

Regularity and Passivity for Jump Rules in Linear Switched Systems

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Abstract—A wide class of linear switched systems (LSS) can be represented by a sequence of modes each one described by a set of differential algebraic equations (DAEs). LSS can exhibit discontinuities in the state evolution, also called jumps, when the state at the end of a mode is not consistent with the DAEs of the successive mode. Then the problem of defining a proper state jump rule arises when an inconsistent initial condition is given. Regularity and passivity conditions provide two conceptually different jump maps respectively. In this paper, after proving some preliminary result on the jump analysis within the regularity framework, it is shown the equivalence of regularity-based and passivity-based jump rules. A switched capacitor electrical circuit is used to numerically confirm the theoretical result.

Index Terms—Switched systems, Energy-based jump rule, Regularity, Wong sequences, Inconsistent initial conditions.

I. INTRODUCTION

An interesting class of linear switched systems (LSS) consists of linear electrical circuits with ideal switches. The electrical circuit framework is interesting for LSS analysis because realistic assumptions can be made on the system model on one side, while on the other hand the LSS resulting from modeling electrical networks represent a quite general class of LSS.

An economic way of modeling linear electrical circuits with ideal switches is via a standard state-space system with constant matrices and virtual inputs and outputs, which are complementary and encode the mode of the position of the switches [1], i.e. open circuit or short circuit. In particular, in each switching configuration some of the “outputs” are set to zero leading to a differential algebraic equation (DAE). It is well known, that for a DAE not all initial values lead to a (classical) solution and the behavior of an electrical circuit in response to an inconsistent initial value has been studied by many researchers.

A good overview of the different jump-rules is given in [1] and under the assumption of a certain passivity concept it is shown that the following three jump rules are equivalent:

- 1) Minimization of the energy.
- 2) Conservation of the charge/flux.
- 3) The Laplace transform approach.

Another approach to model electrical circuits with switches is via switched DAEs [2] and under a certain regularity assumption one can prove that for any switching signal

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and any initial condition the switched DAE has a unique solution (but within a distributional solution framework). In particular, the regularity assumption implies unique jumps at the switching times.

The jump-rule based on regularity was not studied in [1] while passivity was not considered in [2]; but, of course, electrical circuits which are passive *and* regular play an important role and the question arises whether the passivity-based jump rule is equal to the regularity based jump rule. The answer is yes and is our main result (Theorem 6) we would like to present.

The paper is structured as follows. In Section II we present our system class of switched linear systems and how each mode can be rewritten as a DAE. The energy-based jump rule is introduced in Section III and afterwards the regularity-based jump rule is introduced. Our main result is given in Section V. We illustrate the theoretical results with a small example.

We use the following notation: For $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ we let $\text{col}(x, u) \in \mathbb{R}^{n+m}$ be the vector which results in stacking x over u . For a matrix $M \in \mathbb{R}^{p \times q}$ and a subspace $\mathcal{S} \subseteq \mathbb{R}^q$ the subspace $M\mathcal{S} \subseteq \mathbb{R}^p$ denotes the image space of \mathcal{S} under M , i.e. $M\mathcal{S} := \{ Ms \in \mathbb{R}^p \mid s \in \mathcal{S} \}$; for $\mathcal{S} \subseteq \mathbb{R}^p$ the subspace $M^{-1}(\mathcal{S}) \subseteq \mathbb{R}^q$ denotes the preimage of \mathcal{S} under M , i.e. $M^{-1}(\mathcal{S}) := \{ x \in \mathbb{R}^q \mid Mx \in \mathcal{S} \}$. For $v \in \mathbb{R}^n$ and a set of integers $\mathcal{N} \subseteq \{1, 2, \dots, n\}$, with $v_{\mathcal{N}}$ we refer to the vector obtained from v by considering only the components in the set \mathcal{N} . Similarly, if $M \in \mathbb{R}^{n_1 \times n_2}$ is a matrix, with $M_{\mathcal{N}_1 \mathcal{N}_2}$ we refer to the matrix obtained from M by considering only the rows in the set $\mathcal{N}_1 \subseteq \{1, 2, \dots, n_1\}$ and the columns in the set $\mathcal{N}_2 \subseteq \{1, 2, \dots, n_2\}$. The symbol \bullet stands for the whole set of rows or columns.

