Observability of linear differential-algebraic systems – a survey

Thomas Berger, Timo Reis and Stephan Trenn

Abstract We investigate different concepts related to observability of linear constant coefficient differential-algebraic equations. Regularity, which, loosely speaking, guarantees existence and uniqueness of solutions for any inhomogeneity, is not required in this article. Concepts like impulse observability, observability at infinity, behavioral observability, strong and complete observability are described and defined in the time-domain. Special emphasis is placed on a normal form under output injection, state space and output space transformation. This normal form together with duality is exploited to derive Hautus type criteria for observability. We also discuss geometric criteria, Kalman decompositions and detectability. Some new results on stabilization by output injection are proved.

Keywords Differential-algebraic equations · Observability · Controllability · Duality · Output injection · Hautus Test · Kalman decomposition · Wong sequences

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Thomas Berger · Timo Reis
Fachbereich Mathematik, Universität Hamburg,
Bundesstraße 55, 20146 Hamburg, Germany
e-mail: thomas.berger@uni-hamburg.de, timo.reis@math.uni-hamburg.de

Stephan Trenn
Fachbereich Mathematik, Technische Universität Kaiserslautern,
Postfach 3049, 67653 Kaiserslautern, Germany
e-mail: trenn@mathematik.uni-kl.de

1 Introduction

Observability is, roughly speaking, the property of a system that the state can be reconstructed from the knowledge of input and output. The precise concept however depends on the specific framework, as quite a number of different concepts of observability are present today.

As many crucial concepts in mathematical systems theory, observability goes back to KALMAN [44–46], who introduced the notion of observability more than fifty years ago for finite-dimensional linear systems governed by ordinary differential equations (ODEs) . Observability was defined via the property that the initial value of the state is uniquely determined by input and output trajectories. What is particularly nice about observability is a *duality principle*. An ODE system is observable if, and only if, a certain artificial system obtained by taking the transposes of the involved matrices is controllable.

The theory of observability has been an essential ingredient for LUENBERGER's achievements on observer design [58–60], which is, on the other hand, an essential ingredient for the design of dynamic controllers. The idea behind controller design is amazingly simple: The observer reconstructs the state and this reconstructed state is fed back to the system.

A further milestone in mathematical systems theory was the *theory of behaviors* introduced by WILLEMS [70, 84], where systems of differential equations of possibly higher order are considered. The novelty of this approach was to treat inputs, states, and outputs alike; in particular, the behavioral model allows for different choices of inputs and outputs. Nevertheless, or even maybe because of this, the behavioral approach provides a deep understanding of nearly all tasks of modern systems theory. Indeed, the essential systems theoretic concepts of controllability and observability are defined in a way that they coincide with the respective properties of ODE systems: Behavioral controllability is defined via concatenability of trajectories [70, Def. 5.2.2], whereas observability uses a split of the dynamic variables into two kinds, namely *external* and *internal variables* [70, Def. 5.3.2]. For ODE systems, the external variables are inputs and outputs, whereas the internal variables are the states. Behavioral observability means that the external variables uniquely determine the internal variables. The behavioral approach already reveals a certain lack of duality between controllability and observability: While controllable systems with additional equations of the form $0 = 0$ stay controllable in the behavioral sense, their dual may contain free variables and is not observable in general. This does not come as a surprise, especially in view of WILLEMS' remark in [84]:

> "...controllability and observability are prima facie not dual concepts. Controllability is an intrinsic concept of the behavior of a dynamical system, while observability remains representation dependent".

The type of systems to be analyzed in the present article is "in between" ODE and behavioral systems: We consider linear constant coefficient descriptor systems given by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\,, \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times l}$. A matrix pencil $sE - A \in \mathbb{R}[s]^{l\times n}$ is called *regular*, if $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$; otherwise it is called *singular*. In the present paper, we put special emphasis on the singular case.

We distinguish between *input* $u : \mathbb{R} \to \mathbb{R}^m$, *output* $y : \mathbb{R} \to \mathbb{R}^p$, and *(generalized) state* $x : \mathbb{R} \to \mathbb{R}^n$. One should keep in mind that in the singular case u might be constrained and some of the state variables may play the role of an input. Note that, strictly speaking, $x(t)$ is in general not a state in the sense that the free system (i.e., $u \equiv 0$) can be initialized with an arbitrary state $x(0) = x_0 \in \mathbb{R}^n$ [48, Sec. 2.2]. We will, however, speak of the state $x(t)$ for sake of brevity, especially since $x(t)$ contains the full information about the system at time t.

We recall that in DAE systems (1) the algebraic constraints may lead to consistency conditions on the input and cause non-existence of solutions to certain initial value problems. Furthermore, solutions may not be unique due to underdetermined parts. There is a vast amount of literature on the solution theory of DAEs; here we refer to the recent depiction of DAEs in a systems theoretic framework in [16], where also several application areas are mentioned and a comprehensive list of literature is given.

Though DAEs are a subclass of behavioral systems, the study of behavioral observability is not fully satisfactory in the DAE case: The reason is that there might be purely algebraic variables which do not exert influence on the output. An observability concept which also covers this effect is in particular indispensable for the *minimal realization* problem by differential-algebraic systems [32, Sec. 2.6]. This need has led to the notions of *impulse observability* and *observability at infinity* [3, 13, 24, 26, 31–33, 40, 43, 53, 77, 82]. However, a rigorous definition of these concepts is a delicate issue: In various publications, the theoretical claim that an inconsistent initial value causes Dirac impulses in the state was used to define impulse observability (which is actually the reason for the choice of the name) [31,32,40,43]. In particular, this leads to the consideration of *distributional solutions*. However, this approach contains a grave paradox: The initial value is the evaluation of the state at initial time (which can always be chosen to be zero here because of time-invariance); SCHWARTZ' celebrated theory of distributions [76] however does not allow for evaluations at certain time points. Loosely speaking, distributions are only defined by means of their average behavior along compactly supported, infinitely often differentiable functions. In the present article we also aim to circumvent this paradox by focussing on the smaller class of *piecewise-smooth distributions* as introduced in [77, 78]. This class indeed allows for evaluation at specific time points, and therefore it is apt to consider inconsistently initialized DAEs and rigorously define accordant observability concepts.

A survey article [16] on controllability of DAE systems appeared in the same series "Surveys on Differential-Algebraic Equations" within the "Differential-Algebraic Equations Forum". The present article on observability is the counterpart of that survey. The structure of the present paper is similar to [16]: We introduce different observability concepts using the solution behavior and thereafter we give characterizations by means of properties of the involved matrices. We further analyze duality to the respective controllability concepts.

As in [16], many of our considerations utilize certain (normal) forms. Besides the Weierstraß and Kronecker canonical forms for matrix pencils (see [50, 83] and the classical book [36] by GANTMACHER), we also use a form that we call "output injection (OI) normal form", which is a normal form under state space and output space transformation and output injection. Loosely speaking, the OI normal form is the transpose of the feedback canonical form derived by LOISEAU, ÖZÇALDIRAN, MALABRE and KARCANIAS in [56].

The paper is organized as follows:

The solution framework for the present article is introduced in this section. Besides weak solutions (which are basically solutions in a function setting), we consider distributional solutions of linear DAEs. The collection of solutions is called *behavior*. In particular we consider the behavior arising from *initial trajectory problems* which is, loosely speaking, the set of those solutions which satisfy the DAE only for times $t \geq 0$. The relation between the introduced behavior notions is discussed.

This section contains the definition of all observability notions which are treated in the present article, such as behavioral, impulse, strong and complete observability as well as observability at infinity. We further introduce corresponding concepts of relevant state (RS) observability. Loosely speaking, the latter concepts correspond to observability of the part of the state which is uniquely determined by input, output and initial values. The RS observability notions will later turn out to be weaker than the respective conventional observability notions and to be equivalent to them, if the system is regular. All the observability concepts are introduced by means of time-domain properties. That is, they are defined by means of the (distributional) behavior of the underlying system. We also present some basic properties.

We introduce an "output injection (OI) normal form", which is a special form under output injection and coordinate transformation of state and output. We further show that all considered observability concepts from Section 3 are invariant under this type of transformation. This allows for an analysis of the observability concepts by means of a system being in this form. Since, in particular, the OI normal form consists of decoupled parts, this analysis leads to a test of the respective observability properties by means of certain "prototypes".

5 Duality of observability and controllability p.31

It is well known from systems theory for ODEs that controllability and observability are dual in a certain sense. More precisely, an ODE system is observable if, and only if, the control system obtained by transposition is controllable. Here we analyze duality for the introduced observability concepts and behavioral, impulse, strong and complete controllability as well as controllability at infinity as considered in [16]. It turns out that there is a certain lack of duality. However, we show that the aforementioned controllability concepts are dual to the respective relevant state observability notions.

6 Algebraic criteria p.33

Duality and the OI normal form enable us to give short proofs of equivalent criteria for the observability concepts which are in particular generalizations of the Hautus test. Most characterizations are well known and we discuss the relevant literature.

7 Geometric criteria p.39

We present some geometric viewpoints of DAE systems using so-called *restricted Wong sequences*. This leads to further equivalent criteria for the observability concepts from Section 3.

8 Kalman decomposition p.42

We consider different types of Kalman decompositions for DAE systems. We show that a combined Kalman decomposition for controllability and observability is possible as well as a refined pure observability decomposition.

9 Detectability and stabilization by output injection p.45

Finally, we introduce some notions related to detectability for DAE systems. Criteria of Hautus type and duality to stabilizability concepts from [16] are derived. We further prove some new results concerning the stabilization by output injection.

We close the introduction with the nomenclature used in this paper:

$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$	the set of integers, natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, resp.
$\ell(\alpha), \lvert\alpha\rvert$	length $\ell(\alpha) = l$ and absolute value $\lvert\alpha\rvert = \sum_{i=1}^{l} \alpha_i$ of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$
$\mathbb{C}_+ (\mathbb{C}_-)$	open set of complex numbers with positive (negative) real part, resp.
$\overline{\mathbb{C}_+}$	closed set of complex numbers with non-negative real part
$\mathbb{R}[s]$	the ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$	the quotient field of $\mathbb{R}[s]$

$R^{n\times m}$	the set of $n\times m$ matrices with entries in a ring R
$\mathbf{Gl}_n(R)$	the group of invertible matrices in $R^{n\times n}$
$\sigma(M)$	the spectrum of $M\in\mathbb{R}^{n\times m}$
$\|x\|$	$=\sqrt{x^\top x}$, the Euclidean norm of $x\in\mathbb{R}^n$
$M\mathscr{S}$	$=\{\, Mx\in\mathbb{R}^m \mid x\in\mathscr{S}\,\}$, the image of $\mathscr{S}\subseteq\mathbb{R}^n$ under $M\in\mathbb{R}^{m\times n}$
$M^{-1}\mathscr{S}$	$=\{\, x\in\mathbb{R}^m \mid Mx\in\mathscr{S}\,\}$, the pre-image of $\mathscr{S}\subseteq\mathbb{R}^m$ under M
$\mathscr{C}^\infty(\mathscr{T};\mathbb{R}^n)$	the set of infinitely differentiable functions $f:\mathscr{T}\to\mathbb{R}^n$
$\mathscr{AC}(\mathbb{R};\mathbb{R}^n)$	the set of absolutely continuous functions $f:\mathbb{R}\to\mathbb{R}^n$
$\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$	the set of locally Lebesgue integrable functions $f:\mathbb{R}\to\mathbb{R}^n$, where $\int_{K\cap\mathscr{T}}\|f(t)\|\,\mathrm{d}t<\infty$ for all compact $K\subseteq\mathbb{R}$
$\mathscr{D}'$	the set of distributions on $\mathbb{R}$
$\dot f$ $(f^{(i)})$	the (i-th) distributional derivative of $f\in\mathscr{D}'$, $i\in\mathbb{N}_0$
$f_{\mathscr{D}'}$	the distribution induced by the function $f\in\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R})$
δ_t, δ	the Dirac impulse at $t\in\mathbb{R}$ and $\delta=\delta_0$
$f\overset{\text{a.e.}}{=}g$	means that $f,g\in\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ are equal “almost everywhere”, i.e., $f(t)=g(t)$ for almost all $t\in\mathbb{R}$
$\operatorname{ess\,sup}_I\|f\|$	the essential supremum of the measurable function $f:\mathscr{T}\to\mathbb{R}^n$ over $I\subseteq\mathscr{T}$
f_I	the restriction of the function $f:\mathbb{R}\to\mathbb{R}^n$ to $I\subseteq\mathbb{R}$, i.e., $f_I(t)=f(t)$ for $t\in I$ and $f_I(t)=0$ otherwise

We further use the following abbreviations in this article:

DAE	differential-algebraic equation,
ITP	initial trajectory problem, see pp. 8,
ODE	ordinary differential equation,
OI	output injection, see pp. 20,
RS	relevant state, see pp. 14.

2 Weak and distributional solutions

We consider linear DAE systems of the form (1) with $E,A\in\mathbb{R}^{l\times n}$, $B\in\mathbb{R}^{l\times m}$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times l}$. The set of these systems is denoted by $\Sigma_{l,n,m,p}$ and we write $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$.

A trajectory $(x,u,y):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p$ is said to be a *(weak) solution* of (1) if, and only if, it belongs to the *behavior* of (1):

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{ (x,u,y)\in\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{n+m+p}) \;\middle|\; \begin{array}{l} Ex\in\mathscr{AC}(\mathbb{R};\mathbb{R}^n) \text{ and } (x,u,y) \\ \text{fulfills (1) for almost all } t\in\mathbb{R} \end{array} \right\}.$$

Recall that $Ex\in\mathscr{AC}(\mathbb{R};\mathbb{R}^l)$ implies continuity of Ex (but x itself may be discontinuous). For studying inconsistent initial values and impulsive effects we will also consider distributional behaviors which are formally introduced in due course.

For the analysis of DAE systems in $\Sigma_{l,n,m,p}$ we assume that the states, inputs and outputs of the system are fixed a priori by the designer, i.e., the realization is given (but maybe not appropriate). This is different from other approaches based on the behavioral setting, see [27], where only the free variables in the system are viewed as inputs; this may require a reinterpretation of states as inputs and of inputs as states. In the present paper we will assume that such a reinterpretation of variables has already been done or is not feasible, and the given DAE system is fix.

Next we consider solutions of (1) in the distributional sense. We primarily do formal and arithmetical calculations in the space of distributions; the latter is usually denoted by $\mathscr{D}'$ because it is defined as a dual of a certain test function space $\mathscr{D}$. For a deeper introduction to the mathematical (in particular, analytical) background we refer to [74, Chap. 6]. Distributions are generalized functions and allow differentiation of arbitrary order. A key role is played by the *Dirac impulse* (also called the δ *distribution*) δ_t, which corresponds to evaluation of a test function at $t\in\mathbb{R}$.

The *distributional behavior* consists of the *distributional solutions*, i.e.,

$$\mathfrak{B}^{\mathscr{D}'}_{[E,A,B,C,D]} = \left\{ (x,u,y)\in(\mathscr{D}')^{n+m+p} \;\middle|\; \begin{array}{r} E\dot{x}=Ax+Bu \\ y=Cx+Du \end{array} \right\}.$$

Note that $\mathfrak{B}_{[E,A,B,C,D]}$ can be canonically embedded into $\mathfrak{B}^{\mathscr{D}'}_{[E,A,B,C,D]}$. We also consider a special subspace of the distributions which features further properties. To this end we utilize the distributional solution framework as introduced in [77, 78], namely the space of *piecewise-smooth distributions*

$$\mathscr{D}'_{\text{pw}\mathscr{C}^\infty} = \left\{ \sum_{i\in\mathbb{Z}} \left(\left(\alpha^i{}_{[t_i,t_{i+1})}\right)_{\mathscr{D}'} + D_{t_i} \right) \;\middle|\; \begin{array}{c} \{\, t_i\in\mathbb{R} \mid i\in\mathbb{Z} \,\} \text{ is locally finite,} \\ \forall i\in\mathbb{Z}:\; t_i<t_{i+1} \,\wedge\, \alpha^i\in\mathscr{C}^\infty(\mathbb{R};\mathbb{R}) \\ \wedge\, D_{t_i}\in\operatorname{span}\left\{ \delta^{(k)}_{t_i} \;\middle|\; k\in\mathbb{N}_0 \right\} \end{array} \right\}.$$

We clearly have that $\mathscr{D}'_{\text{pw}\mathscr{C}^\infty}$ is a subspace of $\mathscr{D}'$ which is invariant under differentiation, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{D}'_{\text{pw}\mathscr{C}^\infty}=\mathscr{D}'_{\text{pw}\mathscr{C}^\infty}$. Note that $\mathscr{D}'_{\text{pw}\mathscr{C}^\infty}$ is not a (topologically) closed subspace of $\mathscr{D}'$. The behavior corresponding to $\mathscr{D}'_{\text{pw}\mathscr{C}^\infty}$ is

$$\mathfrak{B}^{\mathscr{D}'_{\text{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]} = \mathfrak{B}^{\mathscr{D}'}_{[E,A,B,C,D]} \cap (\mathscr{D}'_{\text{pw}\mathscr{C}^\infty})^{n+m+p}.$$

Note that $\mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]} \not\subseteq \mathfrak{B}_{[E,A,B,C,D]}$ and $\mathfrak{B}_{[E,A,B,C,D]} \not\subseteq \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}$.

