THE QUASI-KRONECKER FORM FOR MATRIX PENCILS

THOMAS BERGER* AND **STEPHAN TRENN†**

Abstract. We study singular matrix pencils and show that the so called Wong sequences yield a quasi-Kronecker form. This form decouples the matrix pencil into an underdetermined part, a regular part and an overdetermined part. This decoupling is sufficient to fully characterize the solution behaviour of the differential-algebraic equations associated with the matrix pencil. Furthermore, the Kronecker canonical form is a simple corollary of our result, hence, in passing by, we also provide a new proof for the Kronecker canonical form. The results are illustrated with an example given by a simple electrical circuit.

Keywords: singular matrix pencil, Kronecker canonical form, differential algebraic equations

1. Introduction. We study (singular) linear matrix pencils

$$sE - A \in \mathbb{K}^{m\times n}[s], \qquad \text{where } \mathbb{K} \text{ is } \mathbb{Q},\ \mathbb{R} \text{ or } \mathbb{C},$$

and the associated differential algebraic equation (DAE)

$$E\dot{x} = Ax + f, \tag{1.1}$$

where f is some inhomogeneity. In the context of DAEs it is natural to call matrix pencils $sE_1 - A_1$ and $sE_2 - A_2$ equivalent and write $sE_1 - A_1 \cong sE_2 - A_2$, if there exist invertible matrices S and T such that

$$S(sE_1 - A_2)T = sE_2 - A_2.$$

Based on this notion of equivalence it is of interest to find the "simplest" matrix pencil within an equivalence class. This problem was solved by Kronecker [12] (see also [8, 14]). Nevertheless, the analysis of matrix pencils is still an active research area (see e.g. the recent paper [10]), mainly because of numerical issues or to find ways to obtain the Kronecker canonical form efficiently (see e.g. [22], [3], [23]).

Our main goal in this paper is to highlight the importance of the *Wong sequences* [24] for the analysis of matrix pencils. The Wong sequences for the matrix pencil $sE - A$ are given by the following sequences of subspaces

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{K}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i) \subseteq \mathbb{K}^n, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i) \subseteq \mathbb{K}^m. \end{aligned}$$

We will show (see Theorem 3.2 and Remark 3.3) that the Wong sequences are sufficient to completely characterize the solution behaviour of the DAE (1.1) including the characterization of consistent initial values as well as constraints on the inhomogeneity f.

The Wong sequences can be traced back to Dieudonné [7], however his focus is only on the first of the two Wong sequences. Bernhard [5] and Armentano [2] used the Wong sequences to carry out a geometric analysis of matrix pencils. In [15] the first Wong sequence is introduced as "fundamental geometric tool in the characterization of the subspace of consistent initial conditions" of a regular DAE. In control theory, modified versions of the first Wong sequence are used to study (A, B)-invariant subspaces, see e.g. [25, Thm. 4.3]. Only a few authors [1, 13, 21] use the Wong sequences in connection with DAEs, in general, it seems that, especially in the DAE community, the relevance of the Wong sequences have been overlooked. We therefore believe that our solvability characterizations solely in terms of the Wong sequences are new.

The Wong sequences directly lead to a *quasi-Kronecker triangular form*, i.e.

$$sE - A \ \cong \ \begin{bmatrix} sE_P - A_P & * & * \\ 0 & sE_R - A_R & * \\ 0 & 0 & sE_Q - A_Q \end{bmatrix},$$

where $sE_R - A_R$ is a regular matrix pencil. $sE_P - A_P$ is the "underdetermined" pencil and $sE_Q - A_Q$ is the "overdetermined" pencil (Theorem 2.3). With only little more effort we can get rid of the off-diagonal blocks and obtain a *quasi-Kronecker form* (Theorem 2.5). From the letter it is easy to obtain the Kronecker canonical form (Corollary 2.7) and hence another contribution of our work is a new proof for the Kronecker canonical

*Institut für Mathematik, Technische Universität Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany, `thomas.berger@tu-ilmenau.de`. Supported by DFG grant Il25/9.

†Institut für Mathematik, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany, `stephan.trenn@mathematik.uni-wuerzburg.de`. Supported by DFG grant Wi1458/10.

form. We have to admit that our proof does not reach the elegance of the proof of Gantmacher [8], however Gantmacher does not provide any geometrical insight. On the other end of the spectrum, Armentano [2] uses the Wong sequences to obtain a similar result as we do (the quasi-Kronecker triangular form), however his approach is purely geometrical so that it is not directly possible to deduce the transformation matrices which are necessary to obtain the quasi-Kronecker (triangular) form. Our result overcomes this disadvantage because it presents geometrical insights and, at the same time, is constructive.

Different to other authors we do not primarily aim to decouple the regular part of the matrix pencil, because 1) the decoupling into three parts which have the solution properties “existence, but non-uniquess” (underdetermined part), “existence and uniqueness” (regular part) and “uniqueness, but possible non-existence” (overdetermined part) seems very natural, and 2) the regular part can be further decoupled if necessary - again with the help of the Wong sequences as we showed in [4].

Another advantage of our approach is that we respect the domain of the entries in the matrix pencil, e.g. if our matrices are real-valued, then all transformations remain real-valued. This is not true for results about the Kronecker canonical form, because, due to possible complex eigenvalues and eigenvectors, even in the case of real-valued matrices it is necessary to allow for complex transformations and complex canonical forms. This is often undesirable, because if one starts with a real-valued matrix pencil one would like to get real-valued results. Therefore, we formulated our results such that they are valid for $\mathbb{K} = \mathbb{Q}$, $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. Especially for $\mathbb{K} = \mathbb{Q}$ it was also necessary to re-check known results, whether their proofs are also valid in $\mathbb{Q}$. We believe that the case $\mathbb{K} = \mathbb{Q}$ is of special importance because this allows the implementation of our approach in exact arithmetic which might be feasible if the matrices are sparse and not too big. In fact, we believe that the construction of the quasi-Kronecker (triangular) form is also possible if the matrix pencil $sE - A$ contains symbolic entries as it is common for the analysis of electrical circuits, where one might just add the symbol R into the matrix instead of a specific value of the corresponding resistor. However, we have not formalized this, but our running example will show that it is no problem to keep symbolic entries in the matrix.

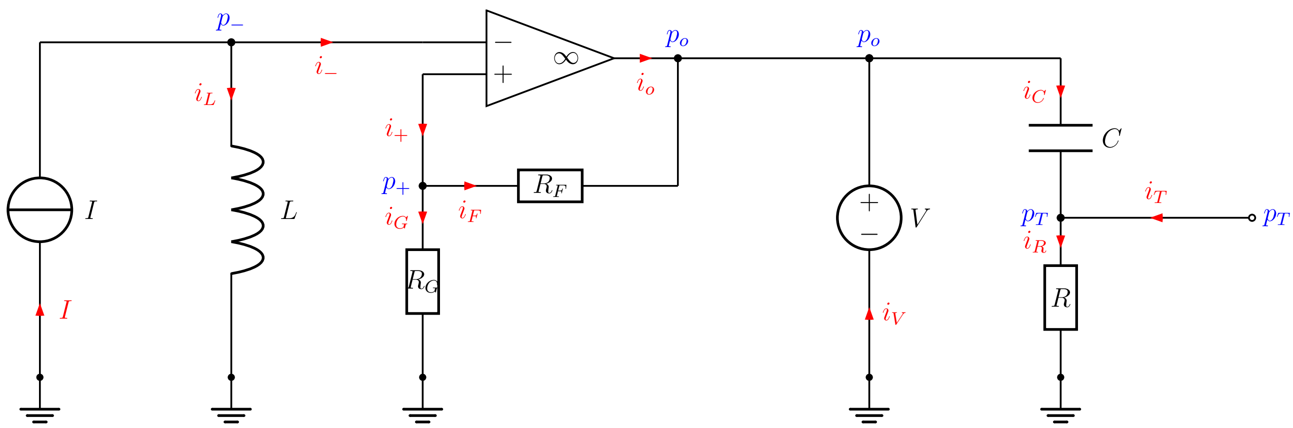

Fig. 1.1. *An electrical circuit with sources and an open terminal used as the origin of the DAE* (1.2). *Used as a running example.*

As a running example we use a DAE arising from an electrical circuit as shown in Figure 1.1. The electrical circuit has no practicable purpose and is for academic analysis only. To obtain the DAE description, let the state variable be given by $x = (p_+, p_-, p_o, p_T, i_L, i_p, i_m, i_G, i_F, i_R, i_o, i_V, i_C, i_T)^\top$ consisting of the node potentials and the currents through the branches. The inhomogeneity is $f = Bu$ with $u = (I, V)^\top$ given by the sources and the matrix B as below. The defining property of an ideal operational amplifier in feedback configuration is given by

$$p_+ = p_- \quad \text{and} \quad i_+ = 0 = i_-.$$

Collecting all defining equations of the circuit we obtain 13 equations for 14 state variables, which can be written as a DAE as follows:

$$\begin{bmatrix} 0&0&0&0&L&0&0&0&0&0&0&0&0&0\\ 0&0&-C&C&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&R_G&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0&R_F&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&R&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0&1&-1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1\\ 0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 1&0\\ 0&0\\ 0&0\\ 0&0\\ 0&1 \end{bmatrix} \begin{pmatrix} I\\ V \end{pmatrix}. \tag{1.2}$$

The coefficient matrices are not square, hence the corresponding matrix pencil $sE - A$ cannot be regular and standard tools cannot be used to analyze this description of the circuit.

The paper is organized as follows. In Section 2 we present our main results, in particular how the Wong sequences directly yield the *quasi-Kronecker triangular form* (Theorem 2.3). The proofs of the main results are carried out in Section 5. Preliminary results are presented and proved in Section 4. After presenting the main results, we show how the quasi-Kronecker form can be used to fully characterize the solution behaviour of the corresponding DAE in Section 3.
We close the introduction with the nomenclature used in this paper.

$\mathbb{N}$		set of natural numbers with zero, $\mathbb{N} = \{0, 1, 2, ...\}$
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$		set of rational, real and complex numbers, resp.
$\mathbb{K}$		either $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$
$\mathbf{Gl}_n(\mathbb{K})$		the set of invertible $n \times n$ matrices over $\mathbb{K}$
$\mathbb{K}[s]$		the ring of polynomials with coefficients in $\mathbb{K}$
$\mathbb{K}(s)$		the quotient field of $\mathbb{K}[s]$
$\mathbb{K}^{m\times n}$		the set of $m \times n$ matrices with entries in $\mathbb{K}$
I or I_n		the identity matrix of size $n \times n$ for $n \in \mathbb{N}$
$A^\top$		the (conjugate) transposed of the matrix $A \in \mathbb{K}^{m\times n}$
$A\mathcal{S}$	$:=$	$\{\ Ax \in \mathbb{K}^m \mid x \in \mathcal{S}\ \}$, the image of the set $\mathcal{S} \subseteq \mathbb{K}^n$ under $A \in \mathbb{K}^{m\times n}$
$A^{-1}\mathcal{S}$	$:=$	$\{\ x \in \mathbb{K}^n \mid Ax \in \mathcal{S}\ \}$, the pre-image of the set $\mathcal{S} \subseteq \mathbb{K}^m$ under $A \in \mathbb{K}^{m\times n}$
$A^{-\top}\mathcal{S}$	$:=$	$(A^\top)^{-1}\mathcal{S}$
$\mathcal{S}^\perp$	$:=$	$\left\{\ x \in \mathbb{K}^n \ \middle|\ \forall s \in \mathcal{S}:\ x^\top s = 0\ \right\}$, the orthogonal complement of the set $\mathcal{S} \subseteq \mathbb{K}^n$
$\operatorname{rank}_\mathbb{C}(\lambda E - A)$		the complex rank of the matrix $(\lambda E - A) \in \mathbb{C}^{m\times n}$, $E, A \in \mathbb{K}^{m\times n}$, for $\lambda \in \mathbb{C}$; $\operatorname{rank}_\mathbb{C}(\infty E - A) := \operatorname{rank}_\mathbb{C} E$
$\operatorname{spec}(sE - A)$	$:=$	$\{\ \lambda \in \mathbb{C} \cup \{\infty\} \mid \operatorname{rank}_\mathbb{C}(\lambda E - A) < n\ \}$, the spectrum of the matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$
$\mathcal{C}^\infty$		the space of smooth (i.e. arbitrarily often differentiable) functions $f : \mathbb{R} \to \mathbb{R}$
$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$		the space of piecewise-smooth distributions as introduced in [19, 20]

2. Main results. As mentioned in the Introduction our approach is based on the Wong sequences which have been introduced in [24] for the analysis of matrix pencils. They can be calculated via a recursive subspace iteration. In a precorser [4] of this paper we used them to determine the quasi-Weierstraß form and it will turn out that they are the appropriate tool to determine a quasi-Kronecker form as well.

Definition 2.1 (Wong sequences [24]). *Consider a matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$. The* Wong sequences *corresponding to $sE - A$ are given by*

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{K}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i) \subseteq \mathbb{K}^n, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i) \subseteq \mathbb{K}^m. \end{aligned}$$

Let $\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i$ and $\mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}_i$ be the limits of the Wong sequences. ⋄

It is easy to see that the Wong sequences are nested, terminate and satisfy

$$\left.\begin{array}{l} \exists\, k^* \in \mathbb{N}\ \forall\, j \in \mathbb{N}\ :\quad \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} =: \mathcal{V}^* = A^{-1}(E\mathcal{V}^*) \supseteq \ker A, \\ \exists\, \ell^* \in \mathbb{N}\ \forall\, j \in \mathbb{N}\ :\quad \mathcal{W}_0 \subseteq \ker E = \mathcal{W}_1 \subsetneq \cdots \subsetneq \mathcal{W}_{\ell^*} = \mathcal{W}_{\ell^*+j} =: \mathcal{W}^* = E^{-1}(A\mathcal{W}^*)\,, \end{array}\right\} \tag{2.1}$$

$$A\mathcal{V}^* \subseteq E\mathcal{V}^* \qquad \text{and} \qquad E\mathcal{W}^* \subseteq A\mathcal{W}^*\,. \tag{2.2}$$

For our example DAE (1.2) we obtain:

$$\mathcal{V}_1 = \operatorname{im}\begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ R&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 1&0&0\\ 0&1&0\\ 0&0&1\\ 0&-1&1\\ -1&1&-1 \end{bmatrix} = \mathcal{V}^* \text{ and } \mathcal{W}_1 = \operatorname{im}\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},\ \mathcal{W}_2 = \operatorname{im}\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} = \mathcal{W}^*$$

We carried out the calculation with Matlab and its Symbolic Tool Box and the following short function for calculating the pre-image:

LISTING 1

Matlab function for calculating a basis of the pre-image $A^{-1}(\operatorname{im} S)$ for some matrices A and S

```
function V=getPreImage(A,S)
[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
else
    error('Both matrices must have same number of rows');
end;
```

Before stating our main result we repeat the result concerning the Wong sequences and *regular* matrix pencils.