II. LINEAR SWITCHED SYSTEMS AND DIFFERENTIAL ALGEBRAIC EQUATIONS

A linear switched system (LSS) within a linear complementarity framework can be described as follows [1], [3]

$$\dot{x} = \bar{A}x + \bar{B}z \quad (1a)$$

$$w = \bar{C}x + \bar{D}z \quad (1b)$$

where $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are matrices of suitable dimensions, $x \in \mathbb{R}^n$ is called *semi-state* and the vectors $z \in \mathbb{R}^p$ and $w \in \mathbb{R}^p$ are *ideal switch variables*, i.e.

$$z_i w_i = 0 \quad (2)$$

for $i = 1, \dots, p$, where the subscript ‘ i ’ indicates the i -th component of the vector. In other words at each time instant

either the component z_i or the corresponding component w_i must be zero.

In order to select a mode of the system it is necessary to define a *switch configuration* through the subset $\pi \subseteq \{1, 2, \dots, p\}$. When π is active the following equations hold

$$w_i = 0 \quad \text{if } i \in \pi \quad (3a)$$

$$z_i = 0 \quad \text{if } i \in \pi^c, \quad (3b)$$

where $\pi^c = \{1, 2, \dots, p\} \setminus \pi$. Say m is the size of the index set π .

By combining (1) and (3), we obtain for each switch configuration π a simplified structure:

$$\dot{x} = \bar{A}x + \bar{B}_{\bullet\pi}z_\pi \quad (4a)$$

$$0 = \bar{C}_{\pi\bullet}x + \bar{D}_{\pi\pi}z_\pi \quad (4b)$$

$$z_{\pi^c} = 0 \quad (4c)$$

$$w_{\pi^c} = \bar{C}_{\pi^c\bullet}x. \quad (4d)$$

The model (4a)–(4b) can be simply rearranged into

$$\dot{x} = Kx + Lu \quad (5a)$$

$$0 = Mx + Nu \quad (5b)$$

where $u = z_\pi$, $K = \bar{A}$, $L = \bar{B}_{\bullet\pi}$, $M = \bar{C}_{\pi\bullet}$, and $N = \bar{D}_{\pi\pi}$.

The model (5) can be rewritten in the form of differential algebraic equations (DAE). Let us define the state vector as $\text{col}(x, u)$. Then (5) can be written as the following DAE

$$E \begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = A \begin{pmatrix} x \\ u \end{pmatrix}, \quad (6)$$

where $E, A \in \mathbb{R}^{(n+m) \times (n+m)}$ are square matrices given by

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} K & L \\ M & N \end{bmatrix} \quad (7)$$

where I is the identity matrix of order n .

III. STATE JUMPS

The solution of an LSS can be obtained by cascading the solutions of the DAE models corresponding to the different modes. In that sense one can restrict the state jump analysis occurrence at the time instant when a generic mode is activated. Then without loss of generality in what follows the state jump at the initial time instant is considered.

In order to define a solution for (5), or equivalently for (6)–(7), one must fix an initial condition, say $\text{col}(x_0, u_0)$. It can be proved that for any initial condition there exists a solution of (6) in the distributional framework [2, Thm 6.5.1] provided the matrix pair (E, A) is regular (see Section IV). The occurrence of a state jump can be formalized by means of the consistency concept.

Definition 1: An initial condition $\text{col}(x_0, u_0)$ is called *consistent* for the DAE (6) if, and only if, there exists an absolutely continuous solution $\text{col}(x(\cdot), u(\cdot))$ of (6) such that $\text{col}(x(0), u(0)) = \text{col}(x_0, u_0)$. Otherwise the initial condition is called *inconsistent*.

Definition 2: The *consistency space* $\mathfrak{C}_{(E,A)}$ is the set of all vectors $\text{col}(x_0, u_0)$ which represent a consistent initial condition for a given DAE.

A. Projection Matrix

Switching between different DAEs each given by (6) in general leads to inconsistent initial values because the consistency space need not to coincide. In order to obtain a solution the state must clearly *jump* to a consistent initial value first. In order to correctly define this state re-initialization a *jump rule* must be chosen which unambiguously maps an inconsistent initial value into a consistent one. Since we are in a linear framework, this jump rule must be linear and also should map consistent initial values on themselves, i.e. the jump rule is given by a *projection matrix* $\Pi \in \mathbb{R}^{(n+m) \times (n+m)}$:

$$\begin{pmatrix} x_0^+ \\ u_0^+ \end{pmatrix} = \Pi \begin{pmatrix} x_0^- \\ u_0^- \end{pmatrix} \quad (8)$$

Definition 3: We define the *jump space* \mathfrak{J}_Π as the set of all vectors which represent a valid re-initialization for the jump rule taken into account, i.e.

$$\mathfrak{J}_\Pi := \left\{ \Delta \in \mathbb{R}^{n+m} \mid \exists \begin{pmatrix} x_0^- \\ u_0^- \end{pmatrix} : \Pi \begin{pmatrix} x_0^- \\ u_0^- \end{pmatrix} = \begin{pmatrix} x_0^- \\ u_0^- \end{pmatrix} - \Delta \right\}$$

