl} \lambda \in \mathbb{C} \text{ is an eigenvalue of } \mathcal{E}^{-1}\mathcal{A} & \iff & \det(A - \lambda E) = 0 \\ \infty \text{ is an eigenvalue of } \mathcal{E}^{-1}\mathcal{A} & \iff & 0 \text{ is an eigenvalue of } \mathcal{A}^{-1}\mathcal{E} \\ 0 \text{ is an eigenvalue of } \mathcal{A}^{-1}\mathcal{E} & \iff & E \text{ is not invertible.} \end{array}$$

In [19] also chains for relations are considered. In the context of the above example this reads: $v_1, \ldots, v_k \in \mathbb{C}^n \setminus \{0\}$ form a *(Jordan) chain* at eigenvalue $\lambda \in \mathbb{C} \cup \{\infty\}$ if, and only if,

$$\left.\begin{array}{rccccl} \lambda \in \operatorname{spec}(A - E\partial) : & (v_1, \lambda v_1), & (v_2, v_1 + \lambda v_2), & \ldots, & (v_k, v_{k-1} + \lambda v_k) & \in \mathcal{E}^{-1}\mathcal{A} \\ \lambda = \infty : & (0, v_1), & (v_1, v_2), & \ldots, & (v_{k-1}, v_k) & \in \mathcal{E}^{-1}\mathcal{A}. \end{array}\right\} \tag{3.3}$$

Obviously, (3.3) is equivalent to (3.1), but the former may be a more "natural" definition. Linear relations have also been analysed and exploited for matrix pencils in [5]. ⋄

In order to decompose $\mathcal{V}^*$, we have to be more specific with the spaces spanned by generalized eigenvectors at eigenvalues.

Definition 3.3 (Generalized eigenspaces)**.** Let $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$ be a matrix pencil. Then the sequences of *eigenspaces (of $A - E\partial$ at eigenvalue λ)* are defined by $\mathcal{G}_\lambda^0 := \{0\}$ and

$$\forall\, i \in \mathbb{N} : \mathcal{G}_\lambda^{i+1} := \begin{cases} (A - \lambda E)^{-1}(E\mathcal{G}_\lambda^i), & \text{if } \lambda \in \operatorname{spec}(A - E\partial) \\ E^{-1}(A\mathcal{G}_\lambda^i), & \text{if } \lambda = \infty. \end{cases}$$

The *generalized eigenspace* (*of* $A - E\partial$ *at eigenvalue* $\lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\}$) is defined by

$$\mathcal{G}_\lambda := \bigcup_{i \in \mathbb{N}} \mathcal{G}_\lambda^i.$$

For the multiplicities we use the following notion

$$\begin{aligned}
\operatorname{gm}(\lambda) &:= \ \dim \mathcal{G}_\lambda^1 \quad \text{is called the } \textit{geometric multiplicity} \text{ of } \lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\}, \\
\operatorname{am}(\lambda) &:= \text{multiplicity of } \lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\} \text{ as a zero of } \det(A - E\partial) \\
&\quad\ \text{is called the } \textit{algebraic multiplicity} \text{ of } \lambda, \\
\operatorname{am}(\infty) &:= n - \sum_{\lambda \in \operatorname{spec}(A - E\partial)} \operatorname{am}(\lambda) = n - \deg\big(\det(A - E\partial)\big) \text{is called the } \textit{algebraic multiplicity at } \infty.
\end{aligned}$$

$\diamond$

Readily verified properties of the eigenspaces are the following.

Remark 3.4 (Eigenspaces)**.** For any regular $A - E\partial \in \mathbb{C}^{n \times n}[\partial]$ and $\lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\}$ we have:

(i) For each $i \in \mathbb{N}$, $\mathcal{G}_\lambda^i$ is the vector space spanned by the eigenvectors up to order i at λ.

(ii) $\exists\, p^* \in \mathbb{N}\ \forall\, j \in \mathbb{N} \ : \ \mathcal{G}_\lambda^0 \subsetneq \cdots \subsetneq \mathcal{G}_\lambda^{p-1} \subsetneq \mathcal{G}_\lambda^{p^*} = \mathcal{G}_\lambda^{p^*+j}$. $\diamond$

The following result is formulated in terms of the pencil $A - E\partial$, its proof invokes the Quasi-Weierstraß form.

Proposition 3.5 (Eigenvectors and eigenspaces)**.** *Let* $A - E\partial \in \mathbb{C}^{n \times n}[\partial]$ *be regular.*

(i) *Every chain* $(v_1, \ldots, v_k)$ *at any* $\lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\}$ *satisfies, for all* $i \in \{1, \ldots, k\}$, $v_i \in \mathcal{G}_\lambda^i \backslash \mathcal{G}_\lambda^{i-1}$.