THEOREM 2.2 (The regular case, [4]). *Consider a* regular *matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$, i.e. $m = n$ and $\det(sE - A) \in \mathbb{K}[s] \setminus \{0\}$. Let $\mathcal{V}^*$ and $\mathcal{W}^*$ be the limits of the corresponding Wong sequences. Choose any full rank matrices V and W such that $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$. Then $T = [V, W]$ and $S = [EV, AW]^{-1}$ are invertible and put the matrix pencil $sE - A$ into quasi-Weierstraß form*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0\\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0\\ 0 & I \end{bmatrix}\right),$$

where $J \in \mathbb{K}^{n_J\times n_J}$, $n_J \in \mathbb{N}$, and $N \in \mathbb{K}^{n_N\times n_N}$, $n_N = n - n_J$, is a nilpotent matrix. In particular, when choosing T_J and T_N such that $T_J^{-1} J T_J$ and $T_N^{-1} N T_N$ are in Jordan canonical form, then $S' = [EVT_J, AWT_N]^{-1}$ and $T' = [VT_J, WT_N]$ put the regular matrix pencil $sE - A$ into Weierstraß canonical form. $\diamond$

Important consequences of the known result about the Wong sequences in the regular case are

$$\begin{aligned} \mathcal{V}^* \cap \mathcal{W}^* &= \{0\}, & E\mathcal{V}^* \cap A\mathcal{W}^* &= \{0\},\\ \mathcal{V}^* + \mathcal{W}^* &= \mathbb{K}^n, & E\mathcal{V}^* + A\mathcal{W}^* &= \mathbb{K}^n. \end{aligned}$$

These properties do not hold anymore for a general matrix pencil $sE - A$, see Figure 2.1 for an illustration of the situation.

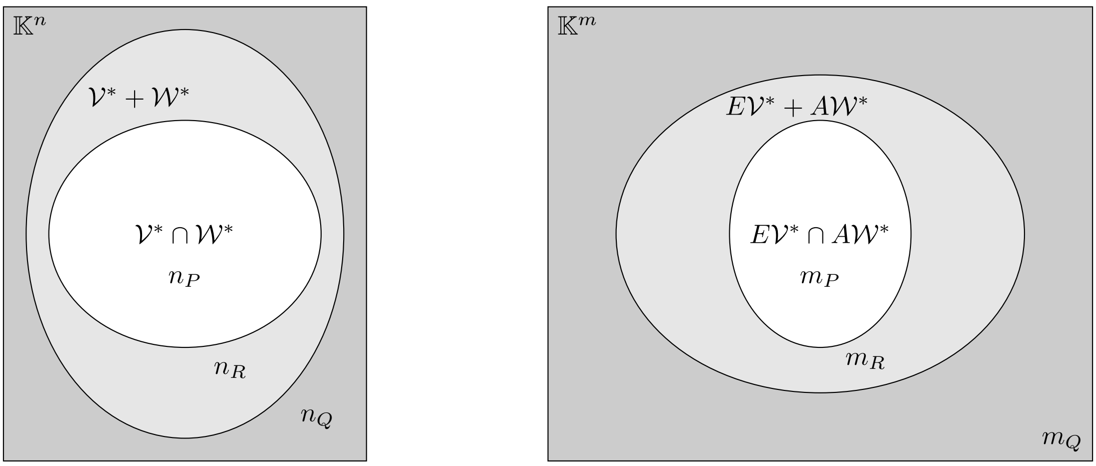

FIG. 2.1. *The relationship of the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences of the matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$ in the general case; the numbers $n_P, n_R, n_Q, m_P, m_R, m_Q \in \mathbb{N}$ denote the (difference of the) dimensions of the corresponding spaces.*

We are now ready to present our first main result which states that the knowledge of the spaces $\mathcal{V}^*$ and $\mathcal{W}^*$ is sufficient to obtain the *quasi-Kronecker triangular form*, which already captures most structural properties of the matrix pencil $sE-A$. With the help of the Wong sequences Armentano [2] already obtained a similar result, however his aim was to obtain a triangular form where the diagonal blocks are in a canonical form. Therefore, his result is more general then ours, however, the price is a more complicated proof and it is also not clear how to obtain the transformation matrices explicitly.

THEOREM 2.3 (Quasi-Kronecker triangular form). *Let $sE-A \in \mathbb{K}^{m\times n}[s]$ and consider the corresponding limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences as in Definition 2.1. Choose any full rank matrices $P_1 \in \mathbb{K}^{n\times n_P}$, $P_2 \in \mathbb{K}^{m\times m_P}$, $R_1 \in \mathbb{K}^{n\times n_R}$, $R_2 \in \mathbb{K}^{m\times m_P}$, $Q_1 \in \mathbb{K}^{n\times n_Q}$, $Q_2 \in \mathbb{K}^{m\times m_Q}$ such that*

$$\begin{aligned} \operatorname{im} P_1 &= \mathcal{V}^* \cap \mathcal{W}^*, & \operatorname{im} P_2 &= E\mathcal{V}^* \cap A\mathcal{W}^*, \\ \mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{im} R_1 &= \mathcal{V}^* + \mathcal{W}^*, & E\mathcal{V}^* \cap A\mathcal{W}^* \oplus \operatorname{im} R_2 &= E\mathcal{V}^* + A\mathcal{W}^*, \\ (\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{im} Q_1 &= \mathbb{R}^n, & (E\mathcal{V}^* + A\mathcal{W}^*) \oplus \operatorname{im} Q_2 &= \mathbb{R}^m. \end{aligned}$$

Then $T_{\mathrm{trian}} = [P_1, R_1, Q_1] \in \mathbf{Gl}_n(\mathbb{K})$ and $S_{\mathrm{trian}} = [P_2, R_2, Q_2]^{-1} \in \mathbf{Gl}_m(\mathbb{K})$ transform $sE-A$ in quasi-Kronecker triangular form*:*

$$(S_{\mathrm{trian}} E T_{\mathrm{trian}}, S_{\mathrm{trian}} A T_{\mathrm{trian}}) = \left(\begin{bmatrix} E_P & E_{PR} & E_{PQ} \\ 0 & E_R & E_{RQ} \\ 0 & 0 & E_Q \end{bmatrix}, \begin{bmatrix} A_P & A_{PR} & A_{PQ} \\ 0 & A_R & A_{RQ} \\ 0 & 0 & A_Q \end{bmatrix} \right), \tag{2.3}$$

where

(i) *$E_P, A_P \in \mathbb{K}^{m_P\times n_P}$, $m_P < n_P$, are such that* $\operatorname{rank}_{\mathbb{C}}(\lambda E_P - A_P) = m_P$ *for all $\lambda \in \mathbb{C}\cup\{\infty\}$,*

(ii) *$E_R, A_R \in \mathbb{K}^{m_R\times n_R}$, $m_R = n_R$, with $sE_R - A_R$* regular, *i.e.* $\det(sE_R - A_R) \not\equiv 0$,

(iii) *$E_Q, A_Q \in \mathbb{K}^{m_Q\times n_Q}$, $m_Q > n_Q$, are such that* $\operatorname{rank}_{\mathbb{C}}(\lambda E_Q - A_Q) = n_Q$ *for all $\lambda \in \mathbb{C}\cup\{\infty\}$.* ⋄

The proof is carried out in Section 5.

REMARK 2.4. *The sizes of the blocks in (2.3) are uniquely given by the matrix pencil $sE-A$ because they only depend on the subspaces constructed by the Wong sequences and not on the choice of bases thereof. It is also possible that $m_P = 0$ (or $n_Q = 0$) which means that there are matrices with no rows (or no columns). On the other hand, if $n_P = 0$, $n_R = 0$ or $m_Q = 0$ then the P-blocks, R-blocks or Q-blocks are not present at all. Furthermore, it is easily seen, that if $sE-A$ fulfills (i), (ii) or (iii) itself then $sE-A$ is already in quasi-Kronecker triangular form with $T_{\mathrm{trian}} = P_1 = I$, $T_{\mathrm{trian}} = R_1 = I$, or $T_{\mathrm{trian}} = Q_1 = I$, and $S_{\mathrm{trian}} = P_2^{-1} = I$, $S_{\mathrm{trian}} = R_2^{-1} = I$, or $S_{\mathrm{trian}} = Q_2^{-1} = I$.* ⋄

In our example (1.2) we have

$$\mathcal{V}^* \cap \mathcal{W}^* = \mathcal{V}^*,\ \mathcal{V}^* + \mathcal{W}^* = \mathcal{W}^*$$

and, with $K := \frac{R_G+R_F}{R_G}$,

$$E\mathcal{V}^* \cap A\mathcal{W}^* = E\mathcal{V}^* = \operatorname{im} \begin{bmatrix} 0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix},\ E\mathcal{V}^* + A\mathcal{W}^* = A\mathcal{W}^* = \operatorname{im} \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1\\ -K&0&-\frac{R_F}{R_G}&1&0&0&-R_F&K&R_F&0&0 \end{bmatrix}$$

Therefore, we can choose

$$[P_1, R_1, Q_1] = \left[\begin{array}{ccc|cccccccccc|c} 0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ R&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&-1&1&0&0&0&0&0&0&0&0&1&0&0\\ -1&1&-1&0&0&0&0&0&0&0&0&0&1&0 \end{array}\right] \text{ and } [P_2, R_2, Q_2] = \left[\begin{array}{c|ccccccccccc|cc} 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&-K&-\frac{R_F}{R_G}&1&0&0&-R_F&K&R_F&0&0&0&1 \end{array}\right].$$

With this choice we obtain the following Kronecker triangular form for our example:

$$
(E, A) \cong \left(\begin{bmatrix}
CR & 0 & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & LK
\end{bmatrix}, \begin{bmatrix}
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & R_G & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & R_F & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \right)
$$

The quasi-Kronecker triangular form is already useful for the analysis of the matrix pencil $sE - A$ and the associated DAE $E\dot{x} = Ax + f$. However, a complete decoupling of the different parts, i.e. a block triangular form, is more satisfying from a theoretical viewpoint and is also a necessary step to obtain the Kronecker canonical form as a corollary. In the next result we show that we can transform any matrix pencil $sE - A$ into a block triangular form, which we call *quasi-Kronecker form* because all the important features of the Kronecker canonical form are captured. In fact, it turns out that the diagonal blocks of the quasi-Kronecker triangular form (2.3) already are the diagonal blocks of the quasi-Kronecker form.

THEOREM 2.5 (Quasi-Kronecker form). *Using the notation from Theorem 2.3 the following equations are solvable for matrices $F_1, F_2, G_1, G_2, H_1, H_2$ of appropriate size:*

$$
\begin{aligned}
0 &= E_{RQ} + E_R F_1 + F_2 E_Q \\
0 &= A_{RQ} + A_R F_1 + F_2 A_Q
\end{aligned} \tag{2.4a}
$$

$$
\begin{aligned}
0 &= E_{PR} + E_P G_1 + G_2 E_R \\
0 &= A_{PR} + A_P G_1 + G_2 A_R
\end{aligned} \tag{2.4b}
$$

$$
\begin{aligned}
0 &= (E_{PQ} + E_{PR} F_1) + E_P H_1 + H_2 E_Q \\
0 &= (A_{PQ} + A_{PR} F_1) + A_P H_1 + H_2 A_Q
\end{aligned} \tag{2.4c}
$$

and for any such matrices let

$$
\begin{aligned}
S &:= \begin{bmatrix} I & -G_2 & -H_2 \\ 0 & I & -F_2 \\ 0 & 0 & I \end{bmatrix}^{-1} S_{\text{trian}} &= [P_2, R_2 - P_2 G_2, Q_2 - P_2 H_2 - R_2 F_2]^{-1} \quad \textit{and} \\
T &:= T_{\text{trian}} \begin{bmatrix} I & G_1 & H_1 \\ 0 & I & F_1 \\ 0 & 0 & I \end{bmatrix} &= [P_1, R_1 + P_1 G_1, Q_1 + P_1 H_1 + R_1 F_1].
\end{aligned}
$$

Then $S \in \mathbf{Gl}_m(\mathbb{K})$ and $T \in \mathbf{Gl}_n(\mathbb{K})$ put $sE - A$ in quasi-Kronecker form

$$
(SET, SAT) = \left(\begin{bmatrix} E_P & 0 & 0 \\ 0 & E_R & 0 \\ 0 & 0 & E_Q \end{bmatrix}, \begin{bmatrix} A_P & 0 & 0 \\ 0 & A_R & 0 \\ 0 & 0 & A_Q \end{bmatrix} \right), \tag{2.5}
$$

where the block diagonal entries are the same as for the quasi-Kronecker triangular form (2.3)*. In particular, the quasi-Kronecker form (without the transformation matrices S and T) can be obtained only with the Wong sequences (i.e. without solving* (2.4)*).* ◇

The proof is carried out in Section 5.