According to (8) it results

$$\mathfrak{C}_{(E,A)} = \text{im } \Pi \quad (9a)$$

$$\mathfrak{J}_\Pi = \text{ker } \Pi \quad (9b)$$

with $\mathfrak{C}_{(E,A)} \oplus \mathfrak{J}_\Pi = \mathbb{R}^{n+m}$. Therefore, for a given projection matrix Π , any initial condition $\text{col}(x_0^-, u_0^-) \in \mathbb{R}^{n+m}$ can be uniquely decomposed as follows

$$\begin{pmatrix} x_0^- \\ u_0^- \end{pmatrix} = \begin{pmatrix} x_0^+ \\ u_0^+ \end{pmatrix} + \begin{pmatrix} \Delta_x \\ \Delta_u \end{pmatrix} \quad (10)$$

with $\text{col}(x_0^+, u_0^+) \in \mathfrak{C}_{(E,A)}$ and $\text{col}(\Delta_x, \Delta_u) \in \mathfrak{J}_\Pi$.

B. Energy-based Jump Rule

A common re-initialization strategy [1] is based on the so-called energy-based jump rule. In order to apply such a rule the passivity concept must be introduced, which can be characterized as follows.

Definition 4: [1, Prop IV.3] A quadruple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is *passive* if, and only if, the following linear matrix inequality

$$\begin{bmatrix} \bar{A}^\top Q + Q\bar{A} & Q\bar{B} - \bar{C}^\top \\ \bar{B}^\top Q - \bar{C} & -(\bar{D} + \bar{D}^\top) \end{bmatrix} \leq 0. \quad (11)$$

has a positive semi-definite solution $Q = Q^\top \geq 0$. We call $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ *strictly passive* if (11) has a *positive definite* solution $Q = Q^\top > 0$.

Assume that the quadruple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ associated to the LSS (1) is strictly passive with a positive definite solution $Q > 0$ of (11) and consider the system (5) resulting for a generic mode. It can be shown that the energy-based jump rule for the semi-state x is independent of u and can be obtained by solving the following quadratic program with respect to x_0^+ [1, Thm VI.1]:

$$\text{minimize } \frac{1}{2}(x_0^+ - x_0^-)^\top Q(x_0^+ - x_0^-) \quad (12a)$$

$$\text{subject to } Mx_0^+ \in \text{im } N. \quad (12b)$$

In particular (12) has a unique solution which can be written as

$$x_0^+ = \Pi_{\text{pas}}^x x_0^- \quad (13)$$

with

$$\Pi_{\text{pas}}^x = I - Q^{-1}M^T P^T (PMQ^{-1}M^T P^T)^\dagger PM \quad (14)$$

where \dagger refers to the pseudoinverse matrix and P is a matrix such that $\ker P = \text{im } N$.

Moreover, under the strict passivity condition, it is also proved in [1] that

$$\text{im } \Pi_{\text{pas}}^x = M^{-1}(\text{im } N) \quad (15a)$$

$$\ker \Pi_{\text{pas}}^x = L \ker N \quad (15b)$$

in particular, $M^{-1}(\text{im } N)$ is the consistency space for the semi-state x and

$$M^{-1}(\text{im } N) \oplus L \ker N = \mathbb{R}^n. \quad (16)$$

IV. REGULARITY AND WONG SEQUENCES

An alternative way for defining a jump rule is based on the DAE representation (6) and on the concept of regularity.

Definition 5: A matrix pair (E, A) is *regular* if $\det(sE - A)$ is not the zero polynomial.

The regularity property allows to obtain an explicit solution formula for (5) by employing the *quasi-Weierstraß form* as described below.

Theorem 1 ([4]): Given a regular matrix pair (E, A) there exist invertible matrices S, T such that (E, A) is transformed into a quasi-Weierstrass form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \quad (17)$$

where J is some matrix and Z is a nilpotent matrix.

In view of [5] we call (17) *quasi-Weierstrass form* because we do not assume that J and Z are in Jordan canonical form.

A convenient method to find the transformation matrices S, T is to employ the so called Wong sequences:

Definition 6 ([6]): Consider a matrix pair (E, A) , then the following subspace sequences are called *Wong sequences*:

$$\mathcal{V}_{i+1} = A^{-1}(E\mathcal{V}_i) \quad (18a)$$

$$\mathcal{W}_{i+1} = E^{-1}(A\mathcal{W}_i) \quad (18b)$$

with $\mathcal{V}_0 = \mathbb{R}^n$, $\mathcal{W}_0 = \{0\}$.