Any $D \in \mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}$ has a unique representation $D = f_{\mathscr{D}'} + \sum_{t\in T} D_t$, where $T \subseteq \mathbb{R}$ is locally finite and $f \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ is piecewise smooth. The *distributional restriction* to some interval $M \subseteq \mathbb{R}$ (cf. [78, Def. 8]) is given by

$$D_M = (f_M)_{\mathscr{D}'} + \sum_{t\in M\cap T} D_t \in \mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}.$$

Note that the restriction is not well-defined for general distributions [78, Thm. 2.2.2]. The class $\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}$ moreover allows to perform point evaluations in some sense. Namely, for $D \in \mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}$ as above and $t_0 \in \mathbb{R}$, the expressions

$$D(t_0^+) := \lim_{t\searrow t_0} f(t), \qquad D(t_0^-) := \lim_{t\nearrow t_0} f(t)$$

are well-defined, since f is piecewise smooth. Furthermore, the *impulsive part of D at $t_0 \in \mathbb{R}$* is given by

$$D[t_0] := \begin{cases} 0, & \text{if } t_0 \notin T, \\ D_{t_0}, & \text{if } t_0 \in T. \end{cases} \tag{2}$$

An important property of DAEs is the fact that due to the algebraic constraints not all initial values $x_0 \in \mathbb{R}^n$ for $x(0^-)$ are possible (even in the above distributional solution framework). Indeed, we call $x_0 \in \mathbb{R}^n$ a *consistent initial value* if, and only if, there exists $(x,u,y) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}$ with $x(0^-) = x_0$. However, there are many reasons to consider also inconsistent initial values. The problem of inconsistent initial values may be formalized in the framework of *initial trajectory problems* (ITP) and its corresponding ITP-behavior

$$\mathfrak{B}^{\mathrm{ITP}}_{[E,A,B,C,D]} = \left\{ (x,u,y) \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^{n+m+p} \;\middle|\; \begin{array}{r} (E\dot{x})_{[0,\infty)} = (Ax+Bu)_{[0,\infty)} \\ y_{[0,\infty)} = (Cx+Du)_{[0,\infty)} \end{array} \right\},$$

i.e., the DAE is supposed to hold only on the interval $[0,\infty)$ and there are no explicit constraints in the past[1]. Clearly, $\mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]} \subseteq \mathfrak{B}^{\mathrm{ITP}}_{[E,A,B,C,D]}$, i.e., any "consistent" solution $(x,u,y) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}$ is also an ITP-solution, but it should be noted that in general

$$\left\{ (x,u,y)_{[0,\infty)} \;\middle|\; \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]} \right\} \neq \left\{ (x,u,y)_{[0,\infty)} \;\middle|\; \mathfrak{B}^{\mathrm{ITP}}_{[E,A,B,C,D]} \right\},$$

because ITP-solutions may exhibit impulsive terms $x[0]$ induced by inconsistent initial values, which are not present in consistent solutions. In the ODE-case, $E = I$, this

[1] For singular DAEs it is however *not true* that all $x(0^-) \in \mathbb{R}^n$ are feasible for an ITP. For example, the overdetermined DAE $\dot{x} = 0$, $0 = x$ has no ITP solution with $x(0^-) \neq 0$, because then $x(0^+) = 0$ and $0 = \dot{x}[0] = (x(0^+) - x(0^-))\delta_0$ are conflicting.

distinction vanishes, that is on $[0,\infty)$ the two behaviors $\mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[I,A,B,C,D]}$ and $\mathfrak{B}^{\mathrm{ITP}}_{[I,A,B,C,D]}$ are identical.

A different approach (motivated somewhat by the Laplace transform) handles inconsistent initial values by the consideration of the following behavior parametrized by the "initial value" $z_0 \in \mathbb{R}^l$

$$\mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]} := \left\{ (x,u,y) \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^{n+m+p} \;\middle|\; \begin{array}{l} E\dot{x} = Ax + Bu + \delta z_0 \\ y = Cx + Du \end{array} \right\}.$$

Indeed, for ODE systems, the addition of δz_0 corresponds to an initialization $x(0^+) = z_0$ (under the assumption that $x(0^-) = 0$). Note that the behavior $\mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]}$ can be seen as a variant of $\mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}$ where an additional impulsive input δz_0 is present. We will need all of the above distributional solution spaces to define different notions of observability.

Before we begin the investigation of the different observability definitions and their characterizations, we like to provide a better understanding of the three different distributional solution spaces and their relationship with each other.

First, we highlight a fundamental property of general homogeneous DAEs $\mathscr{E}\dot{z} = \mathscr{A}z$ with $\mathscr{E},\mathscr{A} \in \mathbb{R}^{r\times s}$ which follows easily from the definition of restriction in $\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}$:

$$\begin{aligned} &\left\{ z_{(-\infty,0)} \;\middle|\; z \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^s,\ \mathscr{E}\dot{z} = \mathscr{A}z \right\} \\ &\qquad = \left\{ z_{(-\infty,0)} \;\middle|\; z \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^s, (\mathscr{E}\dot{z})_{(-\infty,0)} = (\mathscr{A}z)_{(-\infty,0)} \right\}, \end{aligned} \tag{3}$$

in other words any solution given on $(-\infty,0)$ can be extended to a global solution. This "causality" property is now essential to prove the following result which allows to decouple inhomogeneous DAEs.

Lemma 2.1. *Let $\mathscr{E},\mathscr{A} \in \mathbb{R}^{r\times s}$ and $f \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^r$. Then*

$$\begin{aligned} \left\{ z \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^s \;\middle|\; \mathscr{E}\dot{z} = \mathscr{A}z + f_{[0,\infty)} \right\} = \left\{ z \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^s \;\middle|\; \mathscr{E}\dot{z} = \mathscr{A}z \right\} + \\ \left\{ z \in (\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty})^s \;\middle|\; z_{(-\infty,0)} = 0,\ \mathscr{E}\dot{z} = \mathscr{A}z + f_{[0,\infty)} \right\}. \end{aligned}$$

Proof. The subspace inclusion $\supseteq$ is clear. To show the converse let x be a solution of $\mathscr{E}\dot{z} = \mathscr{A}z + f_{[0,\infty)}$, then z satisfies $(\mathscr{E}\dot{z})_{(-\infty,0)} = (\mathscr{A}z + f_{[0,\infty)})_{(-\infty,0)} = (\mathscr{A}z)_{(-\infty,0)}$. By causality (3) we find a solution $\tilde{z}$ of $\mathscr{E}\dot{z} = \mathscr{A}z$ with $\tilde{z}_{(-\infty,0)} = z_{(-\infty,0)}$. Then $\hat{z} := z - \tilde{z}$ satisfies $\hat{z}_{(-\infty,0)} = 0$ and $\mathscr{E}\dot{\hat{z}} = \mathscr{A}z + f_{[0,\infty)} - \mathscr{A}\tilde{z} = \mathscr{A}\hat{z} + f_{[0,\infty)}$. This shows that $z = \tilde{z} + \hat{z}$ can be decomposed as claimed. ☐

Note that Lemma 2.1 is a generalization of the well known property of linear ODEs that the influence from the initial value on the solution can be decoupled from the influence of the inhomogeneity. However, for DAEs the initial condition $z(0) = 0$

is not feasible for general inhomogeneous DAEs (with fixed inhomogeneity), that is why we restrict the influence of the inhomogeneity to the interval $[0,\infty)$, because then a zero initial value (in the past) is feasible.

We are now able to present the relationship between the ITP-behaviors (which allows for inconsistent initial values implicitly) and the δz_0- behavior which introduces an initial value explicitly.

Lemma 2.2. *For $z_0 \in \mathbb{R}^l$ define*

$$\left[\mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]} \ominus \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}\right] := \left\{ \, (x,u,y) \in \mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]} \,\middle|\, (x,u,y)_{(-\infty,0)} = 0 \, \right\}.$$

Then

$$\mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]} = \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]} + \left[\mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]} \ominus \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}\right].$$

Furthermore, for all $x_0 \in \mathbb{R}^n$:

$$\begin{aligned} &\left\{ \, (x,u,y)_{[0,\infty)} \,\middle|\, (x,u,y) \in \mathfrak{B}^{\mathrm{ITP}}_{[E,A,B,C,D]} \,\wedge\, x(0^-) = x_0 \, \right\} \\ &\qquad = \left\{ \, (x,u,y)_{[0,\infty)} \,\middle|\, (x,u,y) \in \left[\mathfrak{B}^{\delta E x_0}_{[E,A,B,C,D]} \ominus \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}\right] \, \right\}, \end{aligned}$$

i.e., the response on $[0,\infty)$ to the (potentially inconsistent) initial value x_0 within the ITP-framework is the same as the response of the DAE with the additional input $\delta E x_0$ and zero initial condition.

Proof. The first equality follows directly from Lemma 2.1 with $z = (x,u,y)$ and $f_{[0,\infty)} = \delta z_0$, the second equality was already shown in [79, Thm. 5.3]. □

Remark 2.3. Note that $\mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]}$ is not a vector space for $z_0 \neq 0$. It might even be empty (for instance, consider $E = A = B = C = D = 0 \in \mathbb{R}$ and $z_0 = 1$). Lemma 2.2 shows that it is an affine linear space. More precisely, it is a shifted version of the distributional behavior $\mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}$, where z_0 takes the role of an initial value in a certain sense. However, the following linearity property holds for any $z_0^1, z_0^2 \in \mathbb{R}^l$:

$$\begin{aligned} &(x_1,u_1,y_1) \in \mathfrak{B}^{\delta z_0^1}_{[E,A,B,C,D]} \wedge (x_2,u_2,y_2) \in \mathfrak{B}^{\delta z_0^2}_{[E,A,B,C,D]} \\ &\qquad\qquad \Rightarrow \quad (x_1+x_2, u_1+u_2, y_1+y_1) \in \mathfrak{B}^{\delta(z_0^1+z_0^2)}_{[E,A,B,C,D]}. \end{aligned}$$

At this point it is not yet clear, why we have introduced the solution set $\mathfrak{B}^{\delta z_0}_{[E,A,B,C,D]}$ but it will turn out that this is fruitful for defining some of the observability concepts.

3 Observability concepts

Classically, observability is defined as the absence of indistinguishable states (see the textbook [80]) or, in a behavioral setting [70], as the absence of nontrivial solutions which generate a trivial output.

In contrast to the observability notions for systems given by ODEs, there are many conceptually different observability definitions for DAE systems (even in the regular case). We first present the most intuitive observability notions and will later present and discuss the remaining observability concepts.

3.1 Behavioral, impulse and strong observability

For the definition of behavioral observability, we follow [70, Def. 5.3.2] and for impulse observability we are inspired by [77, Def. 5.2.1].

Definition 3.1. The system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is called

(a) *behaviorally observable*

$$:\Longleftrightarrow \forall (x^1,u,y),(x^2,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}: \quad x^1 \overset{\text{a.e.}}{=} x^2,$$

(b) *impulse observable*

$$:\Longleftrightarrow \forall (x^1,u,y),(x^2,u,y) \in \mathfrak{B}^{\text{ITP}}_{[E,A,B,C,D]}: \quad x^1[0] = x^2[0],$$

where $D[0]$ is the impulsive part of $D \in \mathscr{D}'_{\text{pw}\mathscr{C}^\infty}$ at $t=0$, see (2).

(c) *strongly observable*

$$:\Longleftrightarrow \forall (x^1,u,y),(x^2,u,y) \in \mathfrak{B}^{\text{ITP}}_{[E,A,B,C,D]}: \quad (x^1)_{[0,\infty)} = (x^2)_{[0,\infty)}.$$

The intuition behind these observability notions is as follows: In general, a system is called observable if the knowledge of the external signals allows the reconstruction of the inner state. This idea is directly formalized by the behavioral observability definition. Note that the forthcoming observability characterization will yield that the system $[E,A,B,C,D]$ is behaviorally observable (defined for weak solutions) if, and only if, it is behaviorally observable in a distributional solution framework, i.e.,

$$\forall (x_1,u,y),(x_2,u,y) \in \mathfrak{B}^{\mathscr{D}'_{\text{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}: \quad x_1 = x_2.$$

Most physical systems are turned on at some time (i.e., the system does not run infinitely long already) and it is well known that DAE systems (in contrast to ODE systems) exhibit new phenomena in response to inconsistent initial values. In particular, inconsistent initial values may lead to Dirac impulses in the solution and an important question is, whether these Dirac impulses in the state variable can uniquely be determined from the measurement of the external signals. This property is formalized by the impulse observability definition.

Example 3.2. Consider the DAE

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x, \quad y = Cx.$$

The only solution (also in a distributional solution framework) is $x \equiv \binom{0}{0}$, in particular $\binom{0}{0}$ is the only consistent initial value and the DAE is behaviorally observable. The ITP with initial value $x(0^-) = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$ leads to the impulsive term $x[0] = \begin{pmatrix} 0 \\ x_0^1 \delta_0 \end{pmatrix}$. Hence, $C = [0,1]$ makes the DAE impulse observable (because then $y[0] = x_0^1 \delta_0$ uniquely determines $x[0]$), while $C = [1,0]$ makes the DAE not impulse-observable (because the impulse in $x[0]$ is not visible in the output y).

The following result is an immediate consequence of Definition 3.1.

Proposition 3.3. *The system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is strongly observable if, and only if, it is behaviorally and impulse observable.*

Linearity of the system (1) implies that $\mathfrak{B}_{[E,A,B,C,D]}$ and $\mathfrak{B}^{\text{ITP}}_{[E,A,B,C,D]}$ are vector spaces. As an immediate consequence, we can characterize the previously introduced notions by the following slightly simpler properties.

Lemma 3.4 (Distinction from zero)**.** *The system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is*

(a) behaviorally observable

$$\iff \quad \forall (x,0,0) \in \mathfrak{B}_{[E,A,B,C,D]} : \quad x \stackrel{\text{a.e.}}{=} 0,$$

(b) impulse observable

$$\iff \quad \forall (x,0,0) \in \mathfrak{B}^{\text{ITP}}_{[E,A,B,C,D]} : \quad x[0] = 0,$$

(c) strongly observable

$$\iff \quad \forall (x,0,0) \in \mathfrak{B}^{\text{ITP}}_{[E,A,B,C,D]} : \quad x_{[0,\infty)} = 0.$$

Corollary 3.5. *The DAE system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is behaviorally, impulse, or strongly observable if, and only if, the DAE system $[E,A,0_{l\times 0},C,0_{p\times 0}]$ with corresponding DAE*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax, \quad y = Cx$$

has the respective property.

The above result justifies to restrict our attention in the following to the system class

$$\mathscr{O}_{l,n,p} := \left\{ \, [E,A,C] \;\middle|\; [E,A,0_{l\times 0},C,0_{p\times 0}] \in \Sigma_{l,n,0,p} \, \right\}$$

with the corresponding behaviors

$$\mathfrak{B}_{[E,A,C]} := \mathfrak{B}_{[E,A,0_{l\times 0},C,0_{p\times 0}]}, \qquad \mathfrak{B}^{\mathrm{ITP}}_{[E,A,C]} := \mathfrak{B}^{\mathrm{ITP}}_{[E,A,0_{l\times 0},C,0_{p\times 0}]}$$

and the question whether a zero output implies a trivial state (behavioral observability) or an impulse free response to any inconsistent initial value (impulse observability). Analogously, we set

$$\mathfrak{B}^{\mathscr{D}'}_{[E,A,C]} := \mathfrak{B}^{\mathscr{D}'}_{[E,A,0_{l\times 0},C,0_{p\times 0}]} \qquad \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,C]} := \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,0_{l\times 0},C,0_{p\times 0}]},$$
$$\mathfrak{B}^{\delta z_0}_{[E,A,C]} := \mathfrak{B}^{\delta z_0}_{[E,A,0_{l\times 0},C,0_{p\times 0}]}.$$

Note that we allow $p = 0$, i.e., DAE systems without an output. On a first glance this might look meaningless in the context of observability, however, the DAE $0 = x$ (for example) is behaviorally and impulse observable, although there is no output. This is also related to the fact, that adding or removing zero output equations $y = 0$ does not change the observability properties.

3.2 Observability at infinity and complete observabilty

Now we introduce two observability notions which will later on prove to be stronger than impulse and strong observability, resp. To this end we seek a definition in terms of "observability of excitations" which is related to input observability as in [41]. The idea is, that a Dirac impulse at time $t = 0$ is applied to the systems equations weighted by some constants represented by a vector $z_0 \in \mathbb{R}^l$.

Definition 3.6. The system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is called

(a) observable at infinity

$$:\Longleftrightarrow\ \forall z_0^1, z_0^2 \in \mathbb{R}^l : \left[(x^1,u,y) \in \mathfrak{B}^{\delta z_0^1}_{[E,A,B,C,D]} \wedge (x^2,u,y) \in \mathfrak{B}^{\delta z_0^2}_{[E,A,B,C,D]} \wedge Ex^1 = Ex^2 \right.$$
$$\left. \Rightarrow\quad z_0^1 = z_0^2 \wedge x^1[0] = x^2[0] \right],$$

(b) completely observable

$$:\Longleftrightarrow\ \forall z_0^1, z_0^2 \in \mathbb{R}^l : \left[(x^1,u,y) \in \mathfrak{B}^{\delta z_0^1}_{[E,A,B,C,D]} \wedge (x^2,u,y) \in \mathfrak{B}^{\delta z_0^2}_{[E,A,B,C,D]} \right.$$
$$\left. \Rightarrow\quad z_0^1 = z_0^2 \wedge x^1[0] = x^2[0] \right].$$

It is obvious that complete observability implies observability at infinity. The forthcoming observability characterizations will further yield that a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is completely observable if, and only if, it is behaviorally observable and observable at infinity.

By using that for all $z_0^1, z_0^2 \in \mathbb{R}^l$ we have from Remark 2.3 that

$$\mathfrak{B}^{\delta z_0^1}_{[E,A,B,C,D]} + \mathfrak{B}^{\delta z_0^1}_{[E,A,B,C,D]} = \mathfrak{B}^{\delta(z_0^1+z_0^2)}_{[E,A,B,C,D]},$$

we can conclude that observability at infinity and complete observability can be characterized by the conditions from Definition 3.6 in which z_0^2, u and y are trivial (cf. Lemma 3.4).

Lemma 3.7 (Distinction from zero II)**.** *The system* $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ *is*

(a) observable at infinity

$$\iff \forall z_0 \in \mathbb{R}^l : \left[(x,0) \in \mathfrak{B}^{\delta z_0}_{[E,A,C]} \ \wedge \ Ex = 0 \ \Rightarrow \ z_0 = 0 \ \wedge \ x[0] = 0 \right],$$

(b) completely observable

$$\iff \forall z_0 \in \mathbb{R}^l : \left[(x,0) \in \mathfrak{B}^{\delta z_0}_{[E,A,C]} \ \Rightarrow \ z_0 = 0 \ \wedge \ x[0] = 0 \right].$$

An immediate consequence is that we can again restrict out attention to systems in $\mathscr{O}_{[E,A,C]}$.

Example 3.8. Consider the DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = x + \delta z_0, \quad y = Cx.$$

If $C = I_2$, then $y = 0$ implies $x = 0$ and thus $z_0 = 0$, i.e., the DAE is completely observable. If we choose $C = [0,1]$, then $x_2 = y = 0$ implies $z_0 = \left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix}\right)$ and a solution exists even for $z_1 \neq 0$. Therefore, the DAE is not completely observable. However, if additionally $Ex = 0$, then $x_1 = 0$ and thus $z_1 = 0$, so we have observability at infinity. If we choose $C = 0$, then $y = 0$ and $Ex = 0$ imply $y = 0$, but for $z_0 = \left(\begin{smallmatrix} 0 \\ z_2 \end{smallmatrix}\right)$ with $z_2 \neq 0$ a solution is given by $x = \left(\begin{smallmatrix} 0 \\ -z_2\delta \end{smallmatrix}\right)$, whence the DAE is not observable at infinity.