REMARK 2.6. *Matrix equations of the form*

$$
\begin{aligned}
0 &= M + PX + YQ \\
0 &= R + SX + YT
\end{aligned}
$$

for given matrices M, P, Q, R, S, T of appropriate size can be written equivalently as a standard linear system

$$
\begin{bmatrix} I \otimes P & Q^\top \otimes I \\ I \otimes S & T^\top \otimes I \end{bmatrix} \begin{pmatrix} \operatorname{vec}(X) \\ \operatorname{vec}(Y) \end{pmatrix} = - \begin{pmatrix} \operatorname{vec}(M) \\ \operatorname{vec}(R) \end{pmatrix},
$$

where $\otimes$ denotes the Kronecker product of matrices and $\operatorname{vec}(H)$ *denotes the vectorization of the matrix H obtained by stacking all columns of H over each other.* ◇

For our example (1.2) we already know the quasi-Kronecker form, because as mentioned in Theorem 2.5 the diagonal blocks are the same as for the quasi-Kronecker triangular form. However, we do not yet know the final transformation matrices which yield the quasi-Kronecker form. Therefore, we have to find solutions of (2.4):

$$F_1 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix},\ F_2 = \begin{bmatrix} 0 & -\frac{1}{K}\\ 0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0 \end{bmatrix},\qquad \begin{array}{ll} G_1 = \begin{bmatrix} 0&0&\frac{1}{R}&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}, & H_1 = \begin{bmatrix}0\\0\\0\end{bmatrix},\\ G_2 = \begin{bmatrix} \frac{-1}{R_G} & \frac{-1}{R_G} & 0&0&0&-1&\frac{1}{R_G}&1&-1&0 \end{bmatrix}, & H_2 = \begin{bmatrix}0&0\end{bmatrix}. \end{array}$$

The transformation matrices S and T which put our example into a quasi-Kronecker form are then

$$T = \begin{bmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
R&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&\frac{1}{R}&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&1&0&0&0&0&0&0&0&0&1&0&0\\
-1&1&-1&0&0&\frac{-1}{R}&0&0&0&0&0&0&1&0
\end{bmatrix} \text{ and } S = \begin{bmatrix}
0&1&\frac{-1}{R_GK}&\frac{-1}{R_GK}&0&0&\frac{-1}{K}&0&0&\frac{1}{K}&-1&0&\frac{1}{R_GK}\\
0&0&\frac{-R_F}{R_GK}&\frac{1}{K}&0&0&\frac{-R_F}{K}&1&0&\frac{R_F}{K}&0&0&\frac{-1}{K}\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
K&0&\frac{R_F}{R_G}&-1&0&0&R_F&-K&0&-R_F&0&0&1
\end{bmatrix}.$$

Finally, an analysis of the matrix pencils $sE_P - A_P$ and $sE_Q - A_Q$ in (2.5) with the property that $\operatorname{rank} \lambda E_P - A_P = m_P$ and $\operatorname{rank} \lambda E_Q - A_Q = n_Q$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$ (see Lemma 4.14 and Corollary 4.15) together with Theorem 2.2 allows now to obtain the Kronecker canonical form as a corollary.

Corollary 2.7. *For every matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$ there exist transformation matrices $S \in \mathbf{Gl}_m(\mathbb{C})$ and $T \in \mathbf{Gl}_n(\mathbb{C})$ such that, for $a, b, c, d \in \mathbb{N}$ and $\varepsilon_1, \ldots, \varepsilon_a, \eta_1, \ldots, \eta_b, \rho_1, \ldots, \rho_c, \sigma_1, \ldots, \sigma_d \in \mathbb{N}$,*

$$S(sE - A)T = \operatorname{diag}(\mathcal{P}_{\varepsilon_1}(s), \ldots, \mathcal{P}_{\varepsilon_a}(s), \mathcal{J}_{\rho_1}(s), \ldots, \mathcal{J}_{\rho_b}(s), \mathcal{N}_{\sigma_1}(s), \ldots, \mathcal{N}_{\sigma_c}(s), \mathcal{Q}_{\eta_1}(s), \ldots, \mathcal{Q}_{\eta_d}(s)),$$

where

$$\mathcal{P}_\varepsilon(s) = s\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{K}^{\varepsilon\times(\varepsilon+1)}[s],\ \ \varepsilon \in \mathbb{N},$$

$$\mathcal{J}_\rho(s) = sI - \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & \lambda \end{bmatrix} \in \mathbb{C}^{\rho\times\rho}[s], \rho \in \mathbb{N}, \lambda \in \mathbb{C},$$

$$\mathcal{N}_\sigma(s) = s\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & 0 \end{bmatrix} - I \in \mathbb{K}^{\sigma\times\sigma}[s], \sigma \in \mathbb{N},$$

$$\mathcal{Q}_\eta(s) = s\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0\\ & & 1 \end{bmatrix} - \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1\\ & & 0 \end{bmatrix} \in \mathbb{K}^{(\eta+1)\times\eta}[s],\ \ \eta \in \mathbb{N}.$$

◇

3. Application of the quasi-Kronecker (triangular) form to DAE solution theory.

In this section we study the DAE (1.1)

$$E\dot{x} = Ax + f$$

corresponding to the matrix pencil $sE - A \in \mathbb{R}^{m\times n}[s]$. Note that we restrict ourselves here to the field $\mathbb{K} = \mathbb{R}$, becaue 1) the vast majority of DAEs arising from modeling physical phenomena are not complex-valued, 2) all the results for $\mathbb{K} = \mathbb{R}$ carry over to $\mathbb{K} = \mathbb{C}$ without modification (the converse is not true in general), 3) the case $\mathbb{K} = \mathbb{Q}$ is rather artificial when considering solutions of the DAE (1.1), because then we had to consider functions $f : \mathbb{R} \to \mathbb{Q}$ or even $f : \mathbb{Q} \to \mathbb{Q}$.

We first have to decide in which (function) space we actually consider the DAE (1.1). To avoid problems with differentiability one suitable choice is the space of smooth functions $\mathcal{C}^\infty$, i.e. we consider smooth inhomogeneities $f \in (\mathcal{C}^\infty)^m$ and smooth $x \in (\mathcal{C}^\infty)^n$. Unfortunately, this excludes the possibility to consider step functions as inhomogeneities which occur rather frequently. It is well known that the solutions of DAEs might involve derivatives of the inhomogeneities, hence jumps in the inhomogeneity might lead to non-existence of solutions due to a lack of differentiability. However, this is not a "structural non-existence" since every smooth approximation of the jump could lead to well defined solutions. Therefore, one might extend the solution space by considering distributions (or generalized functions) as formally introduced by Schwartz [17]. The advantage of this larger solution space is that each distribution is smooth, in particular the unit step function (Heaviside function) has a derivative: the Dirac impulse. Unfortunately, the whole space of distributions is too large, for example it is in general not possible to speak of an initial value, because evaluation of a distribution at a specific time is not defined. To overcome this obstacle we consider the smaller space of *piecewise-smooth distributions* $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ as introduced in [19, 20]. For piecewise-smooth distributions a left- and right-sided evaluation is possible, i.e. for $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ the values $D(t-) \in \mathbb{R}$ and $D(t+) \in \mathbb{R}$ are well defined for all $t \in \mathbb{R}$.

Altogether, we will formulate all results for both solution spaces $\mathcal{S} = \mathcal{C}^\infty$ and $\mathcal{S} = \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$, so that readers who feel uneasy about the distributional solution framework can ignore it.

Before stating our main results concerning the solution theory of the DAE (1.1), we need the following result about polynomial matrices which is proved in Section 4.4. A (square) polynomial matrix $U(s) \in \mathbb{K}^{n\times n}[s]$ is called unimodular if, and only if, it is invertible within the ring $\mathbb{K}^{n\times n}[s]$, i.e. there exists $V(s) \in \mathbb{K}^{n\times n}[s]$ such that $U(s)V(s) = I$.

Lemma 3.1 (Existence of unimodular inverse). *Consider a matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$, $m \neq n$, such that* $\operatorname{rank} \lambda E - A = \min\{m, n\}$ *for all $\lambda \in \mathbb{C}$. Then there exist polynomial matrices $M(s) \in \mathbb{K}^{n\times m}[s]$ and $K(s) \in \mathbb{K}^{n'\times m'}[s]$, $n', m' \in \mathbb{N}$, such that, if $m < n$, $[M(s), K(s)]$ is unimodular and*

$$(sE - A)[M(s), K(s)] = [I_m, 0],$$

or, if $m > n$, $\begin{bmatrix} M(s) \\ K(s) \end{bmatrix}$ is unimodular and

$$\begin{bmatrix} M(s) \\ K(s) \end{bmatrix} (sE - A) = \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

◇

Theorem 3.2 (Complete characterization of solutions of the DAE). *Let $sE - A \in \mathbb{R}^{m\times n}[s]$ and use the notation from Theorem 2.5. Consider the solution space $\mathcal{S} = \mathcal{C}^\infty$ or $\mathcal{S} = \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and let $f \in \mathcal{S}^m$. According to Theorem 2.2 let $S_R, T_R \in \mathbf{Gl}_{n_R}(\mathbb{R})$ be the matrices which transform $sE_R - A_R$ in quasi-Weierstraß form, i.e.*

$$\begin{bmatrix} I & 0 & 0 \\ 0 & S_R & 0 \\ 0 & 0 & I \end{bmatrix} S(sE - A)T \begin{bmatrix} I & 0 & 0 \\ 0 & T_R & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} sE_P - A_P & 0 & 0 & 0 \\ 0 & sI - J & 0 & 0 \\ 0 & 0 & sN - I & 0 \\ 0 & 0 & 0 & sE_Q - A_Q \end{bmatrix} \tag{3.1}$$

and let $(f_P^\top, f_J^\top, f_N^\top, f_Q^\top)^\top := \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & S_R & 0 \\ 0 & 0 & I \end{smallmatrix}\right] Sf$, where the splitting corresponds to the block sizes in (3.1). *According to Lemma 3.1 choose unimodular matrices $[M_P(s), K_P(s)] \in \mathbb{R}^{n_P\times(m_P+(n_P-m_P))}[s]$ and $\left[\begin{smallmatrix} M_Q(s) \\ K_Q(s) \end{smallmatrix}\right] \in \mathbb{R}^{(n_Q+(m_Q-n_Q))\times m_Q}[s]$ such that*

$$(sE_P - A_P)[M_P(s), K_P(s)] = [I, 0] \quad \text{and} \quad \begin{bmatrix} M_Q(s) \\ K_Q(s) \end{bmatrix} (sE_Q - A_Q) = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Then there exist solutions of the DAE $E\dot{x} = Ax + f$ if, and only if,

$$\boxed{K_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q) = 0}.$$

If this is the case, then an initial value $x^0 = T \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & T_R & 0 \\ 0 & 0 & I \end{smallmatrix}\right] \left({x_P^0}^\top, {x_J^0}^\top, {x_N^0}^\top, {x_Q^0}^\top\right)^\top$ is consistent at $t_0 \in \mathbb{R}$, i.e. there exists a solution of the initial value problem

$$E\dot{x} = Ax + f, \quad x(t_0-) = x^0, \tag{3.2}$$

if, and only if,

$$\boxed{x_Q^0 = \left(M_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q)\right)(t_0-)} \quad \text{and} \quad \boxed{x_N^0 = -\left(\sum_{k=0}^{n_N-1} N^k (\tfrac{\mathrm{d}}{\mathrm{d}t})^k (f_N)\right)(t_0-)}. \tag{3.3}$$

If (3.3) *holds, then any solution* $x = T \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & T_R & 0 \\ 0 & 0 & I \end{smallmatrix}\right] \left(x_P{}^\top, x_J{}^\top, x_N{}^\top, x_Q{}^\top\right)^\top$ *of the initial value problem* (3.2) *has the form*

$$\boxed{\begin{array}{rcl} x_P &=& M_P(\frac{\mathrm{d}}{\mathrm{d}t})(f_P) + K_P(\frac{\mathrm{d}}{\mathrm{d}t})(u_{x_P^0}), \\ x_J &=& e^{J(\cdot - t_0)} x_J^0 + e^{J\cdot} \int_{t_0} \left(e^{-J\cdot} f_J\right), \\ x_N &=& -\sum_{k=0}^{n_N-1} N^k (\frac{\mathrm{d}}{\mathrm{d}t})^k (f_N), \\ x_Q &=& M_Q(\frac{\mathrm{d}}{\mathrm{d}t})(f_Q), \end{array}}$$

where $u_{x_P^0} \in \mathcal{S}^{n_P - m_P}$ *is such that the initial condition at* t_0 *for* x_P *is satisfied (which is always possible due to Lemma 4.18), but apart from that arbitrary.* $\diamond$

The proof is carried out in Section 5.

Note that the antiderivative operator $\int_{t_0} : \mathcal{S} \to \mathcal{S},\ f \mapsto F$ as used in Theorem 3.2 is uniquely defined by the two properties $\frac{\mathrm{d}}{\mathrm{d}t} F = f$ and $F(t_0-) = 0$ (for $\mathcal{S} = \mathcal{C}^\infty$ this is well known, whilst for $\mathcal{S} = \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ this is shown in [20, Prop. 3], see also [19, Prop. 2.3.6]).