(ii) *Let* $\lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\}$ *and* $k \in \mathbb{N}\backslash\{0\}$. *Then for any* $v \in \mathcal{G}_\lambda^k \backslash \mathcal{G}_\lambda^{k-1}$, *there exists a unique chain* $(v_1, \ldots, v_k)$ *such that* $v_k = v$.

(iii) *The vectors of any chain* $(v_1, \ldots, v_k)$ *at* $\lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\}$ *are linearly independent.*

(iv)
$$\mathcal{G}_\lambda \ \subseteq \ \begin{cases} \mathcal{V}^*, & \textit{if} \quad \lambda \in \operatorname{spec}(A - E\partial) \\ \mathcal{W}^*, & \textit{if} \quad \lambda = \infty. \end{cases}$$

(v)
$$\forall\, \lambda \in \operatorname{spec}(A - E\partial) \cup \{\infty\} : \ \dim \mathcal{G}_\lambda = \operatorname{am}(\lambda).$$

Proof: Invoking the notation of Theorem 2.6, we first show that

$$\forall\, i \in \mathbb{N} \ : \ \mathcal{G}_\lambda^i = \begin{cases} V \ker(J - \lambda I)^i, & \text{if } \lambda \in \operatorname{spec}(A - E\partial) \\ \mathcal{W}_i = W \ker N^i, & \text{if } \lambda = \infty. \end{cases} \tag{3.4}$$

Suppose $\lambda \in \operatorname{spec}(A - E\partial)$. We prove by induction that

$$\forall\, i \in \mathbb{N} \ : \ \mathcal{G}_\lambda^i \ \subseteq \ V \ker(J - \lambda I)^i. \tag{3.5}$$

The claim is clear for $i = 0$. Suppose (3.5) holds for $i = k-1$. Let $v_k \in \mathcal{G}_\lambda^k \setminus \{0\}$ and $v_{k-1} \in \mathcal{G}_\lambda^{k-1}$ such that $(A - \lambda E)v_k = Ev_{k-1}$. By Proposition 2.4(ii) we may set

$$v_k = V\alpha + W\beta \quad \text{for unique} \quad \alpha \in \mathbb{C}^{n_1},\ \beta \in \mathbb{C}^{n_2}\,.$$

By (2.8), $(A - \lambda E)v_k = Ev_{k-1}$ is equivalent to

$$AW(I - \lambda N)\beta = Ev_{k-1} + EV(\lambda I - J)\alpha\,,$$

and so, since by induction hypothesis

$$v_{k-1} \in \mathcal{G}_\lambda^{k-1} \subseteq V \ker(J - \lambda I)^{k-1} \subseteq \mathcal{V}^* x^0,$$

we conclude

$$W(I - \lambda N)\beta \in A^{-1}(E\mathcal{V}^*) \overset{(2.1)}{=} \mathcal{V}^*.$$

Now Proposition 2.4(ii) yields, since W has full column rank, $(I - \lambda N)\beta = 0$ and hence, since N is nilpotent, $\beta = 0$. It follows from $v_{k-1} \in V \ker(J - \lambda I)^{k-1}$ that there exists $u \in \mathbb{C}^{n_1}$ such that $v_{k-1} = Vu$ and $(J - \lambda I)^{k-1}u = 0$. Then $EV(J - \lambda I)\alpha = EVu$ and Proposition 2.4 (iii) gives, since V has full column rank, $(J - \lambda I)\alpha = u$. Therefore, $v_k = V\alpha$ and $(J - \lambda I)^k\alpha = 0$, hence $v_k \in V \ker(J - \lambda I)^k$ and this completes the proof of (3.5).
Next we prove by induction that

$$\forall\, i \in \mathbb{N} \ :\ \mathcal{G}_\lambda^i \ \supseteq\ V \ker(J - \lambda I)^i. \tag{3.6}$$