3.3 Relevant state observability

A classical result of control theory of linear time-invariant ODE systems is that controllability and observability are dual in a certain sense, see e.g. [80, Sec 3.3]. We will see in Section 5 that for *regular* systems the concepts of behavioral, impulse, strong and complete observability and observability at infinity, are indeed dual to the respective controllability concepts as introduced in [16]. The singular case however exhibits a certain lack of duality. To account for this we introduce the weaker concepts of *relevant state (RS) behavioral, impulse, strong and complete observability and RS observability at infinity*, which will prove to be dual to the respective controllability concepts in Section 5. These concepts refer, as their name suggests, to observability up to "a certain part of the state", i.e., state variables that are not uniquely determined by their past, input and output. The reason is that, from a physical point of view, these states only appear in the model because of "bad design" and the system should not be deemed unobservable because it contains free variables. The definitions are as follows.

Definition 3.9. The system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is called

(a) *RS behaviorally observable*

$$:\Longleftrightarrow \quad \forall (x^1,u,y),(x^2,u,y) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}\ \exists (x^3,u,y) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]} : \\ (x^3)_{(-\infty,0)} = (x^1)_{(-\infty,0)} \ \wedge\ (x^3)_{(0,\infty)} = (x^2)_{(0,\infty)},$$

(b) *RS impulse observable*

$$:\Longleftrightarrow \quad \forall x_0^1,x_0^2 \in \mathbb{R}^n : \\ \Big[(x^1,u,y) \in \mathfrak{B}^{\delta Ex_0^1}_{[E,A,B,C,D]} \ \wedge\ (x^2,u,y) \in \mathfrak{B}^{\delta Ex_0^2}_{[E,A,B,C,D]} \ \wedge\ Ex^1 = Ex^2 \\ \Rightarrow \quad Ex_0^1 = Ex_0^2\Big],$$

(c) *RS strongly observable*

$$:\Longleftrightarrow \quad \forall x_0^1,x_0^2 \in \mathbb{R}^n : \\ \Big[(x^1,u,y) \in \mathfrak{B}^{\delta Ex_0^1}_{[E,A,B,C,D]} \ \wedge\ (x^2,u,y) \in \mathfrak{B}^{\delta Ex_0^2}_{[E,A,B,C,D]} \\ \Rightarrow \quad Ex_0^1 = Ex_0^2\Big],$$

(d) *RS observable at infinity*

$$:\Longleftrightarrow \quad \forall z_0^1,z_0^2 \in \mathbb{R}^l : \\ \Big[(x^1,u,y) \in \mathfrak{B}^{\delta z_0^1}_{[E,A,B,C,D]} \ \wedge\ (x^2,u,y) \in \mathfrak{B}^{\delta z_0^2}_{[E,A,B,C,D]} \ \wedge\ Ex^1 = Ex^2 \\ \Rightarrow \quad z_0^1 = z_0^2\Big],$$

(e) *RS completely observable*

$$:\iff \quad \forall z_0^1, z_0^2 \in \mathbb{R}^l : \\ \Big[(x^1,u,y) \in \mathfrak{B}^{\delta z_0^1}_{[E,A,B,C,D]} \ \wedge\ (x^2,u,y) \in \mathfrak{B}^{\delta z_0^2}_{[E,A,B,C,D]} \\ \Rightarrow \quad z_0^1 = z_0^2 \Big].$$

It is clear that RS strong (complete) observability implies RS impulse observability (RS observability at infinity). The forthcoming observability characterizations will further yield that a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is RS strongly observable if, and only if, it is RS behaviorally observable and RS impulse observable; it is RS completely observable if, and only if, it is RS behaviorally observable and RS observable at infinity.

Remark 3.10. One may wonder why the definition of RS behavioral observability is given in terms of the distributional behavior $\mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,B,C,D]}$ instead of the behavior $\mathfrak{B}_{[E,A,B,C,D]}$. The reason is that the concatenation of two solutions will in general introduce a jump at $t = 0$. For ODEs any concatenation with a jump in the state variable cannot be a solution, but for DAEs this is not true in general. However, the presence of a jump makes it necessary to view the DAE in a distributional solution space; in particular, Dirac impulses at $t = 0$ may occur in the solution in response to the jump. Nevertheless, the definition of RS behavioral observability can also be given in terms of $\mathfrak{B}_{[E,A,B,C,D]}$ as follows

$$\forall (x^1,u,y),(x^2,u,y) \in \mathfrak{B}_{[E,A,B,C,D]} \ \boxed{\exists T > 0}\ \exists (x^3,u,y) \in \mathfrak{B}_{[E,A,B,C,D]} : \\ (x^3)_{(-\infty,0)} \overset{\text{a.e.}}{=} (x^1)_{(-\infty,0)} \ \wedge\ (x^3)_{(T,\infty)} \overset{\text{a.e.}}{=} (x^2)_{(T,\infty)},$$

i.e., the concatenation is not instantaneous. Despite the slight technicalities involved, we find the definition via instantaneous concatenability more appealing because it does not introduce the additional concatenation time $T > 0$.

We can conclude that RS behavioral, impulse, strong and complete observability and RS observability at infinity can be characterized by the conditions from Definition 3.9 in which x_0^2, z_0^2, x^2, u and y are trivial (cf. Lemma 3.4).

Lemma 3.11 (Distinction from zero III)**.** *The system* $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ *is*

(a) RS behaviorally observable

$$\iff \quad \forall (x,0) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,C]} \ \exists (\bar{x},0) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E,A,C]} : \\ x_{(-\infty,0)} = \bar{x}_{(-\infty,0)} \ \wedge\ \bar{x}_{(0,\infty)} = 0,$$

(b) RS impulse observable

$$\iff \quad \forall x_0 \in \mathbb{R}^n : \left[(x,0) \in \mathfrak{B}^{\delta E x_0}_{[E,A,C]} \wedge Ex = 0 \quad \Rightarrow \quad Ex_0 = 0 \right],$$

(c) RS strongly observable

$$\iff \quad \forall x_0 \in \mathbb{R}^n : \left[(x,0) \in \mathfrak{B}^{\delta E x_0}_{[E,A,C]} \quad \Rightarrow \quad Ex_0 = 0 \right],$$

(d) RS observable at infinity

$$\iff \quad \forall z_0 \in \mathbb{R}^l : \left[(x,0) \in \mathfrak{B}^{\delta z_0}_{[E,A,C]} \wedge Ex = 0 \quad \Rightarrow \quad z_0 = 0 \right],$$

(e) RS completely observable

$$\iff \quad \forall z_0 \in \mathbb{R}^l : \left[(x,0) \in \mathfrak{B}^{\delta z_0}_{[E,A,C]} \quad \Rightarrow \quad z_0 = 0 \right].$$

As a consequence from Lemma 3.7 and Lemma 3.11 we can further state the following implications for the so far introduced observability notions.

Corollary 3.12. *The following implications hold true for any system in $\Sigma_{l,n,m,p}$:*

(i) *behaviorally observable $\implies$ RS behaviorally observable,*
(ii) *observable at infinity $\implies$ RS observable at infinity $\implies$ RS impulse observable,*
(iii) *completely observable $\implies$ RS completely observable $\implies$ RS strongly observable.*

Note that it is still not clear (however true) that impulse (strong) observability implies RS impulse (strong) observability. To show this we need the characterizations in terms of the output injection form derived in Section 4.

It will later turn out, see Corollary 4.13, that for regular systems the observability concepts from Subsections 3.1 and 3.2 are equivalent to the respective relevant state observability concepts from Definition 3.9. In view of this, Examples 3.2 and 3.8 provide some illustrative examples for the RS observability concepts.

3.4 Comparison of the concepts with the literature

We compare the relations of the observability concepts introduced in the present paper to existing notions in the literature in the following list of remarks.

(i) The observability concepts are not consistently treated in the literature. While some authors rely on intuitive extensions of the definition known for ODEs [29, 88], others insist on duality to the known controllability concepts [31]. Furthermore, one has to pay attention if it is (tacitly) claimed that $[E^\top, C^\top] \in \mathbb{R}^{l\times(n+p)}$ or $[E^\top, A^\top, C^\top] \in \mathbb{R}^{l\times(2n+p)}$ have full rank. Some of the references introduce observability by means of certain rank criteria for the matrices E,A,C. The connection of the observability concepts to linear algebraic properties of E, A and C are highlighted in Section 6 (and are partly used to derive the following comparisons).

(ii) For *regular systems* the number of different observability concepts reduces to five by Corollary 4.13. We have the following relationships between the observability notions introduced here and the ones given in the literature:

concept	coincides with	called [...] in
behavioral obs.	–	obs. in [29, 88]; R-obs. in [32]; jump obs. in [77]
impulse obs.	[31, 32, 77]	obs. at infinity in [3, 53, 82]
strong obs.	[82]	–
obs. at infinity	[13, 33]	dual normalizability in [32]
complete obs.	[24]	obs. in [31, 32]

(iii) There is also a significant amount of literature dealing with observability for general DAEs; the relationship to the notions introduced here is as follows:

concept	coincides with	called [...] in
behavioral obs.	–	obs. in [70]; right-hand side obs. in [40]; strong almost obs. in [67]
impulse obs.	[24, 26, 40, 43]	obs. at infinity [24, 26][2]
strong obs.	[67]	obs. in [40, 66]
obs. at infinity	–	–
complete obs.	–	obs. in [35]; str. obs. in [66]; str. compl. obs. in [67]
RS behavioral obs.	–	–
RS impulse obs.	–	–
RS strong obs.	–	obs. in [10,67]; weakly obs. in [66]
RS obs. at infinity	–	–

[2] In [24, 26] the notions of impulse observability and observability at infinity are both used for impulse observability.

RS complete obs.	–	strong obs. in [10]; complete obs. in [67][3]

Observability concepts for general discrete time DAE systems have been introduced and investigated in [6, 8, 9].

(iv) Impulse observability and observability at infinity are usually defined by considering distributional solutions of (1) (similar to our definitions), see e.g. [31, 43], sometimes called impulsive modes, see [13, 40, 82]. For regular systems, impulse observability was introduced by VERGHESE ET AL. [82] (called observability at infinity in this work) as observability of the impulsive modes of the system, and later made more precise by COBB [31], see also ARMENTANO [3] (who also calls it observability at infinity) for a more geometric point of view. In [82] the authors do also develop the notion of strong observability as impulse observability with, additionally, "observability in the sense of the regular theory".
The name "observability at infinity" comes from the claim that the system has no infinite unobservable modes: Speaking in terms of rank criteria (see also Section 6) the system $[E,A,C] \in \mathcal{O}_{l,n,p}$ is said to have an unobservable mode at $\frac{\alpha}{\beta}$ if, and only if, $\mathrm{rk}[\alpha E^\top + \beta A^\top, C^\top] < \mathrm{rk}[E^\top, A^\top, C^\top]$ for some $\alpha, \beta \in \mathbb{C}$. If $\beta = 0$ and $\alpha \neq 0$, then the unobservable mode is infinite. Observability at infinity was introduced by ROSENBROCK [73] – although he does not use this phrase – as the absence of infinite output decoupling zeros. Later, COBB [31] compared the concepts of impulse observability and observability at infinity, see [31, Thm. 10]; the notions we use in the present paper go back to the distinction in this work.

(v) Observability concepts with a distributional solution setup have also been considered in [31, 67]. Distributional solutions for time-invariant DAEs have already been considered by COBB [30] and GEERTS [37, 38] and for time-varying DAEs by RABIER and RHEINBOLDT [72], and by KUNKEL and MEHRMANN [52]. In the present paper we use the approach by TRENN [77, 78]. The latter framework is also the basis for several observability concepts for switched DAE systems [**?**].

(vi) Behavioral observability was first defined by YIP and SINCOVEC [88], although merely called observability, as the dual of R-controllability for regular DAEs. They define observability essentially as the state x being computable from the input u, the output y, and the system data E, A, C. This is equivalent to classical observability of the ODE part of the system. Furthermore, it is equivalent to trivial output implying trivial state and hence to behavioral observability. The same approach is followed in [29] and it is emphasized that this "obvious extension of observability is not the dual of complete controllability". We stress that it is not even the dual of R-controllability when it comes to general DAE systems;

[3] Note that although the notion of complete observability is used in [67], it is only introduced by a geometric condition and not by a time domain definition.

however, as it will be shown in Corollary 5.2, the dual of R-controllability is RS behavioral observability.
In the context of the behavioral approach, behavioral observability was introduced in [70], but it is different to RS behavioral observability. These concepts are suitable for generalizations in various directions, see e.g. [28,42,85]. Having found the behavior of the considered system one can take over the definition of RS behavioral observability without the need for any further changes. From this point of view this appears to be the most natural of the observability concepts. However, this concept also seems to be the least regarded in the DAE literature.

(vii) The observability theory of DAE systems can also be treated with the theory of differential inclusions [4, 5] as showed by FRANKOWSKA [35]. However, FRANKOWSKA assumes observability at infinity in order to derive duality between controllability and observability as introduced in [35].

4 Output injection normal form

In this section we recall the concept of output injection for DAE systems and show that it induces an equivalence relation on $\mathscr{O}_{l,n,p}$. Then we state a normal form under this equivalence relation, which we use to characterize the observability concepts introduced in Section 3.

4.1 Output injection equivalence and normal form

Output injection is usually understood as the addition of the output y of the system, weighted by some matrix $L \in \mathbb{R}^{l\times p}$, to the right hand side of the systems equation. Since $y(t) = Cx(t)$, the resulting system has the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= (A+LC)x(t),\\ y(t) &= Cx(t). \end{aligned} \tag{4}$$

Output injection can be understood as an algebraic transformation (more precise: a group operation) within the set $\mathscr{O}_{l,n,p}$:

$$\begin{bmatrix} E \\ A+LC \\ C \end{bmatrix} = \begin{bmatrix} I_l & 0 & 0 \\ 0 & I_l & L \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} E \\ A \\ C \end{bmatrix}.$$

Allowing also for state space and output space transformations leads to the following notion of output injection equivalence.

Definition 4.1 (Output injection equivalence)**.** Two systems $[E_i, A_i, C_i] \in \mathscr{O}_{l,n,p}$, $i = 1,2$, are called *output injection equivalent* (OI equivalent) if, and only if,

$$\begin{aligned}&\exists W \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_p(\mathbb{R}), L \in \mathbb{R}^{l\times p} :\\ &[E_1\,, A_1\,, C_1] = [WE_2T\,, WA_2T - LC_2T\,, VC_2T];\end{aligned} \tag{5}$$

we write

$$[E_1\,, A_1\,, C_1] \overset{W,T,V,L}{\sim_{OI}} [E_2\,, A_2\,, C_2]\,. \tag{6}$$

OI equivalence seems to have been first considered by MORSE [64] for linear ODE systems, and it has already been termed a "nonphysically realizable transformation". For DAE systems, OI equivalence was first exploited by KARCANIAS [47] using the framework introduced by MORSE.

Clearly, multiplying the first equation in the DAE (1) from the left with an invertible matrix W does not change the behaviors introduced in Section 2 at all and a coordinate transformation of the state via T and the output via V does not qualitatively change the behaviors. Provided that the output is zero, its addition to the state equation does certainly not change the behavior as well. This is made precise in the following.

Lemma 4.2 (Behavior and output injection)**.** *If* $[E_1, A_1, C_1], [E_2, A_2, C_2] \in \mathscr{O}_{l,n,p}$ *are OI equivalent for* $W \in \mathbf{Gl}_l(\mathbb{R})$*,* $T \in \mathbf{Gl}_n(\mathbb{R})$*,* $V \in \mathbf{Gl}_p(\mathbb{R})$*,* $L \in \mathbb{R}^{l\times p}$ *as in* (6)*, then we have*

(a) $(x,0) \in \mathfrak{B}_{[E_1,A_1,C_1]} \;\Leftrightarrow\; (Tx,0) \in \mathfrak{B}_{[E_2,A_2,C_2]}$.
(b) $(x,0) \in \mathfrak{B}^{\mathscr{D}'}_{[E_1,A_1,C_1]} \;\Leftrightarrow\; (Tx,0) \in \mathfrak{B}^{\mathscr{D}'}_{[E_2,A_2,C_2]}$.
(c) $(x,0) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E_1,A_1,C_1]} \;\Leftrightarrow\; (Tx,0) \in \mathfrak{B}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}_{[E_2,A_2,C_2]}$.
(d) $(x,0) \in \mathfrak{B}^{\mathrm{ITP}}_{[E_1,A_1,C_1]} \;\Leftrightarrow\; (Tx,0) \in \mathfrak{B}^{\mathrm{ITP}}_{[E_2,A_2,C_2]}$.
(e) $\forall z_0 \in \mathbb{R}^l:\; (x,0) \in \mathfrak{B}^{\delta z_0}_{[E_1,A_1,C_1]} \;\Leftrightarrow\; (Tx,0) \in \mathfrak{B}^{\delta W^{-1} z_0}_{[E_2,A_2,C_2]}$.
In particular, $(x,0) \in \mathfrak{B}^{\delta z_0}_{[E_1,A_1,C_1]}$ *satisfies*

$$E_1 x = 0 \;\Leftrightarrow\; E_2(Tx) = 0.$$

Finally, due to Lemma 3.4, Lemma 3.7 and Lemma 3.11 we can restrict our attention to the solutions which produce a zero output. In summary we have the following result.