We want to use Theorem 3.2 to characterize the solutions of our example DAE (1.2). We first observe that the regular part can be brought into quasi-Weierstraß form $s0 - I$ by pre-multiplying with $S_R = A_R^{-1}$. In particular, the J-part is non-existent, which means that the circuit *contains no classical dynamics.* We choose

$$[M_P(s), K_P(s)] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & CRs & 1 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} M_Q(s) \\ K_Q(s) \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ \hline -LKs & 1 \end{array}\right].$$

Furthermore, $T_R = I$, $f_P = \frac{V}{R_G + R_F}$, $f_J = []$, $f_N = \frac{V}{K}[-1, -1, -K, 0, 0, -\frac{1}{R_G}, -\frac{1}{R_G}, 0, -\frac{1}{R_G}, \frac{1}{R_G}]^\top$, $f_Q = [I, V]^\top$, hence the DAE (1.2) is solvable if, and only if,

$$0 = K_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q) = -LK \tfrac{\mathrm{d}}{\mathrm{d}t} I + V \quad \text{or, equivalently,} \quad \boxed{V = LK \tfrac{\mathrm{d}}{\mathrm{d}t} I},$$

i.e. the voltage source must be proportional to the change of current provided by the current source. In that case, the initial value must fulfill

$$x(0-) = T[*, *, *, -f_N(0-)^\top, M_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q)(0-)]^\top,$$

i.e., recalling $x = (p_+, p_-, p_o, p_T, i_L, i_p, i_m, i_G, i_F, i_R, i_o, i_V, i_C, i_T)^\top$, $i_R(0-)$, $i_o(0-)$, $i_V(0-)$ are arbitrary and $p_-(0-) = \frac{V(0-)}{K}$, $p_+(0-) = \frac{V(0-)}{K}$, $p_o(0-) = V(0-)$, $p_T(0-) = Ri_R(0-)$, $i_L(0-) = I(0-)$, $i_p(0-) = 0$, $i_m(0-) = 0$, $i_G = \frac{V(0-)}{R_F + R_G}$, $i_F = \frac{V(0-)}{R_F + R_G}$, $i_C(0-) = i_V(0-) - i_o(0-) + \frac{V(0-)}{R_F + R_G}$, $i_T(0-) = i_o(0-) - i_R(0-) - i_V(0-) - \frac{V(0-)}{R_F + R_G}$. If these conditions are satisfied, then all solutions of the initial value problem corresponding to our example DAE (1.2) are given by

$$\begin{aligned} x &= T[u_1, u_2, \tfrac{-V}{R_G + R_F} + RC\dot{u}_1 + u_2, \tfrac{V}{K}, \tfrac{V}{K}, V, 0, 0, \tfrac{V}{R_F + R_G}, \tfrac{V}{R_F + R_G}, 0, \tfrac{V}{R_F + R_G}, \tfrac{-V}{R_F + R_G}, I]^\top, \\ &= [\tfrac{V}{K}, \tfrac{V}{K}, V, p_T(u_1), I, 0, 0, \tfrac{V}{R_F + R_G}, \tfrac{V}{R_F + R_G}, i_R(u_1), i_o(u_2), i_V(u_1, u_2), i_C(u_1), i_T(u_1)]^\top, \end{aligned}$$

where $u_1, u_2 \in \mathcal{S}$ are arbitrary, apart from the initial conditions

$$u_1(0-) = i_R(0-) - \tfrac{V(0-)}{R}, \quad \dot{u}_1(0-) = \tfrac{1}{CR}\left(i_V(0-) + \tfrac{V(0-)}{R_F + R_G} - i_o(0-)\right) \quad \text{and} \quad u_2(0-) = i_o(0-)$$

and

$$\begin{array}{lll} p_T(u_1) = V + Ru_1, & i_R(u_1) = u_1 + \frac{V}{R}, & i_o(u_2) = u_2, \\ i_V(u_1, u_2) = u_2 - \frac{V}{R_F + R_G} + CR\dot{u}_1, & i_C(u_1) = CR\dot{u}_1, & i_T(u_1) = -u_1 - \frac{V}{R} - CR\dot{u}_1. \end{array}$$

Remark 3.3. *A similar statement as in Theorem 3.2 is also possible if we only consider the quasi-Kronecker triangular form* (2.3)*, i.e. instead of* (3.1) *we consider*

$$\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & S_R & 0 \\ 0 & 0 & I \end{smallmatrix}\right] S_{\mathrm{trian}}(sE - A) T_{\mathrm{trian}} \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & T_R & 0 \\ 0 & 0 & I \end{smallmatrix}\right] = \begin{bmatrix} sE_P - A_P & sE_{PJ} - A_{PJ} & sE_{PN} - A_{PN} & sE_{PQ} - A_{PQ} \\ 0 & sI - J & 0 & sE_{JQ} - A_{JQ} \\ 0 & 0 & sN - I & sE_{NQ} - A_{NQ} \\ 0 & 0 & 0 & sE_Q - A_Q \end{bmatrix}$$

The corresponding conditions for the Q*-part remain the same, in the condition for the* N*-part the inhomogeneity* f_N *is replaced by* $f_N - (E_{NQ} \frac{\mathrm{d}}{\mathrm{d}t} - A_{NQ})(x_Q)$*, in the* J*-part the inhomogeneity* f_J *is replaced by* $f_J - (E_{JQ} \frac{\mathrm{d}}{\mathrm{d}t} - A_{JQ})(x_Q)$ *and in the* P*-part the inhomogeneity* f_P *is replaced by* $f_P - (E_{PJ} \frac{\mathrm{d}}{\mathrm{d}t} - A_{PJ})(x_J) - (E_{PN} \frac{\mathrm{d}}{\mathrm{d}t} - A_{PN})(x_N) - (E_{PQ} \frac{\mathrm{d}}{\mathrm{d}t} - A_{PQ})(x_Q)$. $\diamond$

4. Useful Lemmas. In this section we collect several lemmas which are needed to prove the main results. Since we use results from different areas we group the lemmas accordingly into subsections.

4.1. Standard results from linear algebra.

Lemma 4.1 (Orthogonal complements and (pre-)images). *For any matrix $M \in \mathbb{K}^{p\times q}$ we have:*

(i) *for all subspaces $\mathcal{S} \subseteq \mathbb{K}^q$ it holds $(M\mathcal{S})^\perp = M^{-\top}(\mathcal{S}^\perp)$.*

(ii) *for all subspaces $\mathcal{S} \subseteq \mathbb{K}^p$ it holds $(M^{-1}\mathcal{S})^\perp = M^{\top}(\mathcal{S}^\perp)$.* ◇

Proof. The following equivalence holds for all $x \in \mathbb{K}^q$:

$$x \in (M\mathcal{S})^\perp \;\Leftrightarrow\; \forall s \in \mathcal{S} : 0 = x^\top M s = (M^\top x)^\top s \;\Leftrightarrow\; M^\top x \in \mathcal{S}^\perp \;\Leftrightarrow\; x \in M^{-\top}(\mathcal{S}^\perp),$$

hence 4.1 is shown. Property 4.1 follows from considering the orthogonal complements and

$$(M^\top \mathcal{S}^\perp)^\perp \overset{\text{(i)}}{=} M^{-1}\mathcal{S}.$$

□

Lemma 4.2 (Rank of matrices). *Let $A, B \in \mathbb{K}^{m\times n}$ with* $\operatorname{im} B \subseteq \operatorname{im} A$*. Then for almost all $c \in \mathbb{K}$:*

$$\operatorname{rank} A = \operatorname{rank}(A + cB),$$

or, equivalently,

$$\operatorname{im} A = \operatorname{im}(A + cB).$$

In fact, $\operatorname{rank} A < \operatorname{rank}(A + cB)$ *can only hold for at most $r =$* $\operatorname{rank} A$ *many values of c.* ◇

Proof. Consider the Smith form ([18]) of A:

$$UAV = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$$

with invertible matrices $U \in \mathbb{K}^{m\times m}$ and $V \in \mathbb{K}^{n\times n}$ and $\Sigma_r = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, $\sigma_i \in \mathbb{K} \setminus \{0\}$, $r = \operatorname{rank} A$. Write

$$UBV = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $B_{11} \in \mathbb{K}^{r\times r}$. Since $\operatorname{im} B \subseteq \operatorname{im} A$ it follows that $B_{21} = 0$ and $B_{22} = 0$. Hence we obtain the following implications:

$$\begin{aligned} \operatorname{rank}(A + cB) < \operatorname{rank} A \;&\Rightarrow\; \operatorname{rank}[\Sigma_r + cB_{11}, cB_{12}] < \operatorname{rank}[\Sigma_r, 0] = r \;\Rightarrow\; \operatorname{rank}(\Sigma_r + cB_{11}) < r \\ &\Rightarrow\; \det(\Sigma_r + cB_{11}) = 0. \end{aligned}$$

Since $\det(\Sigma_r + cB_{11})$ is a polynomial in c of degree at most r but not the zero polynomial (since $\det(\Sigma_r) \neq 0$) it can have at most r zeros. This proves the claim. □

Lemma 4.3 (Dimension formulae). *Let $\mathcal{S} \subseteq \mathbb{K}^n$ be any linear subspace of $\mathbb{K}^n$ and $M \in \mathbb{K}^{m\times n}$. Then*

$$\dim M\mathcal{S} = \dim \mathcal{S} - \dim(\ker M \cap \mathcal{S}).$$

Furthermore, for any two linear subspaces $\mathcal{S}, \mathcal{T}$ of $\mathbb{K}^n$ we have

$$\dim(\mathcal{S} + \mathcal{T}) = \dim \mathcal{S} + \dim \mathcal{T} - \dim(\mathcal{S} \cap \mathcal{T}).$$

◇

Proof. See any textbook on linear algebra. □

4.2. The Wong sequences. The next lemma highlights an important property of the intersection of the limits of the Wong sequences.

Lemma 4.4 (Property of $\mathcal{V}^* \cap \mathcal{W}^*$). *Let $sE - A \in \mathbb{K}^{m\times n}[s]$ and $\mathcal{V}^*$, $\mathcal{W}^*$ be the limits of the corresponding Wong sequences. Then*

$$E(\mathcal{V}^* \cap \mathcal{W}^*) = E\mathcal{V}^* \cap A\mathcal{W}^* = A(\mathcal{V}^* \cap \mathcal{W}^*).$$

$\diamond$

Proof. Clearly, invoking $A\mathcal{V}^* \subseteq E\mathcal{V}^*$ and $E\mathcal{W}^* \subseteq A\mathcal{W}^*$ (see (2.2)),

$$E(\mathcal{V}^* \cap \mathcal{W}^*) \subseteq E\mathcal{V}^* \cap E\mathcal{W}^* \subseteq E\mathcal{V}^* \cap A\mathcal{W}^* \quad \text{and} \quad A(\mathcal{V}^* \cap \mathcal{W}^*) \subseteq A\mathcal{V}^* \cap A\mathcal{W}^* \subseteq E\mathcal{V}^* \cap A\mathcal{W}^*,$$

hence it remains to show the converse subspace relationship. To this end choose $x \in E\mathcal{V}^* \cap A\mathcal{W}^*$, which implies existence of $v \in \mathcal{V}^*$ and $w \in \mathcal{W}^*$ such that

$$Ev = x = Aw,$$

hence

$$v \in E^{-1}\{Aw\} \subseteq E^{-1}(A\mathcal{W}^*) = \mathcal{W}^*, \quad w \in A^{-1}\{Ev\} \subseteq A^{-1}(E\mathcal{V}^*) = \mathcal{V}^*.$$

Therefore $v, w \in \mathcal{V}^* \cap \mathcal{W}^*$ and $x = Ev \in E(\mathcal{V}^* \cap \mathcal{W}^*)$ as well as $x = Aw \in A(\mathcal{V}^* \cap \mathcal{W}^*)$ which concludes the proof. ☐

For the proof of the main result we briefly consider the Wong sequences of the (conjugate) transposed matrix pencil $sE^\top - A^\top$; these are connected to the original Wong sequences as follows.

Lemma 4.5 (Wong-sequences of the transposed matrix pencil). *Consider a matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$ with corresponding limits of the Wong sequences $\mathcal{V}^*$ and $\mathcal{W}^*$. Denote with $\widehat{\mathcal{V}}^*$ and $\widehat{\mathcal{W}}^*$ the limits of the Wong sequences of the (conjugate) transposed matrix pencil $sE^\top - A^\top$. Then the following holds*

$$\widehat{\mathcal{W}}^* = (E\mathcal{V}^*)^\perp \quad \textit{and} \quad \widehat{\mathcal{V}}^* = (A\mathcal{W}^*)^\perp.$$

$\diamond$

Proof. We show that for all $i \in \mathbb{N}$

$$(E\mathcal{V}_i)^\perp = \widehat{\mathcal{W}}_{i+1} \quad \text{and} \quad (A\mathcal{W}_i)^\perp = \widehat{\mathcal{V}}_i, \tag{4.1}$$

from which the claim follows. For $i = 0$ this follows from

$$(E\mathcal{V}_0)^\perp = (\operatorname{im} E)^\perp = \ker E^\top = E^{-\top}(A^\top\{0\}) = \widehat{\mathcal{W}}_1$$

and

$$(A\mathcal{W}_0)^\perp = \{0\}^\perp = \mathbb{R}^m = \widehat{\mathcal{V}}_0.$$

Now suppose that (4.1) holds for some $i \in \mathbb{N}$. Then

$$\begin{aligned} (E\mathcal{V}_{i+1})^\perp &= \left(EA^{-1}(E\mathcal{V}_i)\right)^\perp \\ &\overset{\text{Lem. 4.1(i)}}{=} E^{-\top}(A^{-1}(E\mathcal{V}_i))^\perp \\ &\overset{\text{Lem. 4.1(ii)}}{=} E^{-\top}\left(A^\top(E\mathcal{V}_i)^\perp\right) \\ &= E^{-\top}\left(A^\top\widehat{\mathcal{W}}_{i+1}\right) = \widehat{\mathcal{W}}_{i+2}, \end{aligned}$$

and analogously it follows $(A\mathcal{W}_{i+1})^\perp = \widehat{\mathcal{V}}_{i+1}$, hence we have inductively shown (4.1). ☐

4.3. Singular chains. In this subsection we introduce the notion of *singular chains* for matrix pencils. This notion is inspired by the theory of linear relations, see [16], where they are a vital tool for analyzing the structure of linear relations. We use them here to determine the structure of the intersection $\mathcal{V}^* \cap \mathcal{W}^*$ of the limits of the Wong sequences.

Definition 4.6 (Singular chain). *Let $sE - A \in \mathbb{K}^{m\times n}[s]$. For $k \in \mathbb{N}$ the tuple $(x_0, \ldots, x_k) \in (\mathbb{K}^n)^{k+1}$ is called* singular chain of the matrix pencil $sE - A$ *if, and only if,*

$$0 = Ax_0,\ Ex_0 = Ax_1,\ \ldots,\ Ex_{k-1} = Ax_k,\ Ex_k = 0$$

or, equivalently, the polynomial vector $x(s) = x_0 + x_1 s + \ldots + x_k s^k \in \mathbb{K}^n[s]$ *satisfies* $(sE - A)x(s) = 0$. $\diamond$

Note that with every singular chain $(x_0, x_1, \ldots, x_k)$ also the tuple $(0, \ldots, 0, x_0, \ldots, x_k, 0, \ldots, 0)$ is a singular chain of $sE - A$. Furthermore, with every singular chain, each scalar multiple is a singular chain and for two singular chains of the same length the sum of both is a singular chain. A singular chain $(x_0, \ldots, x_k)$ is called *linearly independent* if the vectors $x_0, \ldots, x_k$ are linearly independent.

LEMMA 4.7 (Linear independency of singular chains). *Let* $sE - A \in \mathbb{K}^{m \times n}[s]$*. For every non-trivial singular chain* $(x_0, x_1, \ldots, x_k)$*,* $k \in \mathbb{N}$*, of* $sE - A$ *there exists* $m \in \mathbb{N}$*,* $m \leq k$*, and a linearly independent singular chain* $(y_0, y_1, \ldots, y_m)$ *with* $\operatorname{span}\{x_0, x_1, \ldots, x_k\} = \operatorname{span}\{y_0, y_1, \ldots, y_m\}$. $\diamond$

Proof. This result is an extension of [16, Lem. 3.1], hence our proof resembles some ideas of the latter.