Theorem 2: [5, Prep 2.4] If the matrix pair (E, A) is regular then the Wong sequences get stationary after finitely many steps. Denote \mathcal{V}^* and \mathcal{W}^* the limits, and choose full rank matrices V, W such that $\mathcal{V}^* = \text{im } V$ and $\mathcal{W}^* = \text{im } W$ then

$$T = [V, W] \quad (19a)$$

$$S = [EV, AW]^{-1}. \quad (19b)$$

yield the quasi-Weierstrass form (17).

It is easily seen that with the help of the quasi-Weierstrass form a unique jump rule based on the regularity assumption

follows. In fact, in [2, Thm 6.5.1] it is proved that the consistency projector is given by

$$\Pi_{\text{reg}} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \quad (20)$$

with the matrix T given by (19a). Note that this projection maps onto \mathcal{V}^* along \mathcal{W}^* , thus it holds

$$\mathfrak{C}_{(E,A)} = \text{im } \Pi_{\text{reg}} = \mathcal{V}^*, \quad (21a)$$

$$\mathfrak{J}_{\Pi_{\text{reg}}} = \ker \Pi_{\text{reg}} = \mathcal{W}^*. \quad (21b)$$

Since the matrix E is assumed to have the special structure (7), in what follows some original preliminary results are obtained.

Firstly it can be proved that, similarly to the energy-based jump rule, the vector u_0 does not influence re-initialization of the semi-state x .

Theorem 3: Consider the DAE (5) and suppose the matrix pair (E, A) is regular. By using the Wong sequences define the spaces $\mathcal{V}^* = \text{im } V$ and $\mathcal{W}^* = \text{im } W$ and the regularity projector (20). Then the following condition holds

$$\Pi_{\text{reg}} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \Pi_{\text{reg}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \quad (22)$$

for all $x_0 \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^m$.

Proof: First notice that

$$\{0\} = \mathcal{W}_0 \subsetneq \mathcal{W}_1 \subsetneq \dots \subsetneq \mathcal{W}_i = \mathcal{W}_{i+1} = \mathcal{W}^*. \quad (23)$$

Moreover

$$\begin{aligned} \mathcal{W}_1 &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} K & L \\ M & N \end{bmatrix} \mathcal{W}_0 \right) \\ &= \ker \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{W}^* = \ker \Pi_{\text{reg}}. \end{aligned} \quad (24)$$

Then

$$\begin{pmatrix} 0 \\ u \end{pmatrix} \in \ker \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \subseteq \ker \Pi_{\text{reg}} \quad (25)$$

which with simple algebraic manipulations implies (22):

$$\Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} = \Pi_{\text{reg}} \left[\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} \right] = \Pi_{\text{reg}} \begin{pmatrix} x \\ 0 \end{pmatrix}. \quad (26)$$

□

Since we are only interested in the semi-state x , in what follows we seek a restriction of the jump map. First we define the restriction of the spaces \mathcal{V}^* and \mathcal{W}^* relative to the variable x .

Theorem 4: Consider the DAE (6) and assume the matrix pair (E, A) is regular. Let \mathcal{V}^* and \mathcal{W}^* be the spaces which terminate the Wong sequences. Define the projector

$$\Pi_x = [I \ 0]. \quad (27)$$

Then the spaces $\mathcal{V}_x^* = \Pi_x \mathcal{V}^*$ and $\mathcal{W}_x^* = \Pi_x \mathcal{W}^*$ fulfil

$$\mathcal{V}_x^* \oplus \mathcal{W}_x^* = \mathbb{R}^n. \quad (28)$$

Proof: In the first part of the proof it is demonstrated that

$$\mathcal{V}_x^* \cap \mathcal{W}_x^* = \{0\}. \quad (29)$$

Let $\bar{x} \in \mathcal{V}_x^* \cap \mathcal{W}_x^*$, then $\exists v \in \mathcal{V}^*, w \in \mathcal{W}^*$ such that

$$\bar{x} = \Pi_x v \quad \wedge \quad \bar{x} = \Pi_x w. \quad (30)$$

Then it follows

$$\Pi_x v = \Pi_x w \quad (31a)$$

$$v - w \in \ker \Pi_x = \ker \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{W}^*. \quad (31b)$$

Therefore since $w \in \mathcal{W}^*$ it is also true that $v \in \mathcal{W}^*$. Finally since $v \in \mathcal{V}^* \cap \mathcal{W}^* = \{0\}$ we obtain

$$\bar{x} = \Pi_x v = 0. \quad (32)$$

We can now proceed by showing that

$$\mathcal{V}_x^* + \mathcal{W}_x^* = \mathbb{R}^n. \quad (33)$$

Choose $x \in \mathbb{R}^n$ and consider the vector $\text{col}(x, 0) \in \mathbb{R}^{n+m}$, then there must exist a unique pair $v \in \mathcal{V}^*$ and $w \in \mathcal{W}^*$ such that