Proposition 4.3 (Invariance under output injection)**.** *On the set* $\mathscr{O}_{l,n,p}$*, behavioral, impulse, strong and complete observability, observability at infinity and the corresponding relevant state RS concepts are all invariant under OI equivalence.*

Proposition 4.3 allows to analyze the observability concepts by means of a normal form under OI equivalence. In order to present such a normal form, we need to introduce the following notation: For $k \in \mathbb{N}$ let

$$N_k = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}, \quad K_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix},\; L_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}.$$

Further, let $e_i^{[k]} \in \mathbb{R}^k$ be the ith canonical unit vector, and, for some multi-index $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, we define

$$\begin{aligned} N_\alpha &= \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_r}) \in \mathbb{R}^{|\alpha| \times |\alpha|}, \\ K_\alpha &= \operatorname{diag}(K_{\alpha_1}, \ldots, K_{\alpha_r}) \in \mathbb{R}^{(|\alpha| - \ell(\alpha)) \times |\alpha|}, \\ L_\alpha &= \operatorname{diag}(L_{\alpha_1}, \ldots, L_{\alpha_r}) \in \mathbb{R}^{(|\alpha| - \ell(\alpha)) \times |\alpha|}, \\ E_\alpha &= \operatorname{diag}(e_{\alpha_1}^{[\alpha_1]}, \ldots, e_{\alpha_r}^{[\alpha_r]}) \in \mathbb{R}^{|\alpha| \times \ell(\alpha)}. \end{aligned}$$

We are now in the position to derive a normal form under OI equivalence for systems $[E,A,C] \in \mathcal{O}_{l,n,p}$. We stress that we use the terminus "normal form" in a colloquial way to distinguish it from the mathematical terminus "canonical form". As to whether the following form is a normal or canonical form is clarified in Remark 4.7.

Theorem 4.4 (Normal form under OI equivalence)**.** *Let $[E,A,C] \in \mathcal{O}_{l,n,p}$. Then there exist $W \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_p(\mathbb{R}), L \in \mathbb{R}^{l \times p}$ such that*

$$[E,A,C] \overset{W,T,V,L}{\sim_{OI}} \left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & N_\kappa^\top & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{o}}} \end{bmatrix}, \begin{bmatrix} N_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & L_\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\overline{o}} \end{bmatrix}, \begin{bmatrix} E_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right], \quad (7)$$

for some multi-indices $\alpha, \beta, \gamma, \varepsilon, \kappa$ and a matrix $A_{\overline{o}} \in \mathbb{R}^{n_{\overline{o}} \times n_{\overline{o}}}$.

Proof. It is easy to see, that $[E_1,A_1,C_1],[E_2,A_2,C_2] \in \mathcal{O}_{n,m,p}$ are OI equivalent if, and only if, $[E_1^\top, A_1^\top, C_1^\top]$ and $[E_1^\top, A_1^\top, C_1^\top]$ with corresponding DAEs

$$E_1^\top \dot{z} = A_1^\top z + C_1^\top u \quad \text{and} \quad E_2^\top \dot{z} = A_2^\top z + C_2^\top u$$

are feedback equivalent in the sense of [16, Def. 3.1]. Hence the transposed feedback normal form derived in [16, Thm. 3.3] is a normal form under OI equivalence. □

Remark 4.5 (Duality for DAEs)**.** It should be noted that although we utilized a "duality" argument in the proof of Theorem 4.4, we have not really defined duality for DAEs or its corresponding behaviors yet. In fact, the proof of Theorem 4.4 just utilizes a normal form for matrix triples and is not related to certain solution concepts for DAEs. Further duality results for DAEs are presented in Section 5.

The interpretation of the OI normal form (7), in terms of solutions of DAEs is as follows: $(x,y) \in \mathfrak{B}_{[E,A,C]}$ if, and only if,

$$\begin{aligned} \left(x_{co}(\cdot)^\top, x_o(\cdot)^\top, x_{uo}(\cdot)^\top, x_u(\cdot)^\top, x_f(\cdot)^\top, x_{\overline{o}}(\cdot)^\top\right)^\top &:= Tx(\cdot), \\ \left(y_{co}(\cdot)^\top, y_{uo}(\cdot)^\top, y_{\overline{o}}(\cdot)^\top\right)^\top &:= Vy(\cdot), \end{aligned}$$

with

$$x_{co}(\cdot) = \begin{pmatrix} x_{co[1]}(\cdot) \\ \vdots \\ x_{co[\ell(\alpha)]}(\cdot) \end{pmatrix}, \quad y_{co}(\cdot) = \begin{pmatrix} y_{co[1]}(\cdot) \\ \vdots \\ y_{co[\ell(\alpha)]}(\cdot) \end{pmatrix}, \quad x_{o}(\cdot) = \begin{pmatrix} x_{o[1]}(\cdot) \\ \vdots \\ x_{o[\ell(\beta)]}(\cdot) \end{pmatrix},$$
$$x_{uo}(\cdot) = \begin{pmatrix} x_{uo[1]}(\cdot) \\ \vdots \\ x_{uo[\ell(\gamma)]}(\cdot) \end{pmatrix}, \quad y_{uo}(\cdot) = \begin{pmatrix} y_{uo[1]}(\cdot) \\ \vdots \\ y_{uo[\ell(\gamma)]}(\cdot) \end{pmatrix}, \quad x_{u}(\cdot) = \begin{pmatrix} x_{u[1]}(\cdot) \\ \vdots \\ x_{u[\ell(\varepsilon)]}(\cdot) \end{pmatrix},$$
$$x_{f}(\cdot) = \begin{pmatrix} x_{f[1]}(\cdot) \\ \vdots \\ x_{f[\ell(\kappa)]}(\cdot) \end{pmatrix}$$

solves the decoupled DAEs

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_{co[i]} = N_{\alpha_i} x_{co[i]}, \quad y_{co[i]} = \left(e_{\alpha_i}^{[\alpha_i]}\right)^\top x_{co[i]}, \quad \text{for } i = 1, \ldots, \ell(\alpha), \tag{8a}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} K_{\beta_i}^\top x_{o[i]} = L_{\beta_i}^\top x_{o[i]}, \quad \text{for } i = 1, \ldots, \ell(\beta), \tag{8b}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} L_{\gamma_i} x_{uo[i]} = K_{\gamma_i} x_{uo[i]}, \quad y_{uo[i]} = \left(e_{\gamma_i}^{[\gamma_i]}\right)^\top x_{uo[i]}, \quad \text{for } i = 1, \ldots, \ell(\gamma), \tag{8c}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} K_{\varepsilon_i} x_{u[i]} = L_{\varepsilon_i} x_{u[i]}, \quad \text{for } i = 1, \ldots, \ell(\varepsilon), \tag{8d}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\kappa_i}^\top x_{f[i]} = x_{f[i]}, \quad \text{for } i = 1, \ldots, \ell(\kappa), \tag{8e}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_{\overline{o}} = A_{\overline{o}} x_{\overline{o}}, \quad y_{\overline{o}} = 0. \tag{8f}$$

An analogous interpretation holds for $(x,u) \in \mathfrak{B}_{[E,A,C]}^{\mathscr{D}'}$ and $(x,u) \in \mathfrak{B}_{[E,A,C]}^{\mathscr{D}'_{\mathrm{pw}\mathscr{C}^\infty}}$. For $(x,u) \in \mathfrak{B}_{[E,A,C]}^{\mathrm{ITP}}$ the equations in (8) have to be restricted to the interval $[0,\infty)$ and for $(x,u) \in \mathfrak{B}_{[E,A,C]}^{\delta z_0}$ an appropriate term $\delta \tilde{z}_0$ has to be added to the respective state space equations in (8).

Remark 4.6 (Regular case)**.** In general, the OI normal form (7) for a *regular* system $[E,A,C] \in \mathscr{O}_{n,n,p}$, that is a system with a regular pencil $sE - A$, is *not regular*. For example, the regular system

$$[E,A,C] = \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, [0,1] \right]$$

has the nonregular OI normal form

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, [0,1] \right]$$

which consists of a 2×1 β-block and a 0×1 γ-block. However, the OI normal form of a regular system cannot have underdetermined DAEs of the form (8d), i.e., $\ell(\varepsilon)=0$, because these DAEs would correspond to underdetermined parts in the original coordinates as well (because the nonexisting output cannot "fix" this nonuniqueness).

Remark 4.7 (Canonical and normal form)**.** To explain the difference between our notions of normal and canonical form, recall the definition of a canonical form: given a group G, a set $\mathscr{S}$, and a group action $\alpha : G\times\mathscr{S}\to\mathscr{S}$ which defines an equivalence relation $s\overset{\alpha}{\sim}s'$ if, and only if, there exists $U\in G$ such that $\alpha(U,s)=s'$. Then a map $\Gamma:\mathscr{S}\to\mathscr{S}$ is called a *canonical form for* α [22] if, and only if,

$$\forall s,s'\in\mathscr{S} \;:\; \Gamma(s)\overset{\alpha}{\sim}s \quad \wedge \quad \Big[s\overset{\alpha}{\sim}s' \Leftrightarrow \Gamma(s)=\Gamma(s')\Big].$$

Therefore, the set $\mathscr{S}$ is divided into disjoint orbits (i.e., equivalence classes) and the mapping Γ picks a unique representative in each equivalence class. In the setup of OI equivalence, the group is $G=\mathbf{Gl}_l(\mathbb{R})\times\mathbf{Gl}_n(\mathbb{R})\times\mathbf{Gl}_p(\mathbb{R})\times\mathbb{R}^{l\times p}$, the considered set is $\mathscr{S}=\mathscr{O}_{l,n,p}$ and the group action

$$\alpha\big((W,T,V,L),[E,A,C]\big)=[WET,WAT+LCT,VCT]$$

corresponds to $\overset{W,T,V,L}{\sim_{OI}}$. However, Theorem 4.4 does not provide a mapping Γ because the matrix $A_{\overline{\sigma}}$ is not uniquely specified. This means that the form (7) is not a unique representative within the equivalence class and hence it is not a canonical form.

However, the OI normal form (7) is very close to a canonical form in the following sense: By a further (in general complex-valued) transformation we may put $A_{\overline{\sigma}}$ into Jordan canonical form. If the entries of the multi-indices $\alpha,\beta,\gamma,\varepsilon,\kappa$ are in non-decreasing order and in the Jordan canonical form of $A_{\overline{\sigma}}$ the Jordan blocks are ordered non-decreasing in size and lexicographically with respect to the corresponding eigenvalues if the blocks have the same size, then the OI normal form (7) is a canonical form.

Summarizing, the form (7) is not a canonical form but can be transformed into a canonical form. We therefore call the form as it stands a *normal form*.

Remark 4.8 (Canonical form under output injection and state feedback)**.** A combination of the OI normal form with the feedback form from [56] (see also [16, Thm. 3.3]) leads to a canonical from of systems $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$ under state space transformation, input space transformation, output space transformation, proportional output injection, proportional state feedback and transformation of the codomain of the state (i.e., left transformation of E,A,B) which was derived in [55]. However, this form is not suitable for either the analysis of controllability or observability, since it is necessary to apply state feedback and output injection simultaneously to obtain the canonical form; but controllability is not invariant under output injection and observability is not invariant under state feedback.

4.2 Characterization of behavioral, impulse and strong observability

Based on the OI normal form we will now present the characterization of behavioral, impulse and strong observability. To this end we first present the observability properties of each of the individual decoupled DAE systems in (8).

Lemma 4.9. *Consider the decoupled DAEs* (8) *resulting from the OI normal form. Then the DAEs*

- (8a) *are always behaviorally, impulse and strongly observable.*
- (8b) *are always behaviorally, impulse and strongly observable.*
- (8c) *are always behaviorally observable; they are impulse and strongly observable if, and only if,* $|\gamma| = \ell(\gamma)$*, i.e.,* $\gamma_i = 1$ *for all* $i = 1, \ldots, \ell(\gamma)$*.*
- (8d) *are neither behaviorally, impulse nor strongly observable.*
- (8e) *are always behaviorally observable; they are impulse and strongly observable if, and only if,* $|\kappa| = \ell(\kappa)$*.*
- (8f) *are never behaviorally and strongly observable and always impulse observable.*

Proof. It suffices to consider behavioral and impulse observability, because the corresponding characterization for strong observability follows trivially from the combination of the characterizations of behavioral and impulse observability.

(8a): The solutions of the ODE with size $k \times k$

$$\dot{x} = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} x$$
$$y = [0, \ldots, 0, 1]x$$

satisfy $x = (y^{(k-1)}, y^{(k-2)}, \ldots, \dot{y}, y)^\top$, hence a zero output implies $x = 0$. For the corresponding ODE-ITP it is easy to see (cf. [77, Thm. 3.3]) that all solutions x exhibit no jumps and no impulses at $t = 0$, hence (irrespectively of the actual output) it holds that $x[0] = 0$.

(8b): DAEs of size $k \times (k-1)$ of the form

$$\begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} x \tag{9}$$

can be interpreted as DAEs of the form (8a) with size $(k-1) \times (k-1)$, where the last state variable x_{k-1} is equal to a zero output. Hence the same arguments as above show behavioral and impulse observability.

(8c): The solutions of the DAE with size $(k-1) \times k$

$$\begin{aligned} \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \dot{x} &= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} x \\ y &= [0, \ldots, 0, 1]x \end{aligned} \tag{10}$$

are given by $x \overset{\text{a.e.}}{=} (y^{(k-1)}, y^{(k-2)}, \ldots, \dot{y}, y)^\top$. Hence a zero output implies a zero state, which shows behavioral observability. If $k = 1$, then the DAE-ITP reduces to the output equation $y_{[0,\infty)} = x_{[0,\infty)}$ for the free (scalar) variable x, in particular, $y = 0$ implies $x[0] = 0$ and the DAE for $k = 1$ is impulse observable. If $k > 1$ we now have $(x_k)_{[0,\infty)} = y_{[0,\infty)}$ and $(x_{k-1})_{[0,\infty)} = (\dot{x}_k)_{[0,\infty)}$. In general $x_k(0^-) \neq 0 = y(0^+) = x_k(0^+)$, hence there will be a jump in x_k at $t = 0$ and consequently a Dirac impulse in x_{k-1}. Therefore, a zero output does not imply that $x[0] = 0$ and we do not have impulse observability.

(8d): The DAE of size $(k-1) \times k$

$$\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} x \tag{11}$$

contains the free variable x_k (unrelated to the output), hence neither $x = 0$ nor $x[0] = 0$ holds true in general and the DAE cannot be behaviorally or impulse observable.

(8e): The solutions of DAEs with size $k \times k$

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \dot{x} = x \tag{12}$$

satisfy $x \overset{\text{a.e.}}{=} 0$, hence we have behavioral observability. If $k = 1$ then the corresponding ITP reads as $x_{[0,\infty)} = 0$; in particular $x[0] = 0$ and impulse observability follows. For $k > 0$ we have $(x_k)_{[0,\infty)} = 0$ and $(x_{k-1})_{[0,\infty)} = (\dot{x}_k)_{[0,\infty)}$. In general $x_k(0^-) \neq 0$ and hence there is a jump in x_k and consequently a Dirac impulse in x_{k-1}, i.e., $x[0] \neq 0$, which shows that the DAE is not impulse observable.

(8f): The ODE (8f) has nontrivial solutions and a zero output, hence it is not behaviorally observable. As already observed for (8a) an ODE-ITP does not exhibit jumps or impulses at the initial time, hence $x[0] = 0$ in any case and we have shown impulse observability. □

4.3 Characterization of observability at infinity and complete observability

Here we analyze observability at infinity and complete observability by means of the OI normal form.

Lemma 4.10. *Consider the decoupled DAEs* (8) *resulting from the OI normal form. Then the DAEs*

- *(8a) are always completely observable and observable at infinity.*
- *(8b) are always completely observable and observable at infinity.*
- *(8c) are completely observable and observable at infinity if, and only if,* $|\gamma| = \ell(\gamma)$*, i.e.,* $\gamma_i = 1$ *for all* $i = 1, \ldots, \ell(\gamma)$.

(8d) are neither observable at infinity nor completely observable.
(8e) are neither observable at infinity nor completely observable.
(8f) are never completely observable and always observable at infinity.

Proof. In the following we use that complete observability implies observability at infinity.

(8a): Any solution of the ODE with size $k \times k$

$$\dot{x} = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & 1 & 0 \end{bmatrix} x + \delta z_0$$
$$y = [0, \dots, 0, 1] x$$

satisfies $x_i = \dot{x}_{i+1}$ on the intervals $(-\infty, 0)$ and $(0, \infty)$ for $i = k-1, \dots, 2, 1$. Hence $y = x_k = 0$ implies $x_i = 0$ on $(-\infty, 0)$ and $(0, \infty)$ for $i = k, k-1, \dots, 1$. It is easy to see that $x[t] = 0$ for all $t \in \mathbb{R}$, hence $\delta z_0 = \dot{x}[0] = (x(0^+) - x(0^-))\delta = 0$, which implies $z_0 = 0$ and $x[0] = 0$.

(8b): Any solution of the DAE with size $k \times (k-1)$ of the form

$$\begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} x + \delta z_0$$

satisfies $x_{k-1} = 0$ on $(-\infty, 0)$ and $(0, \infty)$. From $x_i = \dot{x}_{i+1}$ on these two intervals for $i = k-2, \dots, 2, 1$ it follows that $x = 0$ on $(-\infty, 0)$ and $(0, \infty)$. Hence $\dot{x}_1[0]$ does not contain a Dirac impulse (because x_1 does not have a jump at $t = 0$), and $\dot{x}_1[0] = \delta z_{0,1}$ implies $z_{0,1} = 0$ which in turn implies that $x_1[0] = 0$. Hence, inductively, for $i = 2, 3, \dots, k-1$ we conclude analogously from $\dot{x}_i[0] = x_{i-1}[0] + \delta z_{0,i}$ that $z_{0,i} = 0$ and $x_i[0] = 0$. This gives $x[0] = 0$ and, finally, $0 = x_{k-1}[0] + \delta z_{0,k}$ implies $z_{0,k} = 0$, which shows that also $z_0 = 0$ is necessary for existence of a solution.

(8c): Consider the DAE of size $(k-1) \times k$

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} x + \delta z_0$$
$$y = [0, \dots, 0, 1] x.$$

If $k = 1$, then complete observability follows from $x = y$ (note that there is no z_0 in this case).

Now we consider the case $k \geq 2$: With $z_0 = e_1^{[k-1]} \in \mathbb{R}^{k-1} \setminus \{0\}$, a simple calculation shows that for $x = \delta e_1^{[k]}$ we have $(x, 0) \in \mathfrak{B}^{\delta z_0}_{[L_k, K_k, (e_1^{[k]})^\top]}$. In particular, we have $Ex = 0$ and $x[0] = \delta e_1^{[k]} \neq 0$, whence the system is not observable at infinity.