If $(x_0, x_1, \ldots, x_k)$ is already a linearly independent singular chain nothing is to show, therefore, assume existence of a minimal $m \in \{0, 1, \ldots, k-1\}$ such that $x_{m+1} = \sum_{i=0}^{m} \alpha_i x_i$ for some $\alpha_i \in \mathbb{K}$, $i = 0, \ldots, m$. Consider the chains

$$\begin{array}{r}
\alpha_0\,(0,\ 0,\ \ldots,\ 0,\ 0,\ x_0,\ x_1,\ \ldots,\ x_m,\ x_{m+1},\ \ldots,\ x_{k-1}, x_k) \\
\alpha_1\,(0,\ 0,\ \ldots,\ 0,\ x_0,\ x_1,\ \ldots,\ x_m,\ x_{m+1},\ \ldots,\ x_{k-1},\ x_k,\ 0) \\
\alpha_2\,(0,\ 0,\ \ldots,\ x_0,\ x_1,\ \ldots,\ x_m,\ x_{m+1},\ \ldots,\ x_{k-1},\ x_k,\ 0,\ 0) \\
\vdots \\
\alpha_{m-1}\,(0,\ x_0, x_1,\ \ldots,\ x_{m-2}, x_{m-1},\ x_m,\ x_{m+1},\ \ldots\ x_k,\ 0,\ \ldots,\ 0) \\
\alpha_m\,(x_0, x_1, \ldots, x_{m-2}, x_{m-1},\ x_m,\ x_{m+1},\ \ldots\ x_k,\ 0,\ \ldots,\ 0,\ 0)
\end{array}$$

and denote its sum by $(z_0, z_1, \ldots, z_{k+m})$. Note that by construction $z_m = \sum_{i=0}^{m} \alpha_i x_i = x_{m+1}$. Now consider the singular chain $(v_0, v_1, \ldots, v_{k+m+1}) := (x_0, x_1, \ldots, x_k, 0, \ldots, 0) - (0, z_0, z_1, \ldots, z_{m+k})$ which has the property that $v_{m+1} = x_{m+1} - z_m = 0$. In particular $(v_0, v_1, \ldots, v_m)$ and $(v_{m+2}, v_{m+3} \ldots, v_{k+m+1})$ are both singular chains. Furthermore, (we abbreviate $\alpha_i I$ with α_i)

$$\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_{k+m+1} \end{pmatrix} = \begin{bmatrix}
I & 0 & & & \cdots & & & & 0 \\
-\alpha_m & I & 0 & & & \cdots & & & 0 \\
-\alpha_{m-1} & -\alpha_m & I & 0 & & & \cdots & & 0 \\
& & & \ddots & & & & & \\
-\alpha_1 & -\alpha_2 & \cdots & -\alpha_m & I & & & & \\
-\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_m & I & & & \\
0 & -\alpha_0 & -\alpha_1 & -\alpha_3 & \cdots & -\alpha_m & I & & \\
& & \ddots & & & & & \ddots & \\
0 & \cdots & 0 & -\alpha_0 & -\alpha_1 & -\alpha_3 & \cdots & -\alpha_m & I
\end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

hence $\operatorname{span}\{v_0, v_1, \ldots, v_{k+m+1}\} = \operatorname{span}\{x_0, x_1, \ldots, x_k\} = \operatorname{span}\{v_0, v_1, \ldots, v_k\}$. In particular

$$\operatorname{span}\{v_{k+1}, v_{k+2}, \ldots, v_{k+m+1}\} \subseteq \operatorname{span}\{v_0, v_1, \ldots, v_k\},$$

hence, by applying Lemma 4.2, there exists $c \neq 0$ such that (note that $m < k$)

$$\operatorname{im}[v_0, v_1, \ldots, v_k] = \operatorname{im}([v_0, v_1, \ldots, v_k] + c\ [v_{k+1}, v_{k+2}, \ldots, v_{k+m+1}, 0, \ldots, 0]). \tag{4.2}$$

Therefore, the singular chain

$$(w_0, w_1, \ldots, w_{k-1}) := c\ (v_{m+2}, \ldots, v_k \mid v_{k+1}, v_{k+2}, \ldots, v_{k+m+1}) + (0, \ldots, 0 \mid v_0, v_1, \ldots, v_m)$$

has the property

$$\begin{aligned}
\operatorname{span}\{w_0, w_1, \ldots, w_{k-1}\} &= \operatorname{span}\{v_{m+2}, v_{m+3}, \ldots, v_k\} \\
&\quad + \operatorname{span}\{c v_{k+1} + v_0, c v_{k+2} + v_1, \ldots, c v_{k+m+1} + v_m\} \\
&\overset{v_{m+1}=0}{=} \operatorname{im}([v_0, v_1, \ldots, v_k] + c\ [v_{k+1}, v_{k+2}, \ldots, v_{k+m+1}, 0, \ldots, 0]) \\
&\overset{(4.2)}{=} \operatorname{im}[v_0, v_1, \ldots, v_k] \\
&= \operatorname{span}\{v_0, v_1, \ldots, v_k\} = \operatorname{span}\{x_0, x_1, \ldots, x_k\}.
\end{aligned}$$

Therefore, we have obtained a shorter singular chain which spans the same subspace as the original singular chain. Repeating this procedure until one obtains a linearly independent singular chain proves the claim. $\square$

COROLLARY 4.8 (Basis of the singular chain manifold). *Consider a matrix pencil* $sE - A \in \mathbb{K}^{m \times n}[s]$ *and let the* singular chain manifold *be given by*

$$\mathcal{K} := \left\{\ x \in \mathbb{R}^n\ \middle|\ \exists\, k, i \in \mathbb{N}\ \exists \text{ singular chain } (x_0, \ldots, x_{i-1}, x = x_i, x_{i+1}, \ldots, x_k) \in (\mathbb{K}^n)^{k+1}\ \right\},$$

i.e. $\mathcal{K}$ is the set of all vectors x appearing somewhere in some singular chain of $sE - A$. Then there exists a linearly independent *singular chain $(x_0, x_1, \ldots, x_k)$ of $sE - A$ such that*

$$\mathcal{K} = \operatorname{span}\{x_0, \ldots, x_k\}.$$

$\diamond$

Proof. First note that $\mathcal{K}$ is indeed a linear subspace of $\mathbb{K}^n$, since with every chain its scalar multiple is also a chain and the sum of two chains (extending the chains appropriately with zero vectors) is again a chain.
Let $y^0, y^1, \ldots, y^k$ be any basis of $\mathcal{K}$. By the definition of $\mathcal{K}$, for each $i = 0, 1, \ldots, k$ there exist chains $(y_0^i, y_1^i, \ldots, y_{k_i}^i)$ which contain y^i. Let $(v_0, v_1, \ldots, v_{\hat{k}})$ with $\hat{k} = k_0 + k_1 + \ldots + k_k$ be the chain which results by concatenating the chains $(y_0^i, y_1^i, \ldots, y_{k_i}^i)$. Clearly, $\operatorname{span}\{v_0, \ldots, v_{\hat{k}}\} = \mathcal{K}$, hence Lemma 4.7 yields the claim. $\square$

The following result can, in substance, be found in [2]. However, the proof therein is difficult to follow, involving quotient spaces and additional sequences of subspaces. Our presentation is much more straightforward and simpler.

Lemma 4.9 (Singular chain manifold and the Wong sequences). *Consider a matrix pencil $sE - A \in \mathbb{K}^{m\times n}[s]$ with the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences. Let the singular chain manifold $\mathcal{K}$ be given as in Corollary 4.8, then*

$$\mathcal{V}^* \cap \mathcal{W}^* = \mathcal{K}.$$

$\diamond$

Proof. Step 1: We show $\mathcal{K} \subseteq \mathcal{V}^* \cap \mathcal{W}^*$.
Let $(x_0, \ldots, x_k)$ be a singular chain. Clearly we have $x_0 \in A^{-1}(E\{0\}) = \ker A \subseteq \mathcal{V}^*$ and $x_k \in E^{-1}(A\{0\}) = \ker E \subseteq \mathcal{W}^*$, hence inductively we have, for $i = 0, 1, \ldots, k-1$ and $j = k, k-1, \ldots, 1$

$$x_{i+1} \in A^{-1}(E\{x_i\}) \subseteq A^{-1}(E\mathcal{V}^*) = \mathcal{V}^* \quad \text{and} \quad x_{j-1} \in E^{-1}(A\{x_j\}) \subseteq E^{-1}(A\mathcal{W}^*) = \mathcal{W}^*.$$

Therefore,

$$x_0, \ldots, x_k \in \mathcal{V}^* \cap \mathcal{W}^*.$$

Step 2: We show $\mathcal{V}^* \cap \mathcal{W}^* \subseteq \mathcal{K}$.
Let $x \in \mathcal{V}^* \cap \mathcal{W}^*$. Then, in particular, $x \in \mathcal{W}^*$ and with $l^* \in \mathbb{N}$ such that $\mathcal{W}_{l^*} = \mathcal{W}^*$ there exist $x_1 \in \mathcal{W}_{l^*-1}, x_2 \in \mathcal{W}_{l^*-2}, \ldots, x_{l^*} \in \mathcal{W}_0 = \{0\}$, such that, for $x_0 := x$,

$$Ex_0 = Ax_1,\ Ex_1 = Ax_2,\ \ldots,\ Ex_{l^*-1} = Ax_{l^*}, Ex_{l^*} = 0.$$

Furthermore, since $E(\mathcal{V}^* \cap \mathcal{W}^*) = A(\mathcal{V}^* \cap \mathcal{W}^*)$ there exist $x_{-1}, x_{-2}, \ldots, x_{-(l^*+1)} \in \mathcal{V}^* \cap \mathcal{W}^*$ such that

$$Ax_0 = Ex_{-1},\ Ax_{-1} = Ex_{-2},\ \ldots,\ Ax_{-(l^*-1)} = Ex_{-l^*},\ Ax_{-l^*} = Ex_{-(l^*+1)}.$$

Let $\Delta x_{-(l^*+1)} := -x_{-(l^*+1)} \in \mathcal{V}^* \cap \mathcal{W}^* \subseteq \mathcal{W}^*$ then (with the same argument as above) there exist Δx_{-l^*}, $\Delta x_{-(l^*-1)}, \ldots, \Delta x_{-1} \in \mathcal{W}^*$ such that

$$E\Delta x_{-(l^*+1)} = A\Delta x_{-l^*},\ E\Delta x_{-l^*} = A\Delta x_{-(l^*-1)} \ldots,\ E\Delta x_{-2} = A\Delta x_{-1},\ E\Delta x_{-1} = 0,$$

and thus, defining $\hat{x}_{-i} = x_{-i} + \Delta x_{-i}, i = 1, \ldots, l^*+1$, in particular $\hat{x}_{-(l^*+1)} = 0$, we obtain

$$0 = E\hat{x}_{-(l^*+1)} = A\hat{x}_{-l^*},\ E\hat{x}_{-l^*} = A\hat{x}_{-(l^*-1)}\ \ldots,\ E\hat{x}_{-2} = A\hat{x}_{-1},\ E\hat{x}_{-1} = Ex_{-1} = Ax_0.$$

This shows that $(\hat{x}_{-l^*}, \hat{x}_{-(l^*-1)}, \ldots, \hat{x}_{-1}, x_0, x_1, \ldots, x_{l^*})$ is a singular chain and $x = x_0 \in \mathcal{K}$. $\square$

The last result in this section relates singular chains with the spectrum $\operatorname{spec}(sE - A)$ of the matrix pencil $sE - A$.

Lemma 4.10 (Infinite spectrum implies singular chains). *Let $sE - A \in \mathbb{K}^{m\times n}[s]$. If $\operatorname{spec}(sE - A) = \mathbb{C} \cup \{\infty\}$, then there exists a non-trivial singular chain of $sE - A$.*

Proof. It suffices to observe that Definition 4.6 coincides (modulo a reversed indexing) with the notion of singular chains in [16] applied to the linear relation $E^{-1}A := \{\ (x, y) \in \mathbb{K}^n \times \mathbb{K}^n \mid Ax = Ey\ \}$. Then the claim follows for $\mathbb{K} = \mathbb{C}$ from [16, Thm. 4.4]. The main idea of the proof there is to choose any $m+1$ different eigenvalues and corresponding eigenvectors. This is also possible for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{Q}$, hence the proof in [16] is also valid for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{Q}$. $\square$

4.4. Polynomial matrices.

LEMMA 4.11 (Unimodular extension). *A matrix $P(s) \in \mathbb{K}^{m\times n}[s]$ can be extended to a square unimodular matrix if, and only if,* $\operatorname{rank}_{\mathbb{C}} P(\lambda) = \min\{m,n\}$ *for all* $\lambda \in \mathbb{C}$. ◇

Proof. Necessity is clear, hence it remains to show that under the full rank assumption a unimodular extension is possible. Note that $\mathbb{K}[s]$ is a principal ideal domain, hence we can consider the Smith normal form [18] of $P(s)$ given by

$$P(s) = U(s)\begin{bmatrix}\Sigma_r(s) & 0\\ 0 & 0\end{bmatrix}V(s),$$

where $U(s), V(s)$ are unimodular matrices and $\Sigma(s) = \operatorname{diag}(\sigma_1(s),\ldots,\sigma_r(s))$, $r \in \mathbb{N}$, with non-zero diagonal entries. Note that $\operatorname{rank} P(\lambda) = \operatorname{rank}\Sigma(\lambda)$ for all $\lambda \in \mathbb{C}$, hence the full rank condition implies $r = \min\{m,n\}$ and $\sigma_1(s),\ldots,\sigma_r(s)$ are constant (non-zero) polynomials. For $m = n$ this already shows the claim. For $m > n$, i.e. $P(s) = U(s)\left[\begin{smallmatrix}\Sigma_n(s)\\ 0\end{smallmatrix}\right]V(s)$, the sought unimodular extension is given by

$$[P(s), Q(s)] = U(s)\left[\begin{array}{c|c}\Sigma_n(s) & 0\\ 0 & I\end{array}\right]\begin{bmatrix}V(s) & 0\\ 0 & I\end{bmatrix}$$

and, for $m < n$,

$$\begin{bmatrix}P(s)\\ Q(s)\end{bmatrix} = \begin{bmatrix}U(s) & 0\\ 0 & I\end{bmatrix}\left[\begin{array}{cc}\Sigma_m(s) & 0\\ \hline 0 & I\end{array}\right]V(s).$$

□

Proof of Lemma 3.1. Let $Q(s)$ be any unimodular extension of $sE - A$ according to Lemma 4.11. If $m < n$, choose $[M(s), K(s)] = \begin{bmatrix}sE-A\\ Q(s)\end{bmatrix}^{-1}$ and if $m > n$, let $\left[\begin{smallmatrix}M(s)\\ K(s)\end{smallmatrix}\right] := [sE - A, Q(s)]^{-1}$. □

4.5. Solvability of linear matrix equations. In generalization of the method presented in [6, Sec. 6] we reduce the problem of solvability of (2.4) to the problem of solving a generalized Sylvester equation

$$AXB - CXD = E. \tag{4.3}$$

To this end the following lemma is crucial.