The claim is clear for $i = 0$. Suppose (3.6) holds for $i = k-1$. Let $v_k \in \ker(J - \lambda I)^k$ and $v_{k-1} \in \ker(J - \lambda I)^{k-1}$ such that $(J - \lambda I)v_k = v_{k-1}$. Since EV has full column rank, this is equivalent to $EV(J - \lambda I)v_k = EVv_{k-1}$ which is, by invoking (2.8), equivalent to $(A - \lambda E)Vv_k = EVv_{k-1}$ and then the induction hypothesis yields $Vv_{k-1} \in \mathcal{G}_\lambda^{k-1}$, thus having $Vv_k \in \mathcal{G}_\lambda^k$. This proves (3.6) and completes the proof of (3.4) for finite eigenvalues.
The statement '$\mathcal{W}_i = W \ker N^i$ for all $i \in \mathbb{N}$' follows by Proposition 2.10, and '$\mathcal{G}_\infty^i = \mathcal{W}_i$ for all $i \in \mathbb{N}$' is clear from the definition.
The Assertions (i)–(iv) follow immediately from (3.4) and the respective results of the classical eigenvalue theory, see for example [13, Sec. 12.5, 12.7] and [8, Sec. 4.6].
Assertion (v) is a consequence of Corollary 2.12 (i) and $\mathrm{am}(\infty) = n - \deg\big(\det(A - E\partial)\big) = n_2$. This completes the proof of the proposition. $\square$

An immediate consequence of Proposition 3.5 and (3.4) is the following Theorem 3.6. We stress that our proof relies essentially on the relationship between the eigenspaces of $A - E\partial$ and the eigenspaces of $J - I\partial$ and $N - I\partial$ where J and N are as in (2.7). Alternatively, we could prove Theorem 3.6 by using chains and cyclic subspaces only, however the present proof via the Quasi-Weierstraß form is shorter.

Theorem 3.6 (Decomposition and basis of $\mathcal{V}^*$)**.** *Let $A - E\partial \in \mathbb{C}^{n \times n}[\partial]$ be regular, $\lambda_1, \ldots, \lambda_k$ the pairwise distinct zeros of* $\det(A - E\partial)$ *and use the notation of Theorem 2.6. Then*

$$\forall\, \lambda \in \{\lambda_1, \ldots, \lambda_k\}\ \forall\, j \in \{1, \ldots, \mathrm{gm}(\lambda)\}\ \exists\, n_{\lambda,j} \in \mathbb{N}\backslash\{0\}\ \exists\ \textit{chain}\left(v_{\lambda,j}^1, v_{\lambda,j}^2, \ldots, v_{\lambda,j}^{n_{\lambda,j}}\right)\ \textit{at}\ \lambda\ :$$

$$\mathcal{G}_\lambda = \bigoplus_{j=1}^{\mathrm{gm}(\lambda)} \mathrm{im}\, \underbrace{\left[v_{\lambda,j}^1, \ldots, v_{\lambda,j}^{n_{\lambda,j}}\right]}_{=: V_{\lambda,j} \in \mathbb{C}^{n \times n_{\lambda,j}}}. \tag{3.7}$$

and

$$\boxed{\mathcal{V}^* = \mathcal{G}_{\lambda_1} \oplus \mathcal{G}_{\lambda_2} \oplus \ldots \oplus \mathcal{G}_{\lambda_k}} \quad \textit{and} \quad \boxed{\mathcal{W}^* = \mathcal{G}_\infty}.$$

$\diamond$

In Corollary 3.7 we show that the generalized eigenvectors of a regular matrix pencil $A - E\partial$ at the finite eigenvalues and at the infinite eigenvalue constitute a basis which transforms $A - E\partial$ into the well known Weierstraß form. So the Weierstraß form can be viewed as a generalized Jordan form. This viewpoint is different to the proofs which can be found in papers or textbooks on DAEs, see for example [10, Th. 2.7].

Corollary 3.7 (Weierstraß form)**.** *Let $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$ be regular, $n_1 := \dim \mathcal{V}^*$, $n_2 := n - n_1$ and $\lambda_1, \ldots, \lambda_k$ be the pairwise distinct zeros of $\det(A - E\partial)$. Then we may choose*

$$\begin{aligned} V_f &:= \left[V_{\lambda_1,1}, \ \ldots, \ V_{\lambda_1,\mathrm{gm}(\lambda_1)}, \ V_{\lambda_2,1}, \ \ldots, \ V_{\lambda_2,\mathrm{gm}(\lambda_2)}, \ \ldots, \ V_{\lambda_k,1}, \ \ldots, \ V_{\lambda_k,\mathrm{gm}(\lambda_k)}\right], \\ V_\infty &:= \left[V_{\infty,1}, \ \ldots, \ V_{\infty,\mathrm{gm}(\infty)}\right], \end{aligned}$$

where $V_{\lambda_i,j}$ consists of a chain at λ_i as in (3.7), $j = 1, \ldots, gm(\lambda_i)$, $i = 1, \ldots, k$, resp. For any such V_f, V_∞, the matrices $[V_f, V_\infty]$, $[EV_f, AV_\infty] \in \mathbb{C}^{n\times n}$ are invertible and transform the pencil $A - E\partial$ into Weierstraß form, i.e.

$$\boxed{[EV_f, AV_\infty]^{-1}(A - E\partial)[V_f, V_\infty] = \left(\begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} - \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \partial\right)} \tag{3.8}$$

where both $J \in \mathbb{C}^{n_1\times n_1}$ and $N \in \mathbb{C}^{n_2\times n_2}$ are in Jordan form and N is nilpotent.