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = v + w. \quad (34)$$

It follows $x = \Pi_x v + \Pi_x w \in \mathcal{V}_x^* + \mathcal{W}_x^*$ and thus the claim. \square

Thus, in analogy of (10), any inconsistent initial value x_0^- can be decomposed as follows

$$x_0^- = x_0^+ + \Delta_x \quad (35)$$

with *unique* $x_0^+ \in \mathcal{V}_x^*$ and $\Delta_x \in \mathcal{W}_x^*$. In particular, the jump-rule is given by a projector with image \mathcal{V}_x^* and kernel \mathcal{W}_x^* , i.e. the *regularity-based projector* is given by chosen any two full rank matrices V_x, W_x such that $\mathcal{V}_x^* = \text{im } V_x$, $\mathcal{W}_x^* = \text{im } W_x$ and

$$\Pi_{\text{reg}}^x = [V_x, W_x] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} [V_x, W_x]^{-1}. \quad (36)$$

The following lemma shows that the regularity-based projector for the semi-state x can also be calculated directly with the help of the consistency projector Π_{reg} and the canonical projector $\Pi_x = [I, 0]$.

Lemma 1: If the matrix pair in (6) is regular and Π_{reg} is the corresponding consistency projector, then

$$\Pi_{\text{reg}}^x = \Pi_x \Pi_{\text{reg}} \Pi_x^T \quad (37)$$

The (surprisingly technical) proof is carried out in the Appendix.

The spaces \mathcal{V}_x^* and \mathcal{W}_x^* can also be obtained as the result of subspace iterations as the following result shows.

Theorem 5: Consider the system described by (6). If the matrix pair is regular, then the following algorithm provides the space \mathcal{V}_x^* and \mathcal{W}_x^* after finitely many steps.

$$\mathcal{V}_x^i = \left\{ x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m, \begin{matrix} Kx + Lu \in \mathcal{V}_x^{i-1} \\ Mx + Nu = 0 \end{matrix} \right\} \quad (38a)$$

$$\mathcal{W}_x^i = \left\{ x \in \mathbb{R}^n \mid \begin{matrix} \bar{x} \in \mathcal{W}_x^{i-1} & x = K\bar{x} + L\bar{u} \\ \bar{u} \in \mathbb{R}^m & 0 = M\bar{x} + N\bar{u} \end{matrix} \right\} \quad (38b)$$

with $\mathcal{V}_x^0 = \mathbb{R}^n$ and $\mathcal{W}_x^0 = \{0\}$.

The proof is carried out in the Appendix.

TABLE I
CONSISTENCY AND JUMP SPACES

	Regularity	Passivity
Consistency space	\mathcal{V}_x^*	$M^{-1} \text{im } N$
Jump space	\mathcal{W}_x^*	$L \ker N$

Relevant spaces for the re-initialization of the semi-state x .

V. EQUIVALENCE RESULT

In this section we prove the main result of the paper, i.e. the equivalence between regularity- and energy-based jump rules. Therefore, we first assume that both regularity and strict passivity are satisfied by the system under study. Table I compares the fundamental spaces related to the x -restriction for both the aforementioned theories.

Theorem 6 (Equivalence between jump rules): Consider the DAE (6) with regular matrix pair (E, A) given by (7) for which (K, L, M, N) is strictly passive. Let the regularity-based jump projector Π_{reg}^x be given as in Lemma 1 and let the passivity-based jump-projector Π_{pas}^x be given by (14), then

$$\boxed{\Pi_{\text{reg}}^x = \Pi_{\text{pas}}^x} \quad (39)$$

Proof: The first crucial observation is that we have two expressions for the consistency space for the semi-state x . One is given by the canonical projection \mathcal{V}_x^* of the limit of the first Wong sequence \mathcal{V}^* and the second one as the image of the passivity-jump projector, i.e.

$$\mathcal{V}_x^* = M^{-1} \text{im } N.$$

A more careful analysis of the formula for \mathcal{V}_x^* reveals

$$\begin{aligned} \mathcal{V}_x^1 &= \left\{ x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m, \begin{matrix} Mx + Nu = 0 \end{matrix} \right\} \\ &= \left\{ x \in \mathbb{R}^n \mid Mx \in \text{im } N \right\} \\ &= M^{-1} \text{im } N = \mathcal{V}_x^*. \end{aligned}$$

Hence we can conclude that the \mathcal{V}_x^i -sequence ends after just one step.