LEMMA 4.12. *Let $A, C \in \mathbb{K}^{m\times n}$, $B, D \in \mathbb{K}^{p\times q}$, $E, F \in \mathbb{K}^{m\times q}$ and consider the system of matrix equations with "unknowns" $Y \in \mathbb{K}^{n\times q}$ and $Z \in \mathbb{K}^{m\times p}$*

$$\begin{aligned}0 &= E + AY + ZD,\\ 0 &= F + CY + ZB.\end{aligned} \tag{4.4}$$

Suppose there exists $\lambda \in \mathbb{K}$ and $M_\lambda \in \mathbb{K}^{q\times p}$ such that $M_\lambda(B - \lambda D) = I$, in particular $p \geq q$. Then, for any solution $X \in \mathbb{K}^{n\times p}$ of the matrix equation

$$AXB - CXD = -E - (\lambda E - F)M_\lambda D,$$

the matrices

$$\begin{aligned}Y &= X(B - \lambda D)\\ Z &= -(C - \lambda A)X - (F - \lambda E)M_\lambda\end{aligned}$$

solve (4.4). ◇

Proof. We calculate

$$\begin{aligned}E + AY + ZD &= E + AX(B-\lambda D) - (C-\lambda A)XD - (F-\lambda E)M_\lambda D\\ &= E - AX\lambda D + \lambda AXD - (F-\lambda E)M_\lambda D - E - (\lambda E - F)M_\lambda D\\ &= 0,\end{aligned}$$

$$\begin{aligned}F + CY + ZB &= F + CX(B-\lambda D) - (C-\lambda A)XB - (F-\lambda E)M_\lambda B\\ &= F + CXB - CXB - (F-\lambda E)M_\lambda B - \lambda\left(E + (\lambda E - F)M_\lambda D\right)\\ &= (F-\lambda E) - (F-\lambda E)M_\lambda B - \lambda(\lambda E - F)M_\lambda D\\ &= (F-\lambda E)\big(I_q - M_\lambda(B-\lambda D)\big)\\ &= 0.\end{aligned}$$

□

It is well known [9] that the generalized Sylvester equation (4.3) is solvable if, for all $\lambda \in \mathbb{C}$, $\lambda C - A$ is right invertible, $\lambda B - D$ is left invertible and $\operatorname{spec}(sC - A) \cap \operatorname{spec}(sB - D) = \emptyset$. However, the proof in [9] uses the Kronecker canonical form which we want to obtain as a corollary of our analysis. In our situation we actually do not need to solve an arbitrary generalized Sylvester equation (4.3), because the matrices in (2.4) already have special properties. The following lemma takes this into account.

LEMMA 4.13 (Solvability of the generalized Sylvester equation). *Let $A, C \in \mathbb{K}^{m\times n}$, $m \le n$, $B, D \in \mathbb{K}^{p\times q}$, $p \ge q$, $E \in \mathbb{K}^{m\times q}$ and consider the generalized Sylvester equation (4.3). Assume there exists $\lambda \in \mathbb{K}$, $M_\lambda \in \mathbb{K}^{n\times m}$ and $N_\lambda \in \mathbb{K}^{q\times p}$ such that $(\lambda C - A)M_\lambda = I$ and $N_\lambda(\lambda B - D) = I$. If $\operatorname{spec}(sC - A) \cap \operatorname{spec}(sB - D) = \emptyset$, then (4.3) has a solution $X \in \mathbb{K}^{n\times p}$.* ◇

Proof. Clearly, (4.3) is equivalent to

$$(\lambda C - A)XB - CX(\lambda B - D) = -E, \tag{4.5}$$

and for $Y \in \mathbb{K}^{m\times q}$ let $X = M_\lambda Y N_\lambda$. Then (4.5) reduces to the Sylvester equation

$$YN_\lambda B - CM_\lambda Y = -E \tag{4.6}$$

and each solution Y of (4.6) yields a solution $X = M_\lambda Y N_\lambda$ of (4.3). It is well known (see e.g. [11]), that a disjoint set of eigenvalues of $N_\lambda B$ and CM_λ yields existence of a solution Y of (4.6). Seeking a contradiction, assume existence of $\mu \in \mathbb{C}$ such that $\operatorname{rank}(\mu I - N_\lambda B) < q$ and $\operatorname{rank}(\mu I - CM_\lambda) < m$. Using the defining properties of M_λ and N_λ it follows that

$$\begin{aligned}
\mu I - N_\lambda B &= N_\lambda\big(\mu(\lambda B - D) - B)\big) = N_\lambda\big((\mu\lambda - 1)B - \mu D\big),\\
\mu I - CM_\lambda &= \big(\mu(\lambda C - A) - C)\big)M_\lambda = \big((\mu\lambda - 1)C - \mu A\big)M_\lambda.
\end{aligned}$$

Since $\operatorname{rank} N_\lambda = q$ and $\operatorname{rank} M_\lambda = m$ the assumption that $\operatorname{rank}(\mu I - N_\lambda B) < q$ and $\operatorname{rank}(\mu I - CM_\lambda) < m$ implies

$$\operatorname{rank}\big((\mu\lambda - 1)B - \mu D\big) < q \quad \text{and} \quad \operatorname{rank}\big((\mu\lambda - 1)C - \mu A\big) < m.$$

This implies either, if $\mu \neq 0$, that $\lambda - 1/\mu$ is a common eigenvalue of $sB - D$ and $sC - A$ or, if $\mu = 0$, infinity is a common eigenvalue of $sB - D$ and $sC - A$. In both cases this contradicts the assumption. □

4.6. Kronecker canonical form for full rank pencils. In order to derive the Kronecker canonical form as a corollary of the quasi-Kronecker form we need the following lemma, which shows how to obtain the Kronecker canonical form for the special case of full rank pencils.

LEMMA 4.14 (Kronecker form of full rank rectangular pencil). *Let $sE - A \in \mathbb{K}^{m\times n}[s]$ such that $m < n$ and let $l := n - m$. Then $\operatorname{rank}_\mathbb{C}(\lambda E - A) = m$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$ if, and only if, there exist numbers $\varepsilon_1, \ldots, \varepsilon_l \in \mathbb{N}$ and matrices $S \in \mathbf{Gl}_m(\mathbb{K})$, $T \in \mathbf{Gl}_n(\mathbb{K})$ such that*

$$S(sE - A)T = \operatorname{diag}(\mathcal{P}_{\varepsilon_1}(s), \ldots, \mathcal{P}_{\varepsilon_l}(s)),$$

where $\mathcal{P}_\varepsilon(s)$, $\varepsilon \in \mathbb{N}$, is as in Corollary 2.7. ◇

Proof. Sufficiency is clear, hence it remains to show necessity.

If $m = 0$ and $n > 0$, then nothing is to show since $sE - A$ is already in the "diagonal form" with $k_1 = k_2 = \ldots = k_l = 0$. Hence assume $m > 0$ in the following. The main idea is to reduce the problem to a smaller pencil $sE' - A' \in \mathbb{K}^{m'\times n'}[s]$ with $\operatorname{rank} \lambda E' - A' = m' < n' < n$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. Then we can inductively use the transformation to the desired block diagonal structure for the smaller pencil to obtain the block diagonal structure for the original pencil.

By assumption E does not have full column rank, hence there exists a column operation $T_1 \in \mathbf{Gl}_n(\mathbb{K})$ such that

$$ET_1 = \begin{bmatrix} 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$$

There are two cases now: Either the first column of AT_1 is zero or not. We consider the two cases separately.

Case 1: The first column of AT_1 is zero.

Let $ET_1 =: [0, E']$ and $AT_1 =: [0, A']$. Then, clearly, $\operatorname{rank}_{\mathbb{C}}(\lambda E - A) = \operatorname{rank}_{\mathbb{C}}(\lambda E' - A') = m' := m$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. Furthermore, with $n' := n - 1$, it follows that $n' \geq m'$. Seeking a contradiction, assume $n' = m'$. Then the full rank matrix E' is square and hence invertible. Let $\lambda \in \mathbb{C}$ be any eigenvalue of the matrix ${E'}^{-1}A'$, thus $0 = \det(\lambda I - {E'}^{-1}A') = \det(E')^{-1}\det(\lambda E' - A')$, hence $\operatorname{rank}(\lambda E' - A') < m'$, a contradiction. Altogether, this shows that $sE' - A' \in \mathbb{K}^{m' \times n'}[s]$ is a smaller pencil which satisfies the assumption of the lemma, hence we can inductively use the result of the lemma for $sE' - A'$ with transformation matrices S' and T'. Let $S := S'$ and $T := T_1 \left[\begin{smallmatrix} 1 & 0 \\ 0 & T' \end{smallmatrix}\right]$, then $S(sE - A)T$ has the desired block diagonal structure which coincides with the block structure of $sE' - A'$ apart from one additional $\mathcal{P}_0$ block.

Case 2: The first column of AT_1 is not zero.

Since E has full row rank, the first row of ET_1 cannot be the zero row, hence there exists a second column operation $T_2 \in \mathbf{Gl}_n(n)$ which does not change the fist column such that

$$(ET_1)T_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

The first column of AT_1 and AT_1T_2 are the same and not the zero column, hence there exists a row operation $S_1 \in \mathbf{Gl}_m(\mathbb{K})$ such that

$$S_1(AT_1T_2) = \begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Now let $T_3 \in \mathbf{Gl}_n(\mathbb{K})$ be a column operation which adds multiples of the first column to the remaining columns such that

$$(S_1AT_1T_2)T_3 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Since the first column of $S_1ET_1T_2$ is zero, the column operation T_3 has no effect on the matrix $S_1ET_1T_2$. Let

$$S_1ET_1T_2T_3 =: \begin{bmatrix} 0 & 1 \quad 0 \quad \cdots \quad 0 \\ 0 & \\ \vdots & E' \\ 0 & \end{bmatrix} \quad \text{and} \quad S_1AT_1T_2T_3 =: \begin{bmatrix} 1 & 0 \quad 0 \quad \cdots \quad 0 \\ 0 & \\ \vdots & A' \\ 0 & \end{bmatrix},$$

with $sE' - A' \in \mathbb{K}^{m' \times n'}[s]$ and $m' := m - 1$, $n' := n - 1$, in particular $m' < n'$. Seeking a contradiction, assume $\operatorname{rank}_{\mathbb{C}} \lambda E' - A' < m'$ for some $\lambda \in \mathbb{C} \cup \{\infty\}$. If $\lambda = \infty$ then this implies that E' does not have full row rank which would also imply that E does not have full row rank, which is not the case. Hence we may choose a vector $v' \in \mathbb{C}^{m'}$ such that $v'(\lambda E' - A') = 0$. Let $v := [0, v']S_1$. Then a simple calculation reveals $v(\lambda E - A) = [0, v'(\lambda E' - A')](T_1T_2T_3)^{-1} = 0$, which contradicts full rank of $\lambda E - A$. As in the first case we can now inductively use the result of the lemma for the smaller matrix pencil $sE' - A'$ to obtain transformations S' and T' which put $sE' - A'$ in the desired block diagonal form. With $S := \left[\begin{smallmatrix} 1 & 0 \\ 0 & S' \end{smallmatrix}\right] S_1$ and $T := T_1T_2T_3 \left[\begin{smallmatrix} 1 & 0 \\ 0 & T' \end{smallmatrix}\right]$ we obtain the same block diagonal structure for $sE - A$ as for $sE' - A'$ apart from the first block which is $\mathcal{P}_{\varepsilon_1+1}$ instead of $\mathcal{P}_{\varepsilon_1}$. ☐

Corollary 4.15. *Let $sE - A \in \mathbb{K}^{m \times n}[s]$ be such that $m > n$ and let $l := m - n$. Then $\operatorname{rank}_{\mathbb{C}}(\lambda E - A) = n$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$ if, and only if, there exist numbers $\eta_1, \ldots, \eta_l \in \mathbb{N}$ and matrices $S \in \mathbf{Gl}_m(\mathbb{K})$, $T \in \mathbf{Gl}_n(\mathbb{K})$ such that*

$$S(sE - A)T = \operatorname{diag}(\mathcal{Q}_{\eta_1}(s), \ldots, \mathcal{Q}_{\eta_l}(s)),$$

where $\mathcal{Q}_\eta(s)$, $\eta \in \mathbb{N}$, is as in Corollary 2.7. ◇

4.7. Solutions of DAEs. In order to prove Theorem 3.2 we need the following lemmas, which characterize the solutions of DAEs in the case of full rank pencils. As in Section 3 we restrict ourselves to the case $\mathbb{K} = \mathbb{R}$.