(8d): Consider the DAE of size $(k-1) \times k$

$$\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} x + \delta z_0 \tag{13}$$

If $k = 1$, then $(\delta, 0) \in \mathfrak{B}^{\delta \cot 0}_{[K_0, L_0, 0_{0\times 1}]}$ and hence the system is not observable at infinity in this case. If $k \geq 2$, then for $z_0 = e^{[k-1]}_{k-1} \in \mathbb{R}^{k-1} \setminus \{0\}$ we have that $x = \delta e^{[k]}_k$ fulfills $(x, 0) \in \mathfrak{B}^{\delta z_0}_{[K_k, L_k, 0_{0\times k}]}$. In particular, we have $Ex = 0$ and $x[0] = \delta e^{[k]}_k \neq 0$. Hence, the DAE is not observable at infinity.

(8e): The DAE of size $k \times k$

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix} \dot{x} = x + \delta z_0 \tag{14}$$

has the unique solution

$$x = x[0] = -\sum_{j=0}^{k-1} (N_k^\top)^j \delta^{(j)} z_0.$$

Hence it can never be observable at infinity.

(8f): The ODE of size $k \times k$

$$\begin{aligned} \dot{x} &= Ax + \delta z_0 \\ y &= 0 \end{aligned}$$

has a solution x for any $z_0 \in \mathbb{R}^n$, hence it is never completely observable. The additional constraint $x = 0$ yields $\dot{x} = 0$ and hence $\delta z_0 = \dot{x} - Ax = 0$ which shows observability at infinity. ☐

4.4 Characterization of relevant state observability

Finally we consider the observability notions from Section 4.4. First we focus on RS behavioral, impulse and strong observability.

Lemma 4.11. *Consider the decoupled DAEs* (8) *resulting from the OI normal form. Then the DAEs*

(8a) are always RS behaviorally, RS impulse and RS strongly observable.
(8b) are always RS behaviorally, RS impulse and RS strongly observable.
(8c) are always RS behaviorally observable; they are RS impulse and RS strongly observable if, and only if, $|\gamma| = \ell(\gamma)$, *i.e.,* $\gamma_i = 1$ *for all* $i = 1, \ldots, \ell(\gamma)$.
(8d) are always RS behaviorally observable; they are RS impulse and RS strongly observable if, and only if, $|\varepsilon| = \ell(\varepsilon)$, *i.e.,* $\varepsilon_i = 1$ *for all* $i = 1, \ldots, \ell(\varepsilon)$.
(8e) are always RS behaviorally observable; they are RS impulse and RS strongly observable if, and only if, $|\kappa| = \ell(\kappa)$, *i.e.,* $\kappa_i = 1$ *for all* $i = 1, \ldots, \ell(\kappa)$.
(8f) are never RS behaviorally and RS strongly observable and always RS impulse observable.

Proof. First we consider RS behavioral and RS impulse observability. The statements for RS behavioral observability in (8a)–(8c) and (8e) follow by a combination of Corollary 3.12 and Lemma 4.9. Since observability at infinity implies RS impulse observability by Corollary 3.12, it follows from Lemma 4.10 that (8a), (8b) and (8f) are RS impulse observable.

We prove the remaining statements for RS behavioral and impulse observability:

(8c): If $|\gamma| = \ell(\gamma)$, then the DAE (8c) is RS impulse observable by a combination of Corollary 3.12 and Lemma 4.10. If $|\gamma| > \ell(\gamma)$, then we can use the same counterexample as in the proof of Lemma 4.10 for (8c) by observing that $z_0 = e_1^{[k-1]} = Ee_2^{[k]}$. Hence, the system is not RS impulse observable.

(8c): DAEs with size $(k-1)\times k$ of the form (11) are RS behaviorally observable by the characterization in Remark 3.10 and the fact that, by [70, Thm. 5.2.10], for any two solutions x^1, x^2 we can find some $T > 0$ and some $(x^3, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ with

$$(x^3)_{(-\infty,0)} \overset{\text{a.e.}}{=} (x^1)_{(-\infty,0)} \;\wedge\; (x^3)_{(T,\infty)} \overset{\text{a.e.}}{=} (x^2)_{(T,\infty)}.$$

If $|\varepsilon| = \ell(\varepsilon)$, then the DAE (13) is RS impulse observable by Lemma 3.11 (b) and the fact that $K_\varepsilon = 0 \in \mathbb{R}^{0\times|\varepsilon|}$. If $|\varepsilon| > \ell(\varepsilon)$, then we can use the same counterexample as in the proof of Lemma 4.10 for (8d) by observing that $z_0 = e_{k-1}^{[k-1]} = Ee_{k-1}^{[k]}$. Hence, the system is not RS impulse observable.

(8c): If $|\kappa| = \ell(\kappa)$, then we have RS impulse observability due to $N_\kappa^\top = 0$. If $|\kappa| > \ell(\kappa)$, then there is a DAE of the form (14) with $k \geq 2$. For $z_0 = N_k^\top e_2^{[k]} \in \mathbb{R}^k \setminus \{0\}$ the unique solution of (14) is $x = -\delta e_1^{[k]}$. Since $N_k^\top x = 0$ and $z_0 \neq 0$ the system is not RS impulse observable.

(8c): The ODE (8f) has nontrivial solutions that are uniquely determined by $x(0^+)$, whence it is not RS behaviorally observable.

The characterization of RS strong observability follows from analogous arguments. □

Now we prove the characterizations for RS complete observability and RS observability at infinity.

Lemma 4.12. *Consider the decoupled DAEs* (8) *resulting from the OI normal form. Then the DAEs*

(8a) are always RS completely observable and RS observable at infinity.
(8b) are always RS completely observable and RS observable at infinity.
(8c) are RS completely observable and RS observable at infinity if, and only if, $|\gamma| = \ell(\gamma)$, *i.e.,* $\gamma_i = 1$ *for all* $i = 1,\dots,\ell(\gamma)$.
(8d) are RS completely observable and RS observable at infinity if, and only if, $|\varepsilon| = \ell(\varepsilon)$, *i.e.,* $\varepsilon_i = 1$ *for all* $i = 1,\dots,\ell(\varepsilon)$.
(8e) are neither RS completely observable nor RS observable at infinity.
(8f) are never RS completely observable and always RS observable at infinity.

Proof. The proof is analogous to the proof of Lemma 4.10 with the only difference that for DAEs (8d) in the case $|\varepsilon| = \ell(\varepsilon)$ the system is RS observable at infinity (and hence RS completely observable) since $K_\varepsilon, L_\varepsilon \in \mathbb{R}^{0\times|\varepsilon|}$ and hence there is no z_0 (the number of rows is zero). ☐

4.5 Summary of observability characterizations

The different observability characterizations derived in the previous subsections in terms of the OI normal form are summarized in Table 1.

	$[I_{\alpha_i}, N_{\alpha_i}, (e_{\alpha_i}^{[\alpha_i]})^\top]$	$[K_{\beta_i}^\top, L_{\beta_i}^\top, 0_{0\times\beta_i-1}]$	$[L_{\gamma_i}, K_{\gamma_i}, (e_{\gamma_i}^{[\gamma_i]})^\top]$	$[K_{\varepsilon_i}, L_{\varepsilon_i}, 0_{0\times\varepsilon_i}]$	$[N_{\kappa_i}^\top, I_{\kappa_i}, 0_{0\times\kappa_i}]$	$[I_{n_{\overline{0}}}, A_{\overline{0}}, 0_{q\times\overline{o}}]$
behaviorally observable	✓	✓	✓	×	✓	×
impulse observable	✓	✓	$\Leftrightarrow \gamma_i = 1$	×	$\Leftrightarrow \kappa_i = 1$	✓
strongly observable	✓	✓	$\Leftrightarrow \gamma_i = 1$	×	$\Leftrightarrow \kappa_i = 1$	×
observable at infinity	✓	✓	$\Leftrightarrow \gamma_i = 1$	×	×	✓
completely observable	✓	✓	$\Leftrightarrow \gamma_i = 1$	×	×	×
RS behaviorally observable	✓	✓	✓	✓	✓	×
RS impulse observable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \varepsilon_i = 1$	$\Leftrightarrow \kappa_i = 1$	✓
RS strongly observable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \varepsilon_i = 1$	$\Leftrightarrow \kappa_i = 1$	×
RS observable at infinity	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \varepsilon_i = 1$	×	✓
RS completely observable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \varepsilon_i = 1$	×	×

Table 1: Characterization of the observability concepts in terms of the OI normal form.

We have separated the concepts into two groups of five concepts where the first group consists of the observability notions introduced in Subsections 3.1 and 3.2 and the second group consists of the corresponding relevant state observability notions introduced in Subsection 3.3.

Table 1 together with Lemma 4.2 allows for a characterization of the observability concepts in terms of the OI normal form.

In particular, for regular systems we can conclude the following simplifications from Remark 4.6 and Table 1.

Corollary 4.13. *Consider a* regular *system* $[E,A,C] \in \mathcal{O}_{n,n,p}$. *Then the following equivalences hold for the DAE system:*

(i) *behaviorally observable* $\iff$ *RS behaviorally observable,*

(ii) *impulse observable* $\iff$ *RS impulse observable,*

(iii) *strongly observable* $\iff$ *RS strongly observable,*

(iv) *observable at infinity* $\iff$ *RS observable at infinity,*

(v) *completely observable* $\iff$ *RS completely observable.*

From Table 1 the dependencies between the different observability concepts can easily be concluded and are illustrated in Figure 1.

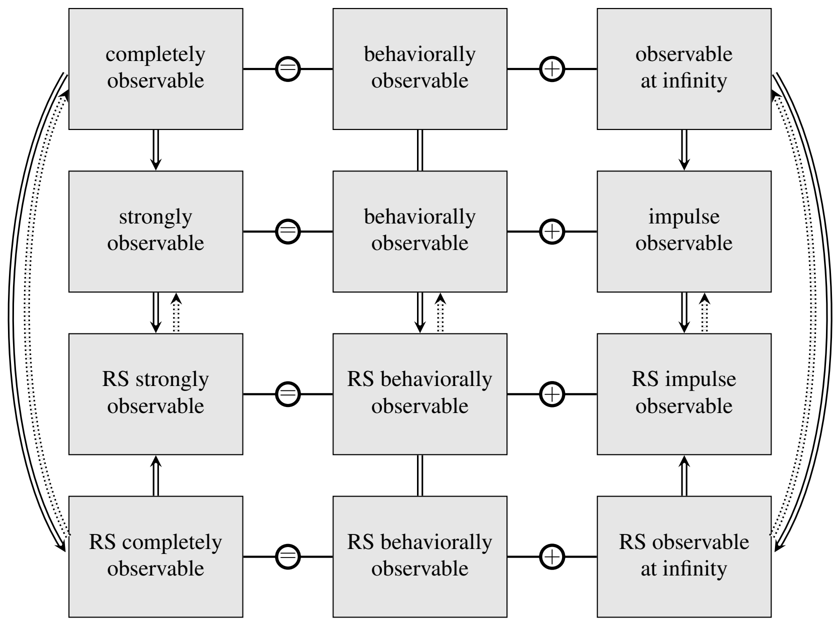

Fig. 1: Relationship between the different observability concepts. For each implication, the converse is false in general; dotted implications indicate the regular case.

5 Duality of observability and controllability

The intuitive definitions of behavioral and impulse observability given in Subsection 3.1 are not satisfying from a duality seeking point of view. Duality means that a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ has a certain observability property if, and only if, the "formal dual" system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}E^\top x(t) &= A^\top x(t) + C^\top u(t) \\ y(t) &= B^\top x(t) + D^\top u(t)\,, \end{aligned} \tag{15}$$

has the corresponding controllability property. Since the controllability properties of the dual system (15) do not depend on $B^\top$ and $D^\top$ it is sufficient to consider the class $\mathscr{C}_{l,n,m}$ of control systems governed by the equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{16}$$

where $E,A \in \mathbb{R}^{l\times n}, B \in \mathbb{R}^{l\times m}$; we write $[E,A,B] \in \mathscr{C}_{l,n,m}$. Each controllability concept (see [14] and the survey [16]) is invariant under the addition of a zero row in $[E,A,B] \in \mathscr{C}_{l,n,m}$ or, equivalently, an equation $0 = 0$ in (16). However, if we consider the dual system $[E^\top,A^\top,B^\top] \in \mathscr{O}_{n,l,m}$, then $E^\top,A^\top,B^\top$ have a common zero column and hence there exists a free state in the system which is not visible at the output. This implies that the system is neither impulse nor behaviorally observable, although $[E,A,B]$ may be both impulse and behaviorally controllable as introduced in [16]. This means that these observability and controllability concepts or not dual.

As already pointed out in Section 3.3, it is not always reasonable to view a state as unobservable which actually does not appear in any of the systems equations; it only appears in the model because of “bad design”. This viewpoint led us to the introduction of the relevant state observability concepts. It allows to provide duality results between the controllability concepts from [16] and the observability concepts from Sections 3.1–3.3. The RS observability concepts cope with “design errors” as mentioned above by preserving the physical meaning of observability. The duality results will provide algebraic characterizations for the observability concepts.

Remark 5.1. The “design errors” mentioned above can be given an interpretation using the behavioral framework. If (16) contains an equation of the form $0 = 0$ or other redundant equations, then it is not minimal in the behavioral sense as introduced in [70, Def. 2.5.24], see also [17]. Minimality is equivalent to $\mathrm{rk}_{\mathbb{R}[s]}[sE-A,B] = l$, see [17] for further characterizations. If this condition is not satisfied, then $\mathrm{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE^\top - A^\top \\ B^\top \end{bmatrix} < l$ and hence the equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E^\top \\ 0 \end{bmatrix} x(t) = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} x(t)$$

does always have an underdetermined part and thus non-unique solutions independent of the properties of the original system $[E,A,B]$. This leads to the “lack of duality” between (for instance) behavioral controllability in the sense of [16] and behavioral observability, in the case of non-minimal systems. Note that if $sE-A$ is regular, then $[E,A,B]$ is always minimal.
If minimality is assumed, then it is easy to check that the controllability concepts from [16] are indeed dual to the observability concepts introduced in the present paper. This can also be deduced from a recent approach by LOMADZE [57] to the definition of the dual of a behavioral system. When the definition of the dual system

given in [57] is applied to DAE systems (1), then the dual is exactly the formal dual system (15).
Summarizing, this justifies to say that a lack of duality does not come from intrinsic system properties, but from a bad (i.e., not minimal) model of the underlying behavior.

Using the OI normal form (which is the "dual" of the feedback form derived in [16]), the characterizations summarized in Table 1 and the respective results in [16] lead to the following duality results between the RS observability and the controllability concepts.

Corollary 5.2 (Duality between observability and controllability)**.** *Let* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *be given. Then we have the following equivalences:*

(a) $[E,A,C]$ *is RS behaviorally observable if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is behaviorally controllable in the sense of [16],*
(b) $[E,A,C]$ *is RS impulse observable if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is impulse controllable in the sense of [16],*
(c) $[E,A,C]$ *is RS strongly observable if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is strongly controllable in the sense of [16],*
(d) $[E,A,C]$ *is RS observable at infinity if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is controllable at infinity in the sense of [16],*
(e) $[E,A,C]$ *is RS completely observable if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is completely controllable in the sense of [16].*

In particular, for regular DAE systems we have duality between the five remaining observability concepts and the corresponding controllability concepts. The duality properties are summarized in Figure 2.

6 Algebraic criteria

Using the duality results derived in Corollary 5.2, in this section we derive algebraic criteria for the observability concepts. These criteria are generalizations of the Hautus test (also called Popov-Belevitch-Hautus test, since they were independently developed by POPOV [71], BELEVITCH [12] and HAUTUS [39]) in terms of rank and kernel criteria on the involved matrices. Most of these conditions are known – we refer to the relevant literature.

Proposition 6.1 (Algebraic criteria for observability)**.** *Let a system* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *be given. Then we have the following:*

$[E,A,C]$ *is*	*if, and only if,*
behaviorally observable	$\forall \lambda \in \mathbb{C}: \ \ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$.

impulse observable	$\ker_{\mathbb{R}} E \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C = \{0\}$.
strongly observable	$\ker_{\mathbb{R}} E \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C = \{0\}$ $\wedge\ \forall \lambda \in \mathbb{C}:\ \ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$.
observable at infinity	$\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} C = \{0\}$.
completely observable	$\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} C = \{0\}$ $\wedge\ \forall \lambda \in \mathbb{C}:\ \ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$.
RS behaviorally observable	$\forall \lambda \in \mathbb{C}:\ \dim \ker_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ C \end{bmatrix} = \dim \ker_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix}$.
RS impulse observable	$\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C = \ker_{\mathbb{R}} E \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C$.
RS strongly observable	$\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C = \ker_{\mathbb{R}} E \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C$ $\wedge\ \forall \lambda \in \mathbb{C}: \ker_{\mathbb{C}} E \cap \ker_{\mathbb{C}} A \cap \ker_{\mathbb{C}} C = \ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C$.
RS observable at infinity	$\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C = \ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} C$.
RS completely observable	$\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C = \ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} C$ $\wedge\ \forall \lambda \in \mathbb{C}: \ker_{\mathbb{C}} E \cap \ker_{\mathbb{C}} A \cap \ker_{\mathbb{C}} C = \ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C$.