Similarly, the calculation of \mathcal{W}_x^1 reveals

$$\mathcal{W}_x^1 = \left\{ x \in \mathbb{R}^n \mid \bar{u} \in \mathbb{R}^m \begin{matrix} x = L\bar{u} \\ 0 = N\bar{u} \end{matrix} \right\} = L \ker N.$$

To show equality of the two jump-rules it remains to be show that $\mathcal{W}_x^* = \mathcal{W}_x^1$.

From (16) it follows that

$$\mathbb{R}^n = M^{-1}(\text{im } N) \oplus L \ker N = \mathcal{V}_x^* \oplus \mathcal{W}_x^1$$

Invoking (28) this implies

$$\dim \mathcal{W}_x^1 = \dim \mathcal{W}_x^*$$

and together with $\mathcal{W}_x^1 \subseteq \mathcal{W}_x^*$ we can conclude $L \ker N = \mathcal{W}_x^1 = \mathcal{W}_x^*$. \square

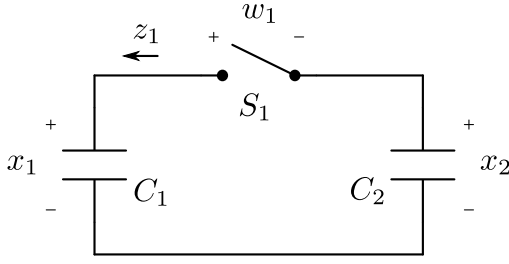


Fig. 1. Two capacitors connected in parallel by a switch.

VI. EXAMPLE

In this section we deploy a canonical switched system, consisting of an electrical network with two capacitors connected together via an ideal switch (Fig. 1).

We first evaluate the projection maps Π_{reg}^x and Π_{pas}^x and then we compare them in order to show numerically the equivalence of the jump rules.

A straightforward analysis of the network lead us to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ -\frac{1}{C_2} \end{bmatrix} z_1 \quad (40a)$$

$$w_1 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \end{bmatrix} z_1 \quad (40b)$$

with either $w_1 = 0$ or $z_1 = 0$.

The system has two different modes depending on the state of the switch S_1 . When it is open the network current z_1 is forced to be null, vice versa when it is closed the voltage w_1 is zero.

Taking this into account, we consider the following scenario.

Initially the switch is open and therefore the voltages across the capacitors remain constant to a certain value $x_0 = \text{col}(x_{01}, x_{02})$. At the time $t = t_0$ the switch closes and the semi-state variables might receive a jump towards a value that is consistent for the new DAE.

The DAEs (6) which describe the system when the switch is closed are given by (40) with $w_1 = 0$ and $u = z_1$.

The consistency and jump spaces are

$$M^{-1} \text{im } N = \ker M = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$L \ker N = \text{im } L = \text{span} \left\{ \begin{pmatrix} 1/C_1 \\ -1/C_2 \end{pmatrix} \right\}$$

The consistency projector relative to the x variable can be easily derived

$$\begin{aligned} \Pi_{\text{reg}}^x &= \begin{bmatrix} 1 & 1/C_1 \\ 1 & -1/C_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/C_1 \\ 1 & -1/C_2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{C_1}{C_1+C_2} & \frac{C_2}{C_1+C_2} \\ \frac{C_1}{C_1+C_2} & \frac{C_2}{C_1+C_2} \end{bmatrix}. \end{aligned} \quad (41)$$

In order to evaluate the energy-based projector we need to solve the LMI problem in (11). To this end, Matlab LMI toolbox comes in handy by assigning numerical values to the network parameters. We chose: $C_1 = 1 \mu F$ and $C_2 = 2 \mu F$.

We find then a solution matrix

$$Q = \begin{bmatrix} 0.006051 & 0.012101 \\ 0.012101 & 0.024203 \end{bmatrix},$$

and choosing a matrix $P = [1]$ we can finally evaluate the projector in (13)

$$\Pi_{\text{pas}}^x = \begin{bmatrix} 0.3333 & 0.6667 \\ 0.3333 & 0.6667 \end{bmatrix}. \quad (42)$$

A straightforward substitution of the numerical values in (41) leads to the same matrix of (42)

VII. CONCLUSIONS

State jumps is an interesting phenomenon which occurs in switched linear systems due to the presence of mode-dependent algebraic constraints. The practical interest of this problem is demonstrated by the wide literature dedicated to the analysis of state jumps in electrical circuits. Different rules can be defined to re-initialize the state at a value which is consistent with the differential algebraic equations of the active mode. In this paper we have shown that the regularity based jump rule derived purely mathematical coincides with a jump rule based on physical considerations (energy minimization).