LEMMA 4.16 (Full row rank pencils). *Let $sE - A \in \mathbb{R}^{m \times n}[s]$ such that $m < n$ and $\operatorname{rank}_{\mathbb{C}}(\lambda E - A) = m$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. According to Lemma 3.1 choose $M(s) \in \mathbb{R}^{n \times m}[s]$ and $K(s) \in \mathbb{R}^{n \times (n-m)}[s]$ such that $(sE - A)[M(s), K(s)] = [I, 0]$ and $[M(s), K(s)]$ is unimodular. Consider the DAE $E\dot{x} = Ax + f$ and the associated solution space $\mathcal{S} = \mathcal{C}^\infty$ or $\mathcal{S} = \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$. Then, for all inhomogeneities $f \in \mathcal{S}^m$, $x \in \mathcal{S}^n$ is a solution if, and only if, there exists $u \in \mathcal{S}^{n-m}$ such that*

$$x = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u).$$

Furthermore, all initial values problems have a solution, i.e. for all $x^0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and all $f \in \mathcal{S}^m$ there exists a solution $x \in \mathcal{S}^n$ such that

$$x(t_0-) = x^0.$$

$\diamond$

Proof. Step 1: We show that $x = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)$ solves $E\dot{x} = Ax + f$ for any $u \in \mathcal{S}^{n-m}$.
This is clear since

$$(E\tfrac{\mathrm{d}}{\mathrm{d}t} - A)\left(M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)\right) = f + 0 = f.$$

Step 2: We show that any solution x of the DAE can be represented as above.
To this end let $u := [0, I][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x \in \mathcal{S}^{n-m}$, which is well-defined due to the unimodularity of $[M(s), K(s)]$. Then

$$f = (E\tfrac{\mathrm{d}}{\mathrm{d}t} - A)x = (E\tfrac{\mathrm{d}}{\mathrm{d}t} - A)[M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]\begin{bmatrix}[I,0][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x \\ [0,I][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x\end{bmatrix} = [I,0][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x,$$

and therefore it follows that

$$M(\tfrac{\mathrm{d}}{\mathrm{d}t})f + K(\tfrac{\mathrm{d}}{\mathrm{d}t})u = [M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]\begin{bmatrix}[I,0][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x \\ [0,I][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x\end{bmatrix} = x.$$

Step 3: We show that every initial value is possible.
Write $K(s) = K_0 + K_1 s + \ldots + K_k s^k$, $k \in \mathbb{N}$, and let $\mathcal{K}$ be the singular chain manifold of $sE - A$ as in Corollary 4.8.
Step 3a: We show $\operatorname{im}[K_0, K_1, \ldots, K_k] = \mathcal{K} = \mathbb{R}^n$.
Remark 2.4 and Lemma 4.9 yields $\mathbb{R}^n = \mathcal{V}^* \cap \mathcal{W}^* = \mathcal{K}$. From $(sE - A)K(s) = 0$ it follows that

$$0 = AK_0, EK_0 = AK_1, \ldots, EK_{k-1} = AK_k, EK_k = 0,$$

hence the i-th column vectors of $K_0, K_1, \ldots, K_k$, $i = 1, \ldots, n-m$, form a singular chain. This shows $\operatorname{im}[K_0, K_1, \ldots, K_k] \subseteq \mathcal{K}$.
For showing the converse inclusion, we first prove $\operatorname{im} K_0 = \ker A$. From $AK_0 = (\lambda E - A)K(\lambda)\big|_{\lambda=0} = 0$ it follows that $\operatorname{im} K_0 \subseteq \ker A$. By unimodularity of $[M(s), K(s)]$ it follows that $K(0) = K_0$ must have full rank, i.e. $\dim \operatorname{im} K_0 = n - m$. Full rank of $(sE - A)$ for all $s \in \mathbb{C}$ also implies full rank of A, hence $\dim \ker A = n - m$ and $\operatorname{im} K_0 = \ker A$ is shown.
Let $(x_0, x_1, \ldots, x_l)$, $l \in \mathbb{N}$, be a singular chain. Then $Ax_0 = 0$, i.e. $x_0 \in \ker A = \operatorname{im} K_0$. Proceeding inductively, assume $x_0, x_1, \ldots, x_i \in \operatorname{im}[K_0, K_1, \ldots, K_i]$, for some $i \in \mathbb{N}$ with $0 \leq i < l$. For notational convenience set $K_j = 0$ for all $j > k$. From $Ax_{i+1} = Ex_i \in \operatorname{im}[EK_0, EK_1, \ldots, EK_i] = \operatorname{im}[AK_1, AK_2, \ldots, AK_{i+1}]$ it follows that $x_{i+1} \in \ker A + \operatorname{im}[K_1, K_2, \ldots, K_{i+1}] = \operatorname{im}[K_0, K_1, \ldots, K_{i+1}]$. This shows that each singular chain is contained in $\operatorname{im}[K_0, K_1, \ldots, K_k]$.
Step 3b: We show existence of $u \in \mathcal{S}^{n-m}$ such that $x(t_0-) = x^0$.
By Step 3a there exist $u_0, u_1, \ldots, u_k \in \mathbb{R}^{n-m}$ such that

$$K_0 u_0 + K_1 u_1 + \ldots + K_k u_k = x^0 - M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f)(t_0-). \tag{4.7}$$

Let

$$u(t) := u_0 + (t - t_0)u_1 + \frac{(t-t_0)^2}{2}u_2 + \ldots + \frac{(t-t_0)^k}{k!}u_k, \quad t \in \mathbb{R}.$$

Then we have that $u \in \mathcal{S}$ and

$$K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)(t_0-) = K_0 u_0 + K_1 u_1 + \ldots + K_k u_k \overset{(4.7)}{=} x^0 - M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f)(t_0-), \tag{4.8}$$

which implies that the solution $x = M(\frac{\mathrm{d}}{\mathrm{d}t})(f) + K(\frac{\mathrm{d}}{\mathrm{d}t})(u)$ satisfies

$$x(t_0-) = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f)(t_0-) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)(t_0-) \overset{(4.8)}{=} x^0.$$

□

Remark 4.17. *A careful analysis of the proof of Lemma 4.16 reveals that for the solution formula the full rank of $\lambda E - A$ for $\lambda = \infty$ is not necessary. The latter is only necessary to show that all initial value problems have a solution.*

Lemma 4.18 (Full column rank pencils). *Let $sE - A \in \mathbb{R}^{m\times n}[s]$ such that $m > n$ and $\operatorname{rank}_{\mathbb{C}}(\lambda E - A) = n$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. According to Lemma 3.1 choose $M(s) \in \mathbb{R}^{n\times m}[s]$ and $K(s) \in \mathbb{R}^{(m-n)\times m}[s]$ such that $\left[\begin{smallmatrix} M(s) \\ R(s) \end{smallmatrix}\right](sE - A) = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} M(s) \\ R(s) \end{smallmatrix}\right]$ is unimodular. Then, for $f \in \mathcal{S}^m$, $x \in \mathcal{S}^n$ is a solution of $E\dot{x} = Ax + f$ if, and only if,*

$$x = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) \quad \wedge \quad K(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) = 0.$$

Furthermore, every component or linear combination of f is restricted in some way, more precisely $K(s)F$ has no zero column for any invertible $F \in \mathbb{R}^{m\times m}$.

Proof. The characterization of the solution follows from the equivalence

$$(E\tfrac{\mathrm{d}}{\mathrm{d}t} - A)x = f \quad \Longleftrightarrow \quad \underbrace{\begin{bmatrix} M(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}(E\tfrac{\mathrm{d}}{\mathrm{d}t} - A)}_{=\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]}x = \begin{bmatrix} M(\frac{\mathrm{d}}{\mathrm{d}t})f \\ K(\frac{\mathrm{d}}{\mathrm{d}t})f \end{bmatrix}.$$

To show that $K(s)F$ does not have any zero column, write $K(s) = K_0 + K_1 s + \ldots + K_k s^k$. Since $(sE^\top - A^\top)K(s)^\top = 0$ it follows with the same arguments as in Step 3a of Lemma 4.16 that $\operatorname{im}[K_0^\top, K_1^\top, \ldots, K_k^\top] = \mathbb{R}^m$. Hence, $\ker[K_0^\top, K_1^\top, \ldots, K_k^\top]^\top = \{0\}$ which shows that the only $v \in \mathbb{R}^m$ with $K_i v = 0$ for all $i = 1, \ldots, k$ is $v = 0$. This shows that $K(s)F$ does not have a zero column for any invertible $F \in \mathbb{R}^{m\times m}$. □

Remark 4.19. *Analogously, as pointed out in Remark 4.17, the condition that $\lambda E - A$ must have full rank for $\lambda = \infty$ is not needed to characterize the solution. It is only needed to show that the inhomogeneity is "completely" restricted.* ◇

5. Proofs of the main results.

Proof of Theorem 2.3: the quasi-Kronecker triangular form. We are now ready to proof our main result about the quasi-Kronecker triangular form. We proceed in several steps.

Step 1: We show the block-triangular form (2.3).
By the choice of P_1, R_1, Q_1 and P_2, R_2, Q_2 it follows immediately that T_{trian} and S_{trian} are invertible. Note that (2.3) is equivalent to the solvability (for given E, A and $P_1, R_1, Q_1, P_2, R_2, Q_2$) of

$$\begin{aligned} EP_1 &= P_2 E_P, & AP_1 &= P_2 A_P, \\ ER_1 &= P_2 E_{PR} + R_2 E_R, & AR_1 &= P_2 A_{PR} + R_2 A_R, \\ EQ_1 &= P_2 E_{PQ} + R_2 E_{RQ} + Q_2 E_Q, & AQ_1 &= P_2 A_{PQ} + R_2 A_{RQ} + Q_2 A_Q. \end{aligned}$$

The solvability of the latter is implied by the following subspace inclusions

$$\begin{aligned} E(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*, & A(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*, \\ E(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*, & A(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*, \\ E\mathbb{K}^n &\subseteq \mathbb{K}^m, & A\mathbb{K}^n &\subseteq \mathbb{K}^m, \end{aligned}$$

which clearly hold due to (2.2).

Step 2: We show (i).
Step 2a: Full row rank of E_P and A_P.
From Lemma 4.4 it follows that

$$\operatorname{im} P_2 E_P = \operatorname{im} EP_1 = \operatorname{im} P_2 \quad \text{and} \quad \operatorname{im} P_2 A_P = \operatorname{im} AP_1 = \operatorname{im} P_2$$

hence, invoking the full column rank of P_2, $\operatorname{im} E_P = \mathbb{K}^{m_P} = \operatorname{im} A_P$, which implies full row rank of E_P and A_P. In particular this shows full row rank of $\lambda E_P - A_P$ for $\lambda = 0$ and $\lambda = \infty$.

Step 2b: Full row rank of $\lambda E_P - A_P$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.
Seeking a contradiction, assume existence of $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{rank}_{\mathbb{C}}(\lambda E_P - A_P) < m_P$. Then there exists $v \in \mathbb{C}^{m_P}$ such that $v^\top(\lambda E_P - A_P) = 0$. Full column rank of $P_2 \in \mathbb{K}^{m \times m_P}$ implies existence of $w \in \mathbb{C}^m$ such that $w^\top P_2 = v^\top$, hence

$$0 = v^\top(\lambda E_P - A_P) = w^\top(\lambda P_2 E_P - P_2 A_P) = w^\top(\lambda E - A)P_1.$$

According to Lemma 4.9 there exists a linear independent singular chain $(x_0, x_1, \ldots, x_k)$ such that

$$\operatorname{span}\{x_0, x_1, \ldots, x_k\} = \operatorname{im} P_1 = \mathcal{V}^* \cap \mathcal{W}^*.$$

Hence

$$\forall i \in \{0, 1, \ldots, k\}: \quad w^\top(\lambda E - A)x_i = 0.$$

Since $Ex_k = 0$ it follows that $w^\top A x_k = 0$ and inductively it follows

$$0 = w^\top(\lambda E x_{i-1} - A x_{i-1}) = w^\top(\lambda A x_i - A x_{i-1}) = -w^\top A x_{i-1}$$

and, therefore,

$$0 = w^\top A P_1 = w^\top P_2 A_P = v^\top A_P.$$

This shows that $A_P \in \mathbb{K}^{m_P \times n_P}$ does not have full row rank over $\mathbb{C}$ which implies also a row rank defect over $\mathbb{K}$. This is the sought contradiction because the full row rank of A_P was already shown in Step 2a.

Step 3: We show (ii).
For notational convenience let $\mathcal{L}^* := \mathcal{V}^* \cap \mathcal{W}^*$.
Step 3a: We show that $m_R = n_R$.
Invoking

$$\ker E \cap \mathcal{L}^* = \ker E \cap \mathcal{V}^*, \quad \ker A \cap \mathcal{L}^* = \ker A \cap \mathcal{W}^*, \tag{5.1}$$

and Lemma 4.3 the claim follows from

$$\begin{aligned}
m_R = \operatorname{rank} R_2 &= \dim(E\mathcal{V}^* + A\mathcal{W}^*) - \dim(E\mathcal{V}^* \cap A\mathcal{W}^*) \\
&= \dim E\mathcal{V}^* + \dim A\mathcal{W}^* - 2\dim(E\mathcal{V}^* \cap A\mathcal{W}^*) \\
&\overset{\text{Lem. 4.4}}{=} \dim \mathcal{V}^* - \dim(\ker E \cap \mathcal{V}^*) + \dim \mathcal{W}^* - \dim(\ker A \cap \mathcal{W}^*) - \dim E\mathcal{L}^* - \dim A\mathcal{L}^* \\
&= \dim \mathcal{V}^* - \dim(\ker E \cap \mathcal{V}^*) + \dim \mathcal{W}^* - \dim(\ker A \cap \mathcal{W}^*) - \dim \mathcal{L}^* \\
&\quad + \dim(\ker E \cap \mathcal{L}^*) - \dim \mathcal{L}^* + \dim(\ker A \cap \mathcal{L}^*) \\
&\overset{(5.1)}{=} \dim \mathcal{V}^* + \dim \mathcal{W}^* - 2\dim \mathcal{L}^* \\
&= \dim(\mathcal{V}^* + \mathcal{W}^*) - \dim(\mathcal{V}^* \cap \mathcal{W}^*) \\
&= \operatorname{rank} R_1 = n_R.
\end{aligned}$$