Proof. Combining Corollary 5.2 and [16, Cor. 4.3] the criteria for RS behavioral, impulse, strong and complete observability and RS observability at infinity follow immediately. From the OI normal form (7) it can be concluded that

$$\ell(\varepsilon) = 0 \ \wedge\ \det A_{\overline{0}} \neq 0 \quad \Longleftrightarrow \quad \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C = \{0\}.$$

Therefore, invoking Table 1, behavioral observability is equivalent to RS behavioral observability together with the condition $\ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C = \{0\}$. Hence, the characterization of RS behavioral observability follows from observing that the conditions $\ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C = \{0\}$ and $\operatorname{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A \\ C \end{smallmatrix}\right] = \operatorname{rk}_{\mathbb{C}} \left[\begin{smallmatrix} \lambda E-A \\ C \end{smallmatrix}\right]$ for all $\lambda \in \mathbb{C}$ are equivalent to $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$ for all $\lambda \in \mathbb{C}$.
Furthermore, it follows from the OI normal form that

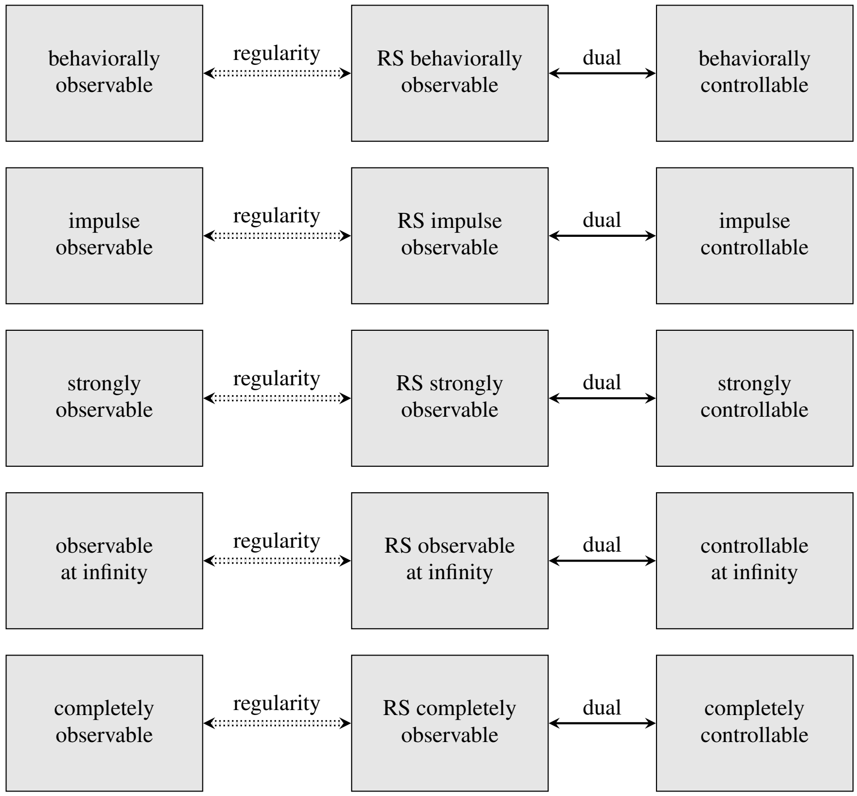

Fig. 2: Illustration of duality between observability and controllability.

$$\ell(\varepsilon)=0 \quad \iff \quad \ell(\varepsilon)=|\varepsilon| \;\wedge\; \ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C=\{0\}. \tag{17}$$

Therefore, invoking Table 1, impulse observability is equivalent to RS impulse observability together with the condition $\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C=\{0\}$, which yields the characterization in the statement of the corollary. The characterization of strong observability then follows from those of behavioral and impulse observability.
Likewise, equation (17) implies that observability at infinity is equivalent to RS observability at infinity together with the condition $\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A \cap \ker_{\mathbb{R}} C=\{0\}$, which yields the characterization in the statement of the corollary. Finally, the characterization for complete observability then follows from those of behavioral observability and observability at infinity. □

In the following we consider further criteria for the observability concepts.

Remark 6.2 (RS observability at infinity)**.** Proposition 6.1 immediately implies that RS observability at infinity is equivalent to

$$\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} C \subseteq \ker_{\mathbb{R}} A.$$

In terms of a rank criterion, this is the same as

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix}. \tag{18}$$

Likewise, observability at infinity is equivalent to the rank condition

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{19}$$

As far as the authors are aware, the conditions (18) and (19) are new for general DAE systems. In the case of regular $sE - A \in \mathbb{R}[s]^{n\times n}$, condition (19) can be found for instance in [31].

Remark 6.3 (RS impulse observability)**.** It follows from Proposition 6.1 that an equivalent characterization for RS impulse observability is that, for one (and hence any) matrix Z with $\operatorname{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} E^\top$, we have

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix}. \tag{20}$$

Likewise, impulse observability is equivalent to

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix} = n. \tag{21}$$

This was first derived in [43]. Furthermore, in [40, 43] it was shown that impulse observability is equivalent to

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + \operatorname{rk}_{\mathbb{R}} E, \tag{22}$$

which is in fact equivalent to (21). If the pencil $sE - A$ is regular, then condition (21) for impulse observability can also be inferred from [32, Thm. 2-3.4].

Remark 6.4 (RS behavioral observability)**.** The algebraic criterion for RS behavioral observability in Proposition 6.1 is equivalent to the fact that the augmented matrix pencil

$$s\mathscr{E} - \mathscr{A} = s \begin{bmatrix} E \\ 0 \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \in \mathbb{R}[s]^{(l+p)\times n}$$

has no eigenvalues. Behavioral observability coincides with observability as defined in [70, Def. 5.3.2] for the larger class of linear differential behaviors, and the rank condition for behavioral observability in Proposition 6.1 has already been derived in [70, Thm. 5.3.3]; the condition has also been derived in [40] where this concept is called right-hand side observability. RS behavioral observability for systems with regular $sE-A$ is considered in [32, Thm. 2-3.2] (called R-observability in this work), where the condition

$$\forall \lambda \in \mathbb{C}: \ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n$$

is derived. This is, for regular $sE-A$, in fact equivalent to the criterion for RS behavioral observability in Proposition 6.1.

Remark 6.5 (RS complete and strong observability)**.** By Table 1, RS complete observability of $[E,A,C] \in \mathscr{O}_{l,n,p}$ is equivalent to $[E,A,C]$ being RS behaviorally observable and RS observable at infinity, whereas RS strong observability of $[E,A,C]$ is equivalent to $[E,A,C]$ being RS behaviorally observable and RS impulse observable.
The algebraic conditions for strong observability in Proposition 6.1 have been first derived in [40] (called observability in this work). On the other hand, as far as the authors are aware, the algebraic criterion for RS complete observability is new for general DAE systems.
For regular systems, the conditions in Proposition 6.1 for complete observability are also derived in [32, Thm. 2-3.1].

The above considerations lead to the following alternative formulation of Proposition 6.1 in terms of rank criteria.

Corollary 6.6. *Let $[E,A,C] \in \mathscr{O}_{l,n,p}$ and Z be a matrix with $\operatorname{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} E^\top$. Then we have the following:*

$[E,A,C]$ *is*	*if, and only if,*
behaviorally observable	$\forall \lambda \in \mathbb{C}: \ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n.$
impulse observable	$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix} = n.$
strongly observable	$\forall \lambda \in \mathbb{C}: \ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix} = n.$
observable at infinity	$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix} = n.$

competely observable	$\forall \lambda \in \mathbb{C}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix} = n.$
RS behaviorally observable	$\forall \lambda \in \mathbb{C}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ C \end{bmatrix}.$
RS impulse observable	$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix}.$
RS strongly observable	$\forall \lambda \in \mathbb{C}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix}.$
RS observable at infinity	$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix}.$
RS completely observable	$\forall \lambda \in \mathbb{C}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix}.$

Remark 6.7 (Kalman criterion for regular systems)**.** For regular systems $[E,A,C] \in \mathscr{O}_{n,n,p}$ the usual Hautus and Kalman criteria for observability can be found in a summarized form e.g. in [32]. Other approaches to derive observability criteria rely on the expansion of $(sE-A)^{-1}$ as a power series in s at $s_0 = 0$, which is only feasible in the regular case. For instance, in [63] the numerator matrices of this expansion, i.e., the coefficients of the polynomial $\operatorname{adj}(sE-A)$, are used to derive a rank criterion for complete observability. Then again, in [49] Kalman rank criteria for complete observability, behavioral observability (called R-observability in this work) and observability at infinity are derived in terms of the coefficients of the power series expansion of $(sE-A)^{-1}$. The advantage of these criteria, especially the last one, is that no transformation of the system needs to be performed as it is usually necessary in order to derive Kalman rank criteria for DAEs, see e.g. [32].
However, simple criteria can be obtained using only a left transformation of little impact: if $\alpha \in \mathbb{R}$ is chosen such that $\det(\alpha E - A) \neq 0$, then the system is completely observable if, and only if, [89, Cor. 2]

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} C \\ C(\alpha E - A)^{-1} E \\ \vdots \\ C\big((\alpha E - A)^{-1} E\big)^{n-1} \end{bmatrix} = n,$$

and it is impulse observable if, and only if, [89, Thm. 5]

$$\ker_{\mathbb{R}}(\alpha E - A)^{-1} E \cap \ker_{\mathbb{R}} C \cap \operatorname{im}_{\mathbb{R}}(\alpha E - A)^{-1} E = \mathbb{R}^n.$$

7 Geometric criteria

In this section we derive geometric criteria for the observability concepts. Geometric theory plays a fundamental role in ODE system theory and was introduced independently by WONHAM and MORSE, and by BASILE and MARRO, see the famous books [11, 87] and also [80]. In [54], Lewis provided a survey of the to date geometric theory of DAEs. As we will do here, he put special emphasis on the two fundamental sequences $(\mathscr{V}_i)_{i\in\mathbb{N}_0}$ and $(\mathscr{W}_i)_{i\in\mathbb{N}_0}$ of subspaces defined as follows:

$$\begin{aligned} \mathscr{V}_0 &:= \mathbb{R}^n, & \mathscr{V}_{i+1} &:= A^{-1}(E\mathscr{V}_i) \cap \ker_{\mathbb{R}} C \subseteq \mathbb{R}^n, & \mathscr{V}^* &:= \bigcap_{i\in\mathbb{N}_0} \mathscr{V}_i, \\ \mathscr{W}_0 &:= \{0\}, & \mathscr{W}_{i+1} &:= E^{-1}(A\mathscr{W}_i) \cap \ker_{\mathbb{R}} C \subseteq \mathbb{R}^n, & \mathscr{W}^* &:= \bigcup_{i\in\mathbb{N}_0} \mathscr{W}_i. \end{aligned}$$

We will call the sequences $(\mathscr{V}_i)_{i\in\mathbb{N}}$ and $(\mathscr{W}_i)_{i\in\mathbb{N}}$ *restricted Wong sequences*. In [15, 18, 19] the Wong sequences for matrix pencils (i.e., $C = 0$) are investigated, the name chosen this way since WONG [86] was the first who used both sequences for the analysis of matrix pencils. In fact, the Wong sequences (with $C = 0$) can be traced back to DIEUDONNÉ [34], who focused on the first of the two Wong sequences. BERNHARD [21] and ARMENTANO [3] used the Wong sequences to carry out a geometric analysis of matrix pencils. They appear also in [1, 2, 51, 75]. The sequences $(\mathscr{V}_i)_{i\in\mathbb{N}}$ and $(\mathscr{W}_i)_{i\in\mathbb{N}}$ are no Wong sequences corresponding to any matrix pencils, that is why we call them *restricted Wong sequences* with respect to the system $[E,A,C] \in \mathscr{O}_{l,n,p}$.

For the investigation of observability of DAE systems, that is when $C \neq 0$, the restricted Wong sequences have been extensively studied by several authors, see e.g. [53, 61, 62, 65, 68, 81] for regular systems and [6, 8–10, 23, 54, 55, 66, 67] for general DAE systems.

For regular systems ÖZÇALDIRAN [65] (see also [66]) showed that $\mathscr{V}^*$ is the supremal (A,E)-invariant subspace contained in $\ker_{\mathbb{R}} C$ and $\mathscr{W}^*$ is the infimal restricted $(E,A;\ker_{\mathbb{R}} C)$-invariant subspace (which is also a subspace of $\ker_{\mathbb{R}} C$); note that by these invariance definitions, $\mathscr{W}^*$ is not the obvious dual to $\mathscr{V}^*$, but by the definition of the restricted Wong sequences this connection becomes more apparent.

The aforementioned invariance concepts, which have also been used in [1,7,53,62], are defined as follows.

Definition 7.1 ((A,E)- and $(E,A;\ker_{\mathbb{R}} C)$-invariance [65])**.** Let $E,A \in \mathbb{R}^{l\times n}$. A subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is called (A,E)*-invariant*, if

$$A\mathscr{V} \subseteq E\mathscr{V}.$$

For $C \in \mathbb{R}^{p\times n}$, a subspace $\mathscr{W} \subseteq \mathbb{R}^n$ is called *restricted* $(E,A;\ker_{\mathbb{R}} C)$*-invariant*, if

$$\mathscr{W} = \ker_{\mathbb{R}} C \cap E^{-1}(A\mathscr{W}).$$

It is easy to verify that the proofs given in [65, Lems. 2.1 & 2.2] remain the same for general $E,A \in \mathbb{R}^{l\times n}$ and (in the notation of [65]) $K = \ker_{\mathbb{R}} C$ for $C \in \mathbb{R}^{p\times n}$ and $B = 0$; this is shown in [7] as well. For $\mathscr{V}^*$ this can be found in [1], see also [62]. We have the following proposition.

Proposition 7.2 (Restricted Wong sequences as invariant subspaces)**.** *Consider* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *and the limits* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *of the restricted Wong sequences. Then the following statements hold true.*

(a) $\mathscr{V}^*$ *is* (A,E)*-invariant with* $\mathscr{V}^* \subseteq \ker_{\mathbb{R}} C$ *and for any* $\mathscr{V} \subseteq \ker_{\mathbb{R}} C$ *which is* (A,E)*-invariant it holds that* $\mathscr{V} \subseteq \mathscr{V}^*$*;*
(b) $\mathscr{W}^*$ *is restricted* $(E,A;\ker_{\mathbb{R}} C)$*-invariant and for any* $\mathscr{W} \subseteq \mathbb{R}^n$ *which is restricted* $(E,A;\ker_{\mathbb{R}} C)$*-invariant it holds that* $\mathscr{W}^* \subseteq \mathscr{W}$*.*

In the following we show how the observability concepts can be characterized in terms of the invariant subspaces $\mathscr{V}^*$ and $\mathscr{W}^*$ by using the OI normal form (7).

Theorem 7.3 (Geometric criteria for observability)**.** *Consider* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *and the limits* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *of the restricted Wong sequences. Then* $[E,A,C]$ *is*

(a) behaviorally observable if, and only if, $\mathscr{V}^* = \{0\}$*;*
(b) impulse observable if, and only if, $\mathscr{W}^* \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) = \{0\}$*;*
(c) strongly observable if, and only if, $(\mathscr{V}^* + \mathscr{W}^*) \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) = \{0\}$*;*
(d) observable at infinity if, and only if, $\mathscr{W}^* = \{0\}$*;*
(e) completely observable if, and only if, $\mathscr{V}^* + \mathscr{W}^* = \{0\}$*;*
(f) RS behaviorally observable if, and only if, $\mathscr{V}^* \subseteq \mathscr{W}^*$*;*
(g) RS impulse observable if, and only if, $A\mathscr{W}^* \cap \operatorname{im}_{\mathbb{R}} E = \{0\}$*.*
(h) RS strongly observable if, and only if, $(E\mathscr{V}^* + A\mathscr{W}^*) \cap \operatorname{im}_{\mathbb{R}} E = \{0\}$*.*
(i) RS observable at infinity if, and only if, $A\mathscr{W}^* = \{0\}$*;*
(j) RS completely observable if, and only if, $E\mathscr{V}^* + A\mathscr{W}^* = \{0\}$*;*

Proof. We prove the assertions by deriving formulas for $\mathscr{V}^*$ and $\mathscr{W}^*$ in terms of the OI normal form (7) and then connect the geometric conditions to the observability concepts by Table 1. We proceed in several steps.

Step 1: Let $[E_1,A_1,C_1],[E_2,A_2,C_2] \in \mathscr{O}_{l,n,p}$ be such that for some $W \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_p(\mathbb{R})$ and $L \in \mathbb{R}^{l\times p}$ it holds

$$[E_1, A_1, C_1] \overset{W,T,V,L}{\sim_{OI}} [E_2, A_2, C_2].$$

We show that the restricted Wong sequences $\mathscr{V}_i^1$, $\mathscr{W}_i^1$ of $[E_1,A_1,C_1]$ and the restricted Wong sequences $\mathscr{V}_i^2$, $\mathscr{W}_i^2$ of $[E_2,A_2,C_2]$ are related by

$$\forall\, i \in \mathbb{N}_0: \; \mathscr{V}_i^1 = T^{-1}\mathscr{V}_i^2 \;\wedge\; \mathscr{W}_i^1 = T^{-1}\mathscr{W}_i^2.$$

We prove the statement by induction. It is clear that $\mathscr{V}_0^1 = T^{-1}\mathscr{V}_0^2$. Assuming that $\mathscr{V}_i^1 = T^{-1}\mathscr{V}_i^2$ for some $i \geq 0$ we find that, by (5),

$$\begin{aligned}
\mathscr{V}_{i+1}^1 &= \ker_{\mathbb{R}} C_1 \cap A_1^{-1}(E_1\mathscr{V}_i^1) \\
&= \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \in \mathscr{V}_i^1 : \; WA_2Tx = WE_2Ty \;\wedge\; VC_2Ty = 0 \right\} \\
&= \left\{ x \in \mathbb{R}^n \;\middle|\; \exists z \in \mathscr{V}_i^2 : \; A_2Tx = E_2z \;\wedge\; C_2z = 0 \right\} \\
&= T^{-1}\left(\ker_{\mathbb{R}} C_2 \cap A_2^{-1}(E_2\mathscr{V}_i^2)\right) = T^{-1}\mathscr{V}_{i+1}^2.
\end{aligned}$$

The statement about $\mathscr{W}_i^1$ and $\mathscr{W}_i^2$ can be proved analogous.