Proof: The existence of V_f and V_∞ satisfying the eigenvector conditions formulated in the corollary follows from Theorem 2.6. In view of (3.2), it follows from the definition of chains that (3.8) holds for some matrices $J \in \mathbb{C}^{n_1\times n_1}$ and $N \in \mathbb{C}^{n_2\times n_2}$ in Jordan form and nilpotent N. $\square$

3.2 Differential algebraic equations - revisited

In the following proposition it is shown that the generalized eigenvectors at the finite eigenvalues of a regular pencil $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$ constitute a basis of solutions of $\ker(A - E\frac{\mathrm{d}}{\mathrm{d}t})$. This is known: see for example [24], [16, Lemma 13.1], [12]. We give a short proof so that the present paper is self contained.

Proposition 3.8 (Chain)**.** *Consider a regular pencil $A - E\partial \in \mathbb{C}^{n\times n}[\partial]$. Then $(v_1, \ldots, v_k)$ is a chain of generalized eigenvectors at $\lambda \in \mathrm{spec}(A - E\partial)$ if, and only if, the functions*

$$x^i(\cdot)\colon \mathbb{R} \to \mathbb{C}^n, \quad t \mapsto x^i(t) := e^{\lambda t}\,[v_1, \ldots, v_i] \left(\frac{t^{i-1}}{(i-1)!}, \ldots, \frac{t}{1}, 1\right)^\top, \qquad i = 1, \ldots, k$$

are linearly independent global solutions of $E\dot{x} = Ax$.

Proof: Note that for $N_i := \left[\begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{smallmatrix}\right] \in \mathbb{R}^{i\times i}$ and V_i as in (3.2) we have, for all $i \in \{1, \ldots, k\}$ and all $t \in \mathbb{R}$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\psi_i(t) = N_i\,\psi_i(t) \quad \text{for} \quad \psi_i(t) := \left(\frac{t^{i-1}}{(i-1)!}, \ldots, \frac{t}{1}, 1\right)^\top$$

and

$$E\tfrac{\mathrm{d}}{\mathrm{d}t}x^i(t) = \lambda e^{\lambda t} E V_i \psi_i(t) + e^{\lambda t} E V_i \tfrac{\mathrm{d}}{\mathrm{d}t}\psi_i(t) = e^{\lambda t}\left[\lambda E V_i + E V_i N_i\right]\psi_i(t)\,. \tag{3.9}$$

We are now ready to prove the proposition. Suppose $(v_1, \ldots, v_k)$ is a chain. Then (3.2) substituted into (3.9) yields

$$\forall\, i \in \{1, \ldots, k\}\ \forall\, t \in \mathbb{R}\ :\ E\tfrac{\mathrm{d}}{\mathrm{d}t}x^i(t) = e^{\lambda t} A V_i \psi_i(t) = A x^i(t)$$

and all $x^i(\cdot)$ are solutions. Linear independence of the solutions follows from $[x^1(0), \ldots, x^k(0)] = [v_1, \ldots, v_k]$ and $v_1, \ldots, v_k$ are linearly independent by Proposition 3.5 (iii).
Suppose next that $x^1(\cdot), \ldots, x^k(\cdot)$ are linearly independent solutions. Then $v_1 = x^1(0), \ldots, v_k = x^k(0)$ are linearly independent and (3.9) gives

$$\forall\, i \in \{1, \ldots, k\}\ \forall\, t \in \mathbb{R}\ :\ E V_i \tfrac{\mathrm{d}}{\mathrm{d}t}\psi_i(t) = (A - \lambda E)\, V_i \psi_i(t)\,.$$

Now differentiation of this equality and inserting $\psi_i(0) = e_i$, $\psi_i^{(1)}(0) = e_{i-1}, \ldots \psi_i^{(i)}(0) = e_1$ yields (3.2) for $i = k$ and hence $(v_1, \ldots, v_k)$ is a chain. $\square$

Proposition 3.8 could be used for analyzing asymptotic solution properties of $E\dot{x} = Ax$, such as stability; but this is not the topic of the present paper.

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