Step 3b: We show that $\det(sE_R - A_R) \not\equiv 0$.
Seeking a contradiction, assume $\det(sE_R - A_R)$ is the zero polynomial. Then $\lambda E_R - A_R$ has a column rank defect for all $\lambda \in \mathbb{C} \cup \{\infty\}$, hence

$$\operatorname{spec}(sE_R - A_R) = \mathbb{C} \cup \{\infty\}.$$

Now, Lemma 4.10 ensures existence of a nontrivial singular chain $(y_0, y_1, \ldots, y_k)$ of the matrix pencil $sE_R - A_R$. We show that there exists a singular chain $(x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_{\hat{k}})$ of $sE - A$ such that $x_i = [P_1, R_1]\left(\begin{smallmatrix} z_i \\ y_i \end{smallmatrix}\right)$ for $i = 0, \ldots, k$. To this end denote some right inverse of A_P (invoking full row rank of A_P as shown in Step 2a) with A_P^+ and let

$$z_0 = -A_P^+ A_{PR} y_0, \quad z_{i+1} = A_P^+(E_P z_i + E_{PR} y_i - A_{PR} y_{i+1}), \; i = 0, \ldots, k,$$

where $y_{k+1} = 0$. Then it follows that

$$Ax_i = A[P_1, R_1]\begin{pmatrix} z_i \\ y_i \end{pmatrix} = AT_{\text{trian}}\begin{pmatrix} z_i \\ y_i \\ 0 \end{pmatrix} = S_{\text{trian}}^{-1}\begin{bmatrix} A_P & A_{PR} & A_{PQ} \\ 0 & A_R & A_{RQ} \\ 0 & 0 & A_Q \end{bmatrix}\begin{pmatrix} z_i \\ y_i \\ 0 \end{pmatrix} = S_{\text{trian}}^{-1}\begin{pmatrix} A_P z_i + A_{PR} y_i \\ A_R y_i \\ 0 \end{pmatrix}$$

and, analogously,

$$Ex_i = S_{\text{trian}}^{-1} \begin{pmatrix} E_P z_i + E_{PR} y_i \\ E_R y_i \\ 0 \end{pmatrix},$$

hence $Ax_0 = 0$ and $Ex_i = Ax_{i+1}$ for $i = 0, \ldots, k$. Note that $x_{k+1} = P_1 z_{k+1}$, hence $x_{k+1} \in \mathcal{V}^* \cap \mathcal{W}^* \subseteq \mathcal{W}^*$ and identically as shown in the fist part of Step 2 of the proof of Lemma 4.9 there exist $x_{k+2}, \ldots, x_{\hat{k}}$, $\hat{k} > k$ such that $Ex_{k+1} = Ax_{k+2}, \ldots, Ex_{\hat{k}-1} = Ax_{\hat{k}}, Ex_{\hat{k}} = 0$ and, therefore, $(x_0, x_1, \ldots, x_{\hat{k}})$ is a singular chain of $sE - A$. Lemma 4.9 implies that $\{x_0, x_1, \ldots, x_{\hat{k}}\} \subseteq \operatorname{im} P_1$, hence $x_i = [P_1, R_1](z_i/y_i)$ implies $y_i = 0$ for all $i \in \{0, \ldots, k\}$, which contradicts non-triviality of $(y_0, \ldots, y_k)$.

Step 4: We show (iii).
We will consider the transposed matrix pencil $sE^\top - A^\top$ with corresponding Wong-sequences and will show that the block $(E_Q^\top, A_Q^\top)$ will play the role of the block (E_P, A_P). Therefore, denote the limits of the Wong-sequences of $sE^\top - A^\top$ by $\widehat{\mathcal{V}}^*$ and $\widehat{\mathcal{W}}^*$. Let

$$\widehat{Q}_1 := ([0,\ 0,\ I_{n_Q}][P_1, R_1, Q_1]^{-1})^\top \quad \text{and} \quad \widehat{Q}_2 := ([0,\ 0,\ I_{m_Q}][P_2, R_2, Q_2]^{-1})^\top,$$

then

$$\widehat{Q}_i^\top Q_i = I \quad \text{and} \quad \operatorname{im} \widehat{Q}_i = (\operatorname{im}[P_i, R_i])^\perp, \quad \text{for } i = 1, 2.$$

In fact, the latter follows from $n - n_Q = n_P + n_R$ and

$$\widehat{Q}_1^\top [P_1, R_1] = [0,\ 0,\ I_{n_Q}] \begin{bmatrix} I_{n_P} & 0 \\ 0 & I_{n_R} \\ 0 & 0 \end{bmatrix} = 0$$

for $i = 1$ and analogously for $i = 2$. We will show in the following that

$$\begin{aligned} E^\top \widehat{Q}_2 &= \widehat{Q}_1 E_Q^\top, & A^\top \widehat{Q}_2 &= \widehat{Q}_1 A_Q^\top, \\ \operatorname{im} \widehat{Q}_2 &= \widehat{\mathcal{V}}^* \cap \widehat{\mathcal{W}}^*, & \operatorname{im} \widehat{Q}_1 &= E^\top \widehat{\mathcal{V}}^* \cap A^\top \widehat{\mathcal{V}}^*, \end{aligned}$$

then the arguments from Step 2 can be applied to $sE_Q^\top - A_Q^\top$ and the claim is shown.
Step 4a: We show $E^\top \widehat{Q}_2 = \widehat{Q}_1 E_Q^\top$ and $A^\top \widehat{Q}_2 = \widehat{Q}_1 A_Q^\top$.
Using (2.3) we obtain

$$\widehat{Q}_2^\top E = \underbrace{\widehat{Q}_2^\top [P_2, R_2, Q_2]}_{=[0\ \ 0\ \ I]} \begin{bmatrix} E_P & E_{PR} & E_{PQ} \\ 0 & E_R & E_{RQ} \\ 0 & 0 & E_Q \end{bmatrix} [P_1, R_1, Q_1]^{-1} = [0\ 0\ E_Q][P_1, R_1, Q_1]^{-1} = E_Q \widehat{Q}_1^\top,$$

hence $E^\top \widehat{Q}_2 = \widehat{Q}_1 E_Q^\top$. Analog arguments show that $A^\top \widehat{Q}_2 = \widehat{Q}_1 A_Q^\top$.
Step 4b: We show $\operatorname{im} \widehat{Q}_2 = \widehat{\mathcal{V}}^* \cap \widehat{\mathcal{W}}^*$.
By construction and Lemma 4.5

$$\operatorname{im} \widehat{Q}_2 = (\operatorname{im}[P_2, R_2])^\perp = (E\mathcal{V}^* + A\mathcal{W}^*)^\perp = (E\mathcal{V}^*)^\perp \cap (A\mathcal{W}^*)^\perp = \widehat{\mathcal{V}}^* \cap \widehat{\mathcal{W}}^*.$$

Step 4c: We show $\operatorname{im} \widehat{Q}_1 = E^\top \widehat{\mathcal{V}}^* \cap A^\top \widehat{\mathcal{V}}^*$.
Lemma 4.5 applied to $(E^\top, A^\top)$ gives

$$(E^\top \widehat{\mathcal{V}}^*)^\perp = \mathcal{W}^* \quad \text{and} \quad (A^\top \widehat{\mathcal{W}}^*)^\perp = \mathcal{V}^*$$

or, equivalently,

$$E^\top \widehat{\mathcal{V}}^* = \mathcal{W}^{*\perp} \quad \text{and} \quad A^\top \widehat{\mathcal{W}}^* = \mathcal{V}^{*\perp}.$$

Hence

$$\operatorname{im} \widehat{Q}_1 = (\operatorname{im}[P_1, R_1])^\perp = (\mathcal{V}^* + \mathcal{W}^*)^\perp = \mathcal{V}^{*\perp} \cap \mathcal{W}^{*\perp} = A^\top \widehat{\mathcal{W}}^* \cap E^\top \widehat{\mathcal{V}}^*.$$

This concludes the proof of our first main result. □

Proof of Theorem 2.5: the quasi-Kronecker form. By the properties of the pencils $sE_P - A_P$, $sE_R - A_R$ and $sE_Q - A_Q$ there exist $\lambda \in \mathbb{K}$, N_λ^P, N_λ^R, M_λ^R and M_λ^Q such that $(\lambda E_P - A_P)N_\lambda^P = I$, $(\lambda E_R - A_R)N_\lambda^R = I$ $M_\lambda^R(\lambda E_R - A_R) = I$ and $M_\lambda^Q(\lambda E_Q - A_Q) = I$. Hence Lemma 4.12 shows that it suffices to consider solvability of the following generalized Sylvester equations

$$E_R X_1 A_Q - A_R X_1 E_Q = -E_{RQ} - (\lambda E_{RQ} - A_{RQ})M_\lambda^Q E_Q \tag{5.2a}$$
$$E_P X_2 A_R - A_P X_2 E_R = -E_{PR} - (\lambda E_{PR} - A_{PR})M_\lambda^R E_R \tag{5.2b}$$
$$E_P X_3 A_Q - A_P X_3 E_Q = -(E_{PQ} + E_{PR}F_1) - (\lambda(E_{PQ} + E_{PR}F_1) - (A_{PQ} + A_{PR}F_1))\, M_\lambda^Q E_Q, \tag{5.2c}$$

where F_1 is any solution of (2.4a), whose existence will follow from solvability of (5.2a). Furthermore, the properties of $sE_P - A_P$ and $sE_Q - A_Q$ imply that $\operatorname{spec}(sE_P^\top - A_P^\top) = \emptyset$ and $\operatorname{spec}(sE_Q - A_Q) = \emptyset$. Hence Lemma 4.13 is applicable to the equations (5.2) (where (5.2b) must be considered in the (conjugate) transposed form) and ensures existence of solutions.
Finally, a simple calculation shows that for any solution of (2.4) the statement of Theorem 2.5 holds. □

Proof of Theorem 3.2: characterization of the solutions of associated DAE. The claim is a simple consequence of Lemma 4.16 and Lemma 4.18 together with the well known solution properties of a DAE corresponding to a regular DAE in (quasi-) Weierstraß form (see e.g. [4]). □

6. Conclusions. We have studied singular matrix pencils $sE - A$ and the associated DAE $E\dot{x} = Ax + f$. With the help of the Wong sequences we were able to transform the matrix pencil into a quasi-Kronecker form. The quasi-Kronecker form decouples the original matrix pencil into three parts: the underdetermined part, the regular part and the overdetermined part. These blocks correspond to different solution behaviour: existence but non-uniqueness (underdetermined part), existence and uniqueness (regular part) and possible non-existence but uniqueness (overdetermined part).

REFERENCES

[1] J. Dwight Aplevich, *Implicit Linear Systems*, no. 152 in Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1991.

[2] Vinícius A. Armentano, *The pencil $(sE - A)$ and controllability-observability for generalized linear systems: a geometric approach*, SIAM J. Control Optim., 24 (1986), pp. 616–638.

[3] Theodorus G. J. Beelen and Paul M. Van Dooren, *An improved algorithm for the computation of Kronecker's canonical form of a singular pencil*, Lin. Alg. Appl., 105 (1988), pp. 9–65.

[4] Thomas Berger, Achim Ilchmann, and Stephan Trenn, *The quasi-Weierstraß form for regular matrix pencils*, Lin. Alg. Appl., (2010). in press, preprint available online, Institute for Mathematics, Ilmenau University of Technology, Preprint Number 09-21.

[5] Pierre Bernhard, *On singular implicit linear dynamical systems*, SIAM J. Control Optim., 20 (1982), pp. 612–633.

[6] King-wah Eric Chu, *The solution of the matrix equations $AXB - CXD = E$ and $(YA - DZ, YC - BZ) = (E, F)$*, Lin. Alg. Appl., 93 (1987), pp. 93–105.

[7] Jean Dieudonné, *Sur la réduction canonique des couples des matrices*, Bull. de la Societé Mathématique de France, 74 (1946), pp. 130–146.

[8] Felix R. Gantmacher, *The Theory of Matrices (Vol. I & II)*, Chelsea, New York, 1959.

[9] Vicente Hernández and Maite Gassó, *Explicit solution of the matrix equation $AXB - CXD = E$*, Lin. Alg. Appl., 121 (1989), pp. 333–344.

[10] Satoru Iwata and Mizuyo Takamatsu, *On the kronecker canonical form of mixed matrix pencils*, SIAM J. Matrix. Anal. & Appl., 32 (2011), pp. 44–71.

[11] Antony Jameson, *Solution of the equation $AX + XB = C$ by inversion of an $m \times m$ or $n \times n$ matrix*, SIAM J. Appl. Math., 16 (1968), pp. 1020–1023.

[12] Leopold Kronecker, *Algebraische Reduction der Schaaren bilinearer Formen*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, (1890), pp. 1225–1237.

[13] Margreet Kuijper, *First-Order Representations of Linear Systems*, Birkhäuser, Boston, 1994.

[14] Peter Kunkel and Volker Mehrmann, *Differential-Algebraic Equations. Analysis and Numerical Solution*, EMS Publishing House, Zürich, Switzerland, 2006.

[15] David H. Owens and Dragutin Lj. Debeljkovic, *Consistency and Liapunov stability of linear descriptor systems: A geometric analysis*, IMA J. Math. Control & Information, (1985), pp. 139–151.

[16] Adrian Sandovici, Henk De Snoo, and Henrik Winkler, *The structure of linear relations in Euclidean spaces*, Lin. Alg. Appl., 397 (2005), pp. 141–169.

[17] Laurent Schwartz, *Théorie des Distributions*, Hermann, Paris, 1957, 1959.

[18] Henry J. S. Smith, *On systems of linear indeterminate equations and congruences*, Philos. Trans. Roy. Soc. London, 151 (1861–1862), pp. 293–326. = Coll. Math. Papers, pp. 367–406.

[19] Stephan Trenn, *Distributional differential algebraic equations*, PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany, 2009.

[20] ———, *Regularity of distributional differential algebraic equations*, Math. Control Signals Syst., 21 (2009), pp. 229–264.

[21] Arjan J. van der Schaft and Johannes M. Schumacher, *The complementary-slackness class of hybrid systems*, Math. Control Signals Syst., 9 (1996), pp. 266–301. 10.1007/BF02551330.

[22] Paul M. Van Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Lin. Alg. Appl., 27 (1979), pp. 103–140.

[23] Andras Varga, *Computation of Kronecker-like forms of a system pencil: applications, algorithms and software*, in Proceedings of the IEEE International Symposium on Computer-Aided Control System Design, Dearborn, MI, USA, September 1996, pp. 77–82.
[24] Kai-Tak Wong, *The eigenvalue problem $\lambda Tx + Sx$*, J. Diff. Eqns., 16 (1974), pp. 270–280.
[25] Walter M. Wonham, *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, New York, 3rd ed., 1985.