APPENDIX

A. Proof Lemma 1

Define $\Pi^* = \Pi_x \Pi_{\text{reg}} \Pi_x^T$.

1. $\Pi^* x = x \Leftrightarrow x \in \mathcal{V}_x^*$

Consider $x \in \mathcal{V}_x^*$, it results

$$\Pi^* x = \Pi_x \Pi_{\text{reg}} \Pi_x^T x = \Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ 0 \end{pmatrix}$$

and using Theorem 3

$$\Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ 0 \end{pmatrix} = \Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} \quad \forall u \in \mathbb{R}^m.$$

Choose now $\bar{u} : \text{col}(x, \bar{u}) \in \mathcal{V}^*$, it follows

$$\Pi^* x = \Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ \bar{u} \end{pmatrix} = \Pi_x \begin{pmatrix} x \\ \bar{u} \end{pmatrix} = x$$

2. $\text{im } \Pi^* = \mathcal{V}_x^*$.

$$\begin{aligned} \text{im } \Pi^* &= \text{im } \Pi_x \Pi_{\text{reg}} \Pi_x^T \\ &= \left\{ \Pi_x \Pi_{\text{reg}} \Pi_x^T x \mid x \in \mathbb{R}^n \right\} \\ &= \left\{ \Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R}^n \right\} \\ &= \left\{ \Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} \mid \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \right\} \\ &= \left\{ \Pi_x \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \mid \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \in \mathcal{V}^* \right\} \\ &= \{ \bar{x} \mid \bar{x} \in \mathcal{V}_x^* \} = \mathcal{V}_x^*. \end{aligned}$$

3. $\ker \Pi^* = \mathcal{W}_x^*$.

“ \supseteq ”

$$\ker \Pi^* = \ker \Pi_x \Pi_{\text{reg}} \Pi_x^T \supseteq \ker \Pi_{\text{reg}} \Pi_x^T$$

but

$$\begin{aligned}
\ker \Pi_{\text{reg}} \Pi_x^T &= \left\{ x \in \mathbb{R}^n \mid \Pi_{\text{reg}} \Pi_x^T x = 0 \right\} \\
&= \left\{ x \in \mathbb{R}^n \mid \Pi_{\text{reg}} \begin{pmatrix} x \\ 0 \end{pmatrix} = 0 \right\} \\
&= \left\{ x \in \mathbb{R}^n \mid \Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} = 0, \forall u \in \mathbb{R}^m \right\} \\
&= \left\{ x \in \mathbb{R}^n \mid \begin{pmatrix} x \\ u \end{pmatrix} \in \ker \Pi_{\text{reg}}, \forall u \in \mathbb{R}^m \right\} \\
&= \Pi_x \ker \Pi_{\text{reg}} = \Pi_x \mathcal{W}^* = \mathcal{W}_x^*.
\end{aligned}$$

“ \subseteq ”

choose $x \in \ker \Pi^* = \ker \Pi_x \Pi_{\text{reg}} \Pi_x^T$, it results

$$\begin{aligned}
\Pi_x \Pi_{\text{reg}} \Pi_x^T x = 0 &\Rightarrow \Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ 0 \end{pmatrix} = 0 \\
\Rightarrow \Pi_x \Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} = 0, \forall u \in \mathbb{R}^m \\
\Rightarrow \Pi_x \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} = 0 &\Rightarrow \bar{x} = 0
\end{aligned}$$

where $\text{col}(\bar{x}, \bar{u}) = \text{col}(0, \bar{u}) = \Pi_{\text{reg}} \text{col}(x, u)$. Now note that

$$\begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} \in \ker \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{W}^* = \ker \Pi_{\text{reg}}$$

therefore

$$\begin{aligned}
\Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} \in \ker \Pi_{\text{reg}}, \quad \forall u \in \mathbb{R}^m \\
\Rightarrow \Pi_{\text{reg}} \Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} = 0,
\end{aligned}$$

and since Π_{reg} is a projector

$$\Pi_{\text{reg}}^2 \begin{pmatrix} x \\ u \end{pmatrix} = \Pi_{\text{reg}} \begin{pmatrix} x \\ u \end{pmatrix} = 0.$$

Finally for all $u \in \mathbb{R}^m$, $\text{col}(x, u) \in \ker \Pi_{\text{reg}}$ and

$$x \in \Pi_x \ker \Pi_{\text{reg}} = \Pi_x \mathcal{W}^* = \mathcal{W}_x^*$$

□

B. Proof Theorem 5

Let us apply the Wong sequences algorithm to the specific matrix pair in (5)

$$\mathcal{V}_i = \begin{bmatrix} K & L \\ M & N \end{bmatrix}^{-1} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}_{i-1} \right).$$