Step 2: By Step 1 we may without loss of generality assume that $[E,A,C]$ is given in OI normal form (7). We make the convention that if $\alpha \in \mathbb{N}^k$ is some multi-index, then $\alpha - 1 := (\alpha_1 - 1, \ldots, \alpha_k - 1)$. It not follows that

$$\forall\, i \in \mathbb{N}_0: \; \mathscr{V}_i = \bigcap_{j=0}^{i-1} \ker_{\mathbb{R}} E_\alpha^\top N_\alpha^j \times \operatorname{im}_{\mathbb{R}} (N_{\beta-1}^\top)^i \times \operatorname{im}_{\mathbb{R}} (N_\gamma^\top)^i \times \mathbb{R}^{|\varepsilon|} \times \operatorname{im}_{\mathbb{R}} (N_\kappa^\top)^i \times \mathbb{R}^{n_{\overline{\sigma}}}, \tag{23}$$

which is immediate from observing that $L_\beta^\top x = K_\beta^\top y$ for some x,y of appropriate dimension yields $x = N_{\varepsilon-1}^\top y$, and $K_\gamma x = L_\gamma y$ with $E_\gamma^\top x = 0$ for some x,y yields $x = N_\gamma^\top y$. Note that in the case $\beta_j = 1$, i.e., we have a 1×0 block, we find that $N_{\beta_j-1}^\top$ is absent, so these relations are consistent.
On the other hand we find that

$$\forall\, i \in \mathbb{N}_0: \; \mathscr{W}_i = \{0\}^{|\alpha|} \times \{0\}^{|\beta|} \times \left(\ker_{\mathbb{R}} (N_\gamma^\top)^i \cap \ker_{\mathbb{R}} E_\gamma^\top\right) \times \ker_{\mathbb{R}} N_\varepsilon^i \times \ker_{\mathbb{R}} (N_\kappa^\top)^i \times \{0\}^{n_{\overline{\sigma}}}. \tag{24}$$

Step 3: From (23) and (24) it follows that

$$\begin{aligned}
\mathscr{V}^* &= \{0\}^{|\alpha|} \times \{0\}^{|\beta|-\ell(\beta)} \times \{0\}^{|\gamma|} \times \mathbb{R}^{|\varepsilon|} \times \{0\}^{|\kappa|} \times \mathbb{R}^{n_{\overline{\sigma}}}, \\
\mathscr{W}^* &= \{0\}^{|\alpha|} \times \{0\}^{|\beta|-\ell(\beta)} \times \ker_{\mathbb{R}} E_\gamma^\top \times \mathbb{R}^{|\varepsilon|} \times \mathbb{R}^{|\kappa|} \times \{0\}^{n_{\overline{\sigma}}}.
\end{aligned}$$

and

$$\begin{aligned}
E\mathscr{V}^* &= \{0\}^{|\alpha|} \times \{0\}^{|\beta|} \times \{0\}^{|\gamma|-\ell(\gamma)} \times \mathbb{R}^{|\varepsilon|-\ell(\varepsilon)} \times \{0\}^{|\kappa|} \times \mathbb{R}^{n_{\overline{\sigma}}}, \\
A\mathscr{W}^* &= \{0\}^{|\alpha|} \times \{0\}^{|\beta|} \times K_\gamma(\ker_{\mathbb{R}} E_\gamma^\top) \times \mathbb{R}^{|\varepsilon|-\ell(\varepsilon)} \times \mathbb{R}^{|\kappa|} \times \{0\}^{n_{\overline{\sigma}}},
\end{aligned}$$

$$\mathrm{im}_{\mathbb{R}} E = \mathbb{R}^{|\alpha|} \times \mathrm{im}_{\mathbb{R}} K_\beta^\top \times \mathbb{R}^{|\gamma|-\ell(\gamma)} \times \mathbb{R}^{|\varepsilon|-\ell(\varepsilon)} \times \mathrm{im}_{\mathbb{R}} N_\kappa^\top \times \mathbb{R}^{n_{\overline{\sigma}}}.$$

The equivalences in (a)–(j) may now be inferred from Table 1. □

Under the additional assumption that $\mathrm{rk}[E^\top, A^\top, C^\top] = n$, the conditions for strong and complete observability as in Theorem 7.3 are derived in [10, 66] (which are called observability and strong observability in these works, resp.). The conditions for strong and complete observability are also derived in [67], as well as those for behavioral and RS strong observability; in [67] the observability concepts are defined within a distributional solution setup and other names are used than in the present work (cf. Subsection 3.4).

8 Kalman decomposition

The famous decomposition of linear ODE control systems derived by Kalman [45] is one of the most important tools in the analysis of these systems. This decomposition has later been generalized to regular DAEs by VERGHESE ET AL. [82], see also [32]. A Kalman decomposition of general discrete-time DAE systems was provided by BANASZUK ET AL. [6] in a very nice way using the restricted/augmented Wong sequences (cf. Section 7 and [14]). They derive the following result.

Theorem 8.1 (Kalman decomposition [6])**.** *For $[E,A,B,C,0] \in \Sigma_{l,n,m,p}$, there exist $S \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that*

$$[SET, SAT, SB, CT] = \left[\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ 0 & E_{22} & 0 & E_{24} \\ 0 & 0 & E_{33} & E_{34} \\ 0 & 0 & 0 & E_{44} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, [0, C_2, 0, C_4] \right], \tag{25}$$

where $E_{ij}, A_{ij} \in \mathbb{R}^{l_i \times n_j}$, $B_i \in \mathbb{R}^{l_i \times m}$, $C_j \in \mathbb{R}^{p \times n_j}$ for $i, j = 1, \ldots, 4$, such that

(i) $\left[\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right] \in \mathscr{C}_{l_1+l_2, n_1+n_2, m}$ *is completely controllable and* $\mathrm{rk} \begin{bmatrix} E_{11} & E_{12} & B_1 \\ 0 & E_{22} & B_2 \end{bmatrix} = l_1 + l_2$.

(ii) $\left[\begin{bmatrix} E_{22} & E_{24} \\ 0 & E_{44} \end{bmatrix}, \begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, [C_2, C_4] \right] \in \mathscr{O}_{l_2+l_4, n_2+n_4, p}$ *is completely observable.*

(iii) $\mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE_{33} - A_{33} & sE_{34} - A_{34} \\ 0 & sE_{44} - A_{44} \end{bmatrix} = n_3 + n_4$.

(iv) $\mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE_{11} - A_{11} & sE_{13} - A_{13} \\ 0 & sE_{33} - A_{33} \end{bmatrix} = l_1 + l_3$.

We like to stress that there are several subtleties of the Kalman decomposition (25) which are highlighted in [16, Rem. 7.2] for a pure controllability decomposition and carry over to the general case.

Proposition 8.2 (Uniqueness of the Kalman decomposition). *Let* $[E,A,B,C,0] \in \Sigma_{l,n,m,p}$ *be given and assume that, for all* $i \in \{1,2\}$*, the systems* $[E_i,A_i,B_i,C_i] = [S_iET_i,S_iAT_i,S_iB,CT_i]$ *with*

$$sE_i - A_i = \begin{bmatrix} sE_{11,i}-A_{11,i} & sE_{12,i}-A_{12,i} & sE_{13,i}-A_{13,i} & sE_{14,i}-A_{14,i} \\ 0 & sE_{22,i}-A_{22,i} & 0 & sE_{24,i}-A_{24,i} \\ 0 & 0 & sE_{33,i}-A_{33,i} & sE_{34,i}-A_{34,i} \\ 0 & 0 & 0 & sE_{44,i}-A_{44,i} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1,i} \\ B_{2,i} \\ 0 \\ 0 \end{bmatrix},$$
$$C_i = [0, C_{2,i}, 0, C_{4,i}]$$

where $E_{fg,i}, A_{fg,i} \in \mathbb{R}^{l_{f,i} \times n_{g,i}}$*,* $B_{f,i} \in \mathbb{R}^{l_{f,i} \times m}$*,* $C_g \in \mathbb{R}^{p \times n_{g,i}}$*,* $f,g = 1,\ldots,4$*, satisfy the conditions (i)–(iv) in Theorem 8.1.*

Then $l_{j,1} = l_{j,2}$ *and* $n_{j,1} = n_{j,2}$ *for all* $j = 1,\ldots,4$*. Moreover, for some* $W_{ij} \in \mathbb{R}^{l_{i,1} \times l_{j,1}}$*,* $T_{ij} \in \mathbb{R}^{n_{i,1} \times n_{j,1}}$ *such that* $\det W_{ii} \neq 0$ *and* $\det T_{ii} \neq 0$*,* $i,j = 1,\ldots,4$*, we have*

$$W_2 W_1^{-1} = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ 0 & W_{22} & 0 & W_{24} \\ 0 & 0 & W_{33} & W_{34} \\ 0 & 0 & 0 & W_{44} \end{bmatrix}, \quad T_1^{-1} T_2 = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ 0 & T_{22} & 0 & T_{24} \\ 0 & 0 & T_{33} & T_{34} \\ 0 & 0 & 0 & T_{44} \end{bmatrix}.$$

Proof. The result can be concluded from [16, Prop. 7.2] applied to $[E,A,B] \in \mathscr{C}_{l,n,m}$ and its dual $[E^\top, A^\top, C^\top] \in \mathscr{C}_{n,l,p}$ (invoking Corollary 5.2). □

Similar to [16, Cor. 7.3] several controllability, stabilizability, observability and detectability properties (and conditions for them) can be inferred for the subsystems appearing in the Kalman decomposition (25); we omit the details here.

The Kalman decomposition (25) is not satisfactory from a behavioral point of view: The trivial DAE $0 = x, y = 0$ given by $[0,I,0,0,0]$ is behaviorally controllable and behaviorally observable, but in the decomposition (25) it is part of the uncontrollable and unobservable subsystem $[E_{33},A_{33},0,0]$. This is an unsatisfactory situation and is due to the fact, that for DAE systems (both regular and singular) certain states can be inconsistent and it does not really make sense to label those controllable or uncontrollable (observable or unobservable, resp.). In the case of controllability decompositions this problem was treated in [20] and the following more detailed Kalman controllability decomposition was proved for $[E,A,B] \in \mathscr{C}_{l,n,m}$:

$$[SET, SAT, SB] = \left[\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ 0 & E_{22} & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} \right],$$

where S and T are invertible matrices and the DAE system given by $[E_{11},A_{11},B_1]$ is completely controllable. Furthermore, E_{22} is invertible and the DAE $[E_{33},A_{33},0]$

is such that it only has the trivial solution. Hence, we now have the decomposition into a completely controllable part, a classical uncontrollable part (given by an ODE) and an inconsistent part (which is behaviorally controllable but contains no completely controllable part). This decomposition seems to be more adequate for the analysis of DAE control systems as it takes into account the special DAE feature of possible inconsistent states which play a special role with respect to controllability. Using duality (see Section 5) we may derive the following analogous observability decomposition.

Theorem 8.3 (Kalman observability decomposition)**.** *For* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *there exist* $S \in \mathbf{Gl}_l(\mathbb{R})$ *and* $T \in \mathbf{Gl}_n(\mathbb{R})$ *such that*

$$[SET, SAT, CT] = \left[\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ 0 & E_{22} & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, [0,0,C_3] \right], \tag{26}$$

where $E_{ij}, A_{ij} \in \mathbb{R}^{l_i \times n_j}$ *for* $i,j = 1,\ldots,3$, $C_3 \in \mathbb{R}^{p \times n_3}$ *such that*

(i) $[E_{11}, A_{11}, 0] \in \mathscr{O}_{l_1,n_1,p}$ *with* $l_1 \leq n_1$ *and* $\operatorname{rk}_{\mathbb{C}}(\lambda E_{11} - A_{11}) = l_1$ *for all* $\lambda \in \mathbb{C}$,
(ii) $[E_{22}, A_{22}, 0] \in \mathscr{O}_{l_2,n_2,p}$ *with* $l_2 = n_2$ *and* E_{22} *is invertible,*
(iii) $[E_{33}, A_{33}, C_3] \in \mathscr{O}_{l_3,n_3,p}$ *is completely observable.*

Remark 8.4.

(i) In the decomposition (26) we have an underdetermined and possibly inconsistent part $[E_{11}, A_{11}, 0]$, a classical unobservable part $[E_{22}, A_{22}, 0]$ and a completely observable part $[E_{33}, A_{33}, C_3]$. Note that furthermore

$$\left[\begin{bmatrix} E_{11} & E_{13} \\ 0 & E_{33} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}, [0, C_3] \right] \text{ is RS behaviorally observable and}$$

$$\left[\begin{bmatrix} E_{22} & E_{23} \\ 0 & E_{33} \end{bmatrix}, \begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix}, [0, C_3] \right] \text{ is observable at infinity.}$$

(ii) Uniqueness of the Kalman observability decompositionn (26) may be analysed similar to Proposition 8.2 and [20, Thm. 3.5]; in particular the block sizes are unique.
(iii) Similar to [20, Thm. 3.3] it is possible to derive the decomposition (26) with the help of the restricted Wong sequences which have been introduced in Section 7. In fact, the subspace decomposition leading to (26) is uniquely determined by the restricted Wong sequences. Also note that, especially in the singular case, the decomposition (26) bears several subtleties which can be analysed similar to [20, Rem. 3.2].
(iv) It is also possible to extend the pure observability decomposition (26) to a Kalman decomposition of the form (25) where additionally the classical (ODE) uncontrollable and unobservable parts are decomposed. However, due to the complexity of such a decomposition we omit it here.

9 Detectability and stabilization by output injection

In this subsection we introduce detectability concepts for DAE systems. We characterize them in terms of the OI normal form and derive duality to the respective stabilizability concepts from [16]. This will enable us to infer algebraic criteria for the detectability concepts and to finally show that stabilization and index reduction can be achieved by output injection.

In general, detectability is a weaker version of observability in the sense that the state x is not exactly determined by the external signals but only asymptotically. In the following, we will use the simplified notation "$x(t) \to 0$ as $t \to \infty$" for $x \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ if, and only if,

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{\tau\in[t,\infty)} \|x(\tau)\| = 0.$$

Definition 9.1. The system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is called

(a) *behaviorally detectable*

$$:\iff \forall\,(x^1,u,y),(x^2,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}: \quad x^1(t)-x^2(t) \to 0 \text{ as } t \to \infty,$$

(b) *RS behaviorally detectable*

$$:\iff \forall\,(x^1,u,y),(x^2,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}\ \exists\,(x^3,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}: \\ x^1_{(-\infty,0)} = x^3_{(-\infty,0)} \ \wedge\ x^2(t)-x^3(t) \to 0 \text{ as } t \to \infty,$$

(c) *strongly detectable*, if it is impulse observable and behaviorally detectable,
(d) *completely detectable*, if it is observable at infinity and behaviorally detectable,
(e) *RS strongly detectable*, if it is RS impulse observable and RS behaviorally detectable,
(f) *RS completely detectable*, if it is RS observable at infinity and RS behaviorally detectable.

The definitions of RS complete and strong detectability are motivated by the corresponding characterizations of RS complete and strong observability (see Figure 1) in terms of RS observability at infinity, RS impulse observability and RS behavioral observability; where the latter is replaced by RS behavioral detectability. Similar as for the observability concepts, the detectability definitions can be simplified due to linearity.

Lemma 9.2. *The system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is*

(a) behaviorally detectable

$$\iff \forall\,(x,0) \in \mathfrak{B}_{[E,A,C]}: \quad x(t) \to 0 \textit{ as } t \to \infty,$$

(b) RS behaviorally detectable

$$\iff \forall (x,0) \in \mathfrak{B}_{[E,A,C]}\ \exists (\bar{x},0) \in \mathfrak{B}_{[E,A,C]} : \quad x_{(-\infty,0)} = \bar{x}_{(-\infty,0)} \wedge \bar{x}(t) \to 0 \text{ as } t \to \infty.$$

Hence we may restrict our attention to systems in $\mathscr{O}_{l,n,p}$ and we can use the OI normal form (7) to obtain (similar to the observability characterizations given in Table 1) the following characterizations of the detectability concepts.

Corollary 9.3 (Detectability and OI normal form)**.** *Let* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *with OI normal form* (7)*. Then* $[E,A,C]$ *is*

(a) behaviorally detectable if, and only if, $\ell(\varepsilon) = 0$ *and* $\sigma(A_{\overline{o}}) \subseteq \mathbb{C}_-$.
(b) RS behaviorally detectable if, and only if, $\sigma(A_{\overline{o}}) \subseteq \mathbb{C}_-$.
(c) strongly detectable if, and only if, $\gamma = (1,\dots,1)$, $\ell(\varepsilon) = 0$, $\kappa = (1,\dots,1)$ *and* $\sigma(A_{\overline{o}}) \subseteq \mathbb{C}_-$.
(d) completely detectable if, and only if, $\gamma = (1,\dots,1)$, $\ell(\varepsilon) = 0$, $\ell(\kappa) = 0$ *and* $\sigma(A_{\overline{o}}) \subseteq \mathbb{C}_-$.
(e) RS strongly detectable if, and only if, $\gamma = (1,\dots,1)$, $\varepsilon = (1,\dots,1)$, $\kappa = (1,\dots,1)$ *and* $\sigma(A_{\overline{o}}) \subseteq \mathbb{C}_-$.
(f) RS completely detectable if, and only if, $\gamma = (1,\dots,1)$, $\varepsilon = (1,\dots,1)$, $\ell(\kappa) = 0$ *and* $\sigma(A_{\overline{o}}) \subseteq \mathbb{C}_-$.

Using the OI normal form, the characterizations in Corollary 9.3 and the respective results for the feedback form derived in [16, Cor. 3.4], we are able to infer duality between detectability and stabilizability as follows.

Corollary 9.4 (Duality of detectability and stabilizability)**.** *Let* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *be given. Then we have the following equivalences:*

(a) $[E,A,C]$ *is RS behaviorally detectable if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is behaviorally stabilizable in the sense of [16].*
(b) $[E,A,C]$ *is RS strongly detectable if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is strongly stabilizable in the sense of [16].*
(c) $[E,A,C]$ *is RS completely detectable if, and only if,* $[E^\top,A^\top,C^\top] \in \mathscr{C}_{n,l,p}$ *is completely stabilizable in the sense of [16].*

In particular, for regular DAE systems, behavioral, strong and complete detectability are dual to behavioral, strong and complete stabilizability. The duality properties are illustrated in Figure 3.

As a consequence of Corollaries 9.3 and 9.4 and [16, Cor. 4.3] we obtain the following algebraic criteria of Hautus type for the detectability concepts from Definition 9.1.

Corollary 9.5 (Algebraic criteria for detectability)**.** *Let* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *and* Z *be a matrix with* $\operatorname{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} E^\top$*. Then we have the following:*

$[E,A,C]$ *is*	*if, and only if,*

behaviorally detectable	$\forall \lambda \in \overline{\mathbb{C}_+}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n.$
RS behaviorally detectable	$\forall \lambda \in \overline{\mathbb{C}_+}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ C \end{bmatrix}.$
strongly detectable	$\forall \lambda \in \overline{\mathbb{C}_+}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix} = n.$
completely detectable	$\forall \lambda \in \overline{\mathbb{C}_+}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix} = n.$
RS strongly detectable	$\forall \lambda \in \overline{\mathbb{C}_+}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ Z^\top A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix}.$
RS completely detectable	$\forall \lambda \in \overline{\mathbb{C}_+}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ C \end{bmatrix} = \operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E \\ A \\ C \end{bmatrix}.$

Remark 9.6. Behavioral detectability was investigated in [32] for regular systems, where it is called detectability. In this case, the algebraic criteria for RS behavioral detectability from Corollary 9.5 have been derived in [32, Thm. 3-1.3].