Considering now the definition of pre-image we obtain

$$\begin{aligned}
\mathcal{V}_i &= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \mid \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \in \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}_{i-1} \right\} \\
&= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \mid \begin{matrix} Kx + Lu \in \Pi_x \mathcal{V}_{i-1} \\ Mx + Nu = 0 \end{matrix} \right\}
\end{aligned}$$

If we restrict this space to x by defining $\mathcal{V}_x^i = \Pi_x \mathcal{V}_i$ we finally have

$$\mathcal{V}_x^i = \left\{ x \in \mathbb{R}^n \mid u \in \mathbb{R}^m, \begin{matrix} Kx + Lu \in \mathcal{V}_x^{i-1} \\ Mx + Nu = 0 \end{matrix} \right\}.$$

Similar calculations for \mathcal{W} space sequence lead to

$$\begin{aligned}
\mathcal{W}_i &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} K & L \\ M & N \end{bmatrix} \mathcal{W}_{i-1} \right) \\
&= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \mid \exists \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \in \mathcal{W}_{i-1}, \right. \\
&\quad \left. \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \mid \exists \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \in \mathcal{W}_{i-1}, \right. \\
&\quad \left. \begin{matrix} x = K\bar{x} + L\bar{u} \\ 0 = M\bar{x} + N\bar{u} \end{matrix} \right\}.
\end{aligned}$$

Define now the restriction $\mathcal{W}_x^i = \Pi_x \mathcal{W}_i$

$$\mathcal{W}_x^i = \left\{ x \in \mathbb{R}^n \mid \begin{matrix} \bar{x} \in \mathcal{W}_x^{i-1} & x = K\bar{x} + L\bar{u} \\ \bar{u} \in \mathcal{W}_u^{i-1} & 0 = M\bar{x} + N\bar{u} \end{matrix} \right\}$$

where $\mathcal{W}_u^i = [0 \ I] \mathcal{W}_i$ has the following property

$$\mathcal{W}_u^i = \begin{cases} \{0\} & \text{if } i = 0 \\ \mathbb{R}^m & \text{if } i > 0 \end{cases}.$$

Moreover

$$\mathcal{W}_x^1 = \Pi_x \mathcal{W}_1 = \Pi_x \ker \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \{0\}$$

and thus, we can redefine the algorithm as it follows

$$\mathcal{W}_x^i = \left\{ x \in \mathbb{R}^n \mid \begin{matrix} \bar{x} \in \mathcal{W}_x^{i-1} & x = K\bar{x} + L\bar{u} \\ \bar{u} \in \mathbb{R}^m & 0 = M\bar{x} + N\bar{u} \end{matrix} \right\}$$

with $\mathcal{W}_x^0 = \{0\}$.

In such way it turns out that $\mathcal{W}_x^i = \Pi_x \mathcal{W}_{i+1}$. The two algorithms defining the \mathcal{V}_x^i and \mathcal{W}_x^i must converge after finitely many steps since they are defined to be just the projection of the full spaces \mathcal{V}_i and \mathcal{W}_i which in turn converge.

□

REFERENCES

- [1] R. Frasca, M. Camlibel, I. Goknar, L. Iannelli, and F. Vasca, “Linear passive networks with ideal switches: Consistent initial conditions and state discontinuities,” *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 57, no. 12, pp. 3138–3151, dec. 2010.
- [2] S. Trenn, “Switched differential algebraic equations,” in *Dynamics and Control of Switched Electronic Systems*, ser. Advances in Industrial Control, F. Vasca and L. Iannelli, Eds. Springer London, 2012, pp. 189–216. [Online]. Available: http://dx.doi.org/10.1007/978-1-4471-2885-4_6
- [3] M. K. Camlibel, W. P. M. H. Heemels, A. J. van der Schaft, and J. M. Schumacher, “Switched networks and complementarity,” *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 50, no. 8, pp. 1036–1046, aug. 2003.
- [4] K. Weierstraß, “Zur Theorie der bilinearen und quadratischen Formen,” *Berl. Monatsb.*, pp. 310–338, 1868.
- [5] T. Berger, A. Ilchmann, and S. Trenn, “The quasi-weierstrass form for regular matrix pencils,” *Linear Algebra and its Applications*, vol. 436, no. 10, pp. 4052–4069, 2012. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S0024379509006776>
- [6] K.-T. Wong, “The eigenvalue problem $\lambda Tx + Sx$,” *J. Diff. Eqns.*, vol. 16, pp. 270–280, 1974.