In the remainder of this section we consider stabilization and index reduction by output injection. As explained in Section 4, a system $[E,A,C] \in \mathcal{O}_{l,n,p}$ can, via output injection with some $L \in \mathbb{R}^{l\times p}$, be turned into a DAE of the form (4), that is a new system $[E,A+LC,C] \in \mathcal{O}_{l,n,p}$. It is our aim to choose L such that this new system is stable in a certain sense and its index is at most one. The *index* $\nu \in \mathbb{N}_0$ of a matrix pencil $sE - A \in \mathbb{R}[s]^{l\times n}$ is defined via its (quasi-)Kronecker form [18,19,36] as in [16, Def. 3.2]: If for some $S \in \mathbf{Gl}_l(\mathbb{R})$ and $T \in \mathbf{Gl}_n(\mathbb{R})$

$$S(sE-A)T = \begin{bmatrix} sI_r - J & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix}, \qquad (27)$$

$$\text{then} \quad \nu = \max\{0, \alpha_1, \ldots, \alpha_{\ell(\alpha)}, \gamma_1, \ldots, \gamma_{\ell(\gamma)}\}.$$

The index is independent of the choice of S,T and can be computed via the Wong sequences corresponding to $sE - A$ as shown in [18, 19].

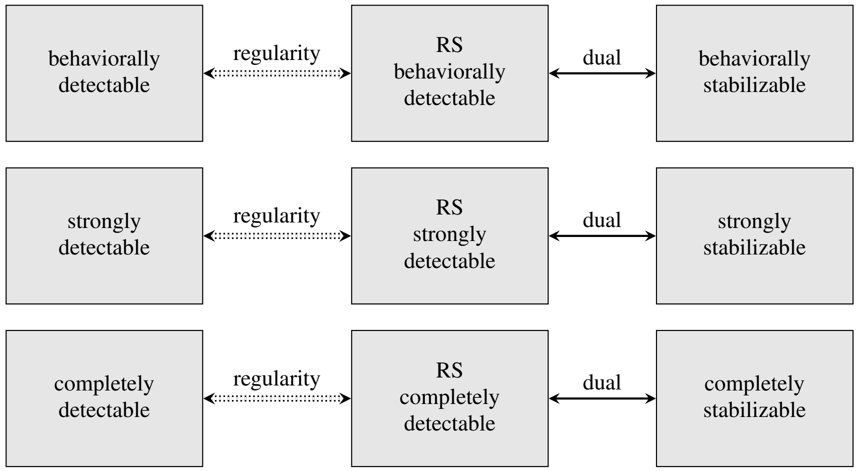

Fig. 3: Illustration of duality between detectability and stabilizability.

The following result can now be inferred from Corollaries 5.2 and 9.4 and [16, Thm. 5.3].

Proposition 9.7 (Stabilization and index reduction)**.** *For a system* $[E,A,C] \in \mathscr{O}_{l,n,p}$ *the following holds true:*

(a) $[E,A,C]$ *is RS impulse observable if, and only if, there exists* $L \in \mathbb{R}^{l\times p}$ *such that the index of* $sE^\top - (A+LC)^\top$ *is at most one.*
(b) $[E,A,C]$ *is RS strongly detectable if, and only if, there exists* $L \in \mathbb{R}^{l\times p}$ *such that the index of* $sE^\top - (A+LC)^\top$ *is at most one and the pair* $[E,A+LC]$ *is behaviorally stabilizable in the sense of [16, Def. 5.1].*

If we consider square systems $[E,A,C] \in \mathscr{O}_{n,n,p}$, then we may obtain an additional stabilization result via behavioral detectability which is false in general in the nonregular case. To this end, we call a system $[E,A,C] \in \mathscr{O}_{l,n,p}$ *behaviorally stable*, if $[E,A,0]$ is behaviorally detectable. From Corollary 9.5 we obtain the characterization

$$[E,A,C] \text{ is behaviorally stable} \quad \Longleftrightarrow \quad \forall \lambda \in \overline{\mathbb{C}_+} : \ \mathrm{rk}_\mathbb{C}(\lambda E - A) = n. \tag{28}$$

Furthermore, under the slightly stronger assumptions of impulse observability and strong detectability, resp., the results of Proposition 9.7 can be improved for square systems. It is then possible to show that the output injection leads to a system which is additionally regular. The equivalence of impulse observability and regularizability with index reduction by output injection (statement (a) of the following theorem) has been proved in [25], see also [24]. For completeness we provide a new proof using the OI normal form. To the best of our knowledge, statements (b) and (c) of the following theorem are new.

Theorem 9.8 (Stabilization and index reduction for square systems)**.** *For a system $[E,A,C] \in \mathscr{O}_{n,n,p}$ the following holds true:*

(a) $[E,A,C]$ is impulse observable if, and only if, there exists $L \in \mathbb{R}^{n\times p}$ such that $sE-(A+LC)$ is regular and its index is at most one.
(b) $[E,A,C]$ is behaviorally detectable if, and only if, there exists $L \in \mathbb{R}^{n\times p}$ such that $sE-(A+LC)$ is regular and $[E,A+LC,C]$ is behaviorally stable.
(c) $[E,A,C]$ is strongly detectable if, and only if, there exists $L \in \mathbb{R}^{n\times p}$ such that $sE-(A+LC)$ is regular, its index is at most one and $[E,A+LC,C]$ is behaviorally stable.

Proof. (a) Without loss of generality, we may assume that $[E,A,C]$ is in OI normal form (7). First let $[E,A,C]$ be impulse observable, and hence it follows from Table 1 that $\gamma=(1,\ldots,1)$, $\ell(\varepsilon)=0$ and $\kappa=(1,\ldots,1)$. Since E and A are square we may further deduce that $\ell(\beta)=\ell(\gamma)$, and therefore

$$E=\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0\\ 0 & K_\beta^\top & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I_{n_{\overline{o}}}\end{bmatrix},\ A=\begin{bmatrix} N_\alpha & 0 & 0 & 0 & 0\\ 0 & L_\beta^\top & 0 & 0 & 0\\ 0 & 0 & 0 & I_{|\kappa|} & 0\\ 0 & 0 & 0 & 0 & A_{\overline{o}}\end{bmatrix},\ C=\begin{bmatrix} E_\alpha^\top & 0 & 0 & 0 & 0\\ 0 & 0 & I_{|\gamma|} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}. \quad (29)$$

It is easy to see that

$$s[K_\beta^\top,0]-[L_\beta^\top,E_\beta]=S\left(s\begin{bmatrix} I_{|\beta-1|} & 0\\ 0 & 0\end{bmatrix}-\begin{bmatrix} N_{\beta-1} & 0\\ 0 & I_{\ell(\beta)}\end{bmatrix}\right)T$$

for some invertible matrices S,T, where $\beta-1=(\beta_1-1,\ldots,\beta_{\ell(\beta)}-1)$. Therefore, the pencil $s[K_\beta^\top,0]-[L_\beta^\top,E_\beta]$ is regular and has index at most one. Choosing

$$L=\begin{bmatrix} 0 & 0 & 0\\ 0 & E_\beta & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

we obtain that

$$sE-(A+LC)=\begin{bmatrix} sI_{|\alpha|}-N_\alpha & 0 & 0 & 0 & 0\\ 0 & sK_\beta^\top-L_\beta^\top & -E_\beta & 0 & 0\\ 0 & 0 & 0 & -I_{|\kappa|} & 0\\ 0 & 0 & 0 & 0 & sI_{n_{\overline{o}}}-A_{\overline{o}}\end{bmatrix}$$

is regular and its index is at most one.
To show the opposite implication let $L \in \mathbb{R}^{n\times p}$ be such that $sE-(A+LC)$ is regular and its index is at most one. Then Proposition 9.7 implies that $[E,A,C]$ is RS impulse observable. To show impulse observability, by Table 1 it remains to show that $\ell(\varepsilon)=0$. Since a OI normal form of $[E,A,C]$ is also a OI normal form

of $[E,A+LC,C]$, it follows from the regularity of $sE-(A+LC)$ and Remark 4.6 that $\ell(\varepsilon)=0$.

(b) Again, we assume that $[E,A,C]$ is in OI normal form (7). First let $[E,A,C]$ be behaviorally detectable, and hence it follows from Corollary 9.3 that $\ell(\varepsilon)=0$ and $\sigma(A_{\overline{\sigma}})\subseteq\mathbb{C}_-$. Since E and A are square we may further deduce that $\ell(\beta)=\ell(\gamma)$, and therefore

$$E=\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0\\ 0 & K_\beta^\top & 0 & 0 & 0\\ 0 & 0 & L_\gamma & 0 & 0\\ 0 & 0 & 0 & N_\kappa^\top & 0\\ 0 & 0 & 0 & 0 & I_{n_{\overline{\sigma}}}\end{bmatrix},\ A=\begin{bmatrix} N_\alpha & 0 & 0 & 0 & 0\\ 0 & L_\beta^\top & 0 & 0 & 0\\ 0 & 0 & K_\gamma & 0 & 0\\ 0 & 0 & 0 & I_{|\kappa|} & 0\\ 0 & 0 & 0 & 0 & A_{\overline{\sigma}}\end{bmatrix},\ C=\begin{bmatrix} E_\alpha^\top & 0 & 0 & 0 & 0\\ 0 & 0 & E_\gamma^\top & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

By [80, Thm. 4.20] there exists $F_\alpha\in\mathbb{R}^{|\alpha|\times\ell(\alpha)}$ such that $\sigma(N_\alpha+F_\alpha E_\alpha^\top)\subseteq\mathbb{C}_-$. Furthermore, choosing

$$F_\beta=\operatorname{diag}\left(e_1^{[\beta_1]},\dots,e_1^{[\beta_{\ell(\beta)}]}\right)$$

we find that, by the same argument as in the proof of [17, Thm. 3.5],

$$s\begin{bmatrix} K_\beta^\top & 0\\ 0 & L_\gamma\end{bmatrix}-\begin{bmatrix} L_\beta^\top & F_\beta E_\gamma^\top\\ 0 & K_\gamma\end{bmatrix}=S\left(s\begin{bmatrix} N_\beta^\top & 0\\ * & N_{\gamma-1}^\top\end{bmatrix}-\begin{bmatrix} I_{|\beta|} & 0\\ 0 & I_{|\gamma-1|}\end{bmatrix}\right)T$$

for some invertible matrices S,T, where $\gamma-1=(\gamma_1-1,\dots,\gamma_{\ell(\gamma)}-1)$. Therefore, the pencil $s\begin{bmatrix} K_\beta^\top & 0\\ 0 & L_\gamma\end{bmatrix}-\begin{bmatrix} L_\beta^\top & F_\beta E_\gamma^\top\\ 0 & K_\gamma\end{bmatrix}$ is regular and the system $\left[\begin{bmatrix} K_\beta^\top & 0\\ 0 & L_\gamma\end{bmatrix},\begin{bmatrix} L_\beta^\top & F_\beta E_\gamma^\top\\ 0 & K_\gamma\end{bmatrix},[0,E_\gamma^\top]\right]$ is behaviorally stable by (28). Choosing

$$L=\begin{bmatrix} F_\alpha & 0 & 0\\ 0 & F_\beta & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

we obtain that

$$\begin{aligned} &sE-(A+LC)\\ &\quad=\begin{bmatrix} sI_{|\alpha|}-(N_\alpha+F_\alpha E_\alpha^\top) & 0 & 0 & 0 & 0\\ 0 & sK_\beta^\top-L_\beta^\top & -F_\beta E_\gamma^\top & 0 & 0\\ 0 & 0 & sL_\gamma-K_\gamma & 0 & 0\\ 0 & 0 & 0 & sN_\kappa^\top-I_{|\kappa|} & 0\\ 0 & 0 & 0 & 0 & sI_{n_{\overline{\sigma}}}-A_{\overline{\sigma}}\end{bmatrix}\end{aligned}$$

is regular and $[E,A+LC,C]$ is behaviorally stable by (28).
To show the opposite implication let $L\in\mathbb{R}^{n\times p}$ be such that $sE-(A+LC)$ is regular and $[E,A+LC,C]$ is behaviorally stable. Seeking a contradiction, assume that $[E,A,C]$ is not behaviorally detectable. Then it follows from Corollary 9.5 that there exist $\lambda\in\overline{\mathbb{C}_+}$ and $x\in\mathbb{C}^n\setminus\{0\}$ such that $\left[\begin{smallmatrix}\lambda E-A\\ C\end{smallmatrix}\right]x=0$. This implies

$$(\lambda E-(A+LC))x=[I_n,-L]\begin{bmatrix}\lambda E-A\\ C\end{bmatrix}x=0,$$

thus $\mathrm{rk}_{\mathbb{C}}(\lambda E-(A+LC))<n$ which contradicts behavioral stability of $[E,A+LC,C]$ by (28).

(c) Again, we assume that $[E,A,C]$ is in OI normal form (7). First let $[E,A,C]$ be strongly detectable, and hence it follows from Corollary 9.3 that $\gamma=(1,\dots,1)$, $\ell(\varepsilon)=0$, $\kappa=(1,\dots,1)$ and $\sigma(A_{\overline{\sigma}})\subseteq\mathbb{C}_-$. Since E and A are square we may further deduce that $\ell(\beta)=\ell(\gamma)$ and (29) holds. Let $F_\alpha\in\mathbb{R}^{|\alpha|\times\ell(\alpha)}$ be such that $\sigma(N_\alpha+F_\alpha E_\alpha^\top)\subseteq\mathbb{C}_-$. Furthermore, let

$$a_j=[a_{j0},\dots,a_{j\beta_j-2},1]^\top\in\mathbb{R}^{\beta_j}$$

with the property that the polynomials

$$p_j(s)=s^{\beta_j}+a_{j\beta_j-1}s^{\beta_j-1}+\ldots+a_{j0}\in\mathbb{R}[s]$$

are Hurwitz for $j=1,\dots,\ell(\beta)$, and let

$$B_\beta=\mathrm{diag}\,(a_1,\dots,a_{\ell(\beta)})\in\mathbb{R}^{|\beta|\times\ell(\beta)}.$$

Consider the system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}[K_\beta^\top,0]\begin{pmatrix}z(t)\\ u(t)\end{pmatrix}=[L_\beta^\top,B_\beta]\begin{pmatrix}z(t)\\ u(t)\end{pmatrix}. \tag{30}$$

We see that the input u is uniquely determined by $u=-E_{\beta-1}^\top z$, where $\beta-1=(\beta_1-1,\dots,\beta_{\ell(\beta)}-1)$ and if $\beta_j=1$ for some j, then the respective x-component does not exist and the equation simply reads $u_j=0$. With $B_{\beta-1}=\mathrm{diag}\,(\tilde a_1,\dots,\tilde a_{\ell(\beta)})$, where $\tilde a_j=[a_{j0},\dots,a_{j\beta_j-2}]^\top$, a permutation of rows in (30) and insertion of u gives

$$\begin{aligned}\dot z(t)&=(N_{\beta-1}-B_{\beta-1}E_{\beta-1}^\top)z(t),\\ u(t)&=E_{\beta-1}^\top z(t).\end{aligned}$$

It is now clear, that the pencil $s[K_\beta^\top,0]-[L_\beta^\top,B_\beta]$ in system (30) is regular and has index at most one. Furthermore, the characteristic polynomial of $N_{\beta-1}+B_{\beta-1}E_{\beta-1}^\top$ (which is a block diagonalization of companion matrices) is given by

$$\det\left(sI-(N_{\beta-1}+B_{\beta-1}E_{\beta-1}^\top)\right)=\prod_{j=1}^{\ell(\beta)}p_j(s),$$

which is Hurwitz, since all $p_j(s)$ are Hurwitz. Therefore, $\left[[K_\beta^\top,0],[L_\beta^\top,B_\beta],[0,I_{|\gamma|}]\right]$ is also behaviorally stable. Choosing

$$L=\begin{bmatrix}F_\alpha & 0 & 0\\ 0 & B_\beta & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

we obtain that

$$sE-(A+LC)=\begin{bmatrix}sI_{|\alpha|}-(N_\alpha+F_\alpha E_\alpha^\top) & 0 & 0 & 0 & 0\\ 0 & sK_\beta^\top-L_\beta^\top & -B_\beta & 0 & 0\\ 0 & 0 & 0 & -I_{|\kappa|} & 0\\ 0 & 0 & 0 & 0 & sI_{n_{\overline{o}}}-A_{\overline{o}}\end{bmatrix}$$

is regular, its index is at most one and $[E,A+LC,C]$ is behaviorally stable by (28).

To show the opposite implication let $L\in\mathbb{R}^{n\times p}$ be such that $sE-(A+LC)$ is regular, its index is at most one and $[E,A+LC,C]$ is behaviorally stable. Then Proposition 9.7 implies that $[E,A,C]$ is RS strongly detectable. To show strong detectability, by Table 1 and Corollary 9.3 it remains to show that $\ell(\varepsilon)=0$. As in a), this follows from the regularity of $sE-(A+LC)$. □

Note that in the proof of necessity in Theorem 9.8 (b) regularity of $sE-(A+LC)$ has not been used explicitly, so one may wonder as to whether this property is necessary here. In fact, it is not: The regularity of $sE-(A+LC)$ is a direct consequence of behavioral stability of $[E,A+LC,C]$ and the fact that E and $A+LC$ are square.

Remark 9.9.

(i) It is a consequence of Theorem 9.8 that impulse observability or behavioral detectability in particular implies that the square system $[E,A,C]\in\mathcal{O}_{n,n,p}$ is regularizable by output injection, i.e., there exists $L\in\mathbb{R}^{n\times p}$ such that $sE-(A+LC)$ is regular. The dual of this concept is regularizability by state feedback and has been well-investigated, see [17] and the references therein.

(ii) Another result on index reduction which is slightly different from both Proposition 9.7 (a) and Theorem 9.8 (a) was derived in [43, Thm. 5]. It is shown that $[E,A,C]\in\mathcal{O}_{l,n,p}$ is impulse observable if, and only if, there exists $L\in\mathbb{R}^{l\times p}$ such that

$$(A+LC)^{-1}(\operatorname{im}_{\mathbb{R}}E)\cap\ker_{\mathbb{R}}E=\{0\},$$

which is slightly stronger than to require that $sE-(A+LC)$ has index at most one; in fact, it is equivalent to the index being at most one and the absence of overdetermined γ-blocks in the quasi-Kronecker form (27).

(iii) Stabilization and index reduction by output injection for regular DAE systems have been investigated in [32]. In particular, under the additional assumption of regularity of $sE-A$, Theorem 9.8 (a) and (b) have been derived in [32, Thm. 3-2.1 & Cor. 3-3.2], resp.

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