Topological solvability and index characterizations for a common DAE power system model

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***Abstract*— For the widely-used power system model consisting of the generator swing equations and the power flow equations resulting in a system of differential algebraic equations (DAEs), we introduce a sufficient and necessary solvability condition for the linearized model. This condition is based on the topological structure of the power system. Furthermore we show sufficient conditions for the linearized DAE-system and a nonlinear version of the model to have differentiation index equal to one.**

***Index Terms*— Differential algebraic equations, power systems, regularity, differentiation index.**

I. INTRODUCTION

The integration of an increasing number of renewable energy sources makes it necessary to analyze and simulate sophisticated dynamical models of power grids. One common model of power grids on the transmission level is the combination of the generator swing equations with the nonlinear power balance equation resulting in a nonlinear differential algebraic equation (DAE). Solvability of nonlinear (as well as linear) DAEs is in general not guaranteed and can often only be checked ad hoc for the specific given DAE when parameter values are known. In this paper we present a characterization of solvability solely in terms of the topology of the power network for the linearized DAE model. For numerical simulation the differentiation index, which we will shortly call index, of a DAE plays a crucial role and we also show that any solvable linearized DAE model of the power system is index one. Furthermore, we extend this index one characterization to a simplified nonlinear DAE model. There are similar investigations concerning the index for circuit models in [12] and [15], but to the authors' best knowledge, these topological methods have so far not been applied on models of power networks.

Nonlinear DAE models of power grids as well as their linearization have been studied frequently in the past decades. One of the first textbooks on the generator equations (the swing equations) is [4]. In this book a comprehensive analysis, derivation and mechanical interpretation can be found (see also [5] for a more general framework). The power flow equations are used to carry out a load flow analysis, which is one of the most frequent routines performed for power system operation and planning. A good introduction, derivation and analysis regarding this topic can be found e.g. in [1]. Finally, the combination of these models, which is also sometimes referred to as swing equations, was used for investigations of e.g. bifurcations in power systems ([6]), observer design ([9], [10]) or cyber-physical security ([7]).

This paper is structured as follows. In the second chapter we introduce the nonlinear model and its linearized version along with a graph theoretical interpretation of the power network. Also definitions of regularity and the index are given. The third chapter contains the main mathematical result, which is a characterization of regularity in terms of the topology of the power network. In the fourth chapter we show that in the linear case the index equals one if the system is regular. Furthermore for a nonlinear version of the model we give a sufficient condition for the index to be equal to one. Afterwards we present an example in chapter five and finally a conclusion.

Throughout this paper we denote for some $n, m \in \mathbb{N}$ the identity matrix by $I_n \in \mathbb{R}^{n\times n}$, a vector by $\alpha = [\alpha_1, \dots, \alpha_n]^\top \in \mathbb{R}^n$ and a Matrix by $A = [A_{ij}]_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$. Furthermore we denote the polynomial ring over a field $\mathbb{F}$ by $\mathbb{F}[s]$ and the field of rational expressions over a field $\mathbb{F}$ by $\mathbb{F}(s)$. Evaluations of $F(s) \in \mathbb{F}[s]$ and $G(s) \in \mathbb{F}(s)$ at some value $\lambda \in \mathbb{C}$ are denoted by $F(\lambda)$ and $G(\lambda)$ respectively.

II. MATHEMATICAL MODEL AND PRELIMINARIES

We consider a power system consisting of synchronous generators and loads, both of which are connected to buses. These buses are interconnected by transmission lines. The generators are represented by constant voltage behind transient reactance models and the loads are assumed to be independent from the system variables and also can be viewed as poorly controllable renewable infeed, such as photovoltaic or wind energy. Transmission lines are represented by the well-known Π-models. The system of equations which we obtain from the combination of these models contains ordinary differential equations (ODEs), describing the dynamic behavior of the generators, and algebraic equations, describing the power flow between the buses. Together the ODEs and the algebraic equations form a system of DAEs.

A. The nonlinear model

Consider a power network consisting of n generators, connected to n generator buses, and m load buses. Note,

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that in the literature it is often assumed that there is no load connected to the generator buses, but for the results in the following chapters this assumption is not required. The $n+m$ buses are ordered such that the generator buses have indices $i = 1, 2, \dots, n$ and the load buses have indices $i = n+1, n+2, \dots, n+m$. The generators are represented by constant voltage behind transient reactance models (see [4], [5]) and the transmission lines by Π-models (see [3], [5]).

For $i = 1, \dots, n$, let $M_i > 0$ be the inertia, $D_i > 0$ the damping coefficient, $Z_i > 0$ the transient reactance, $V_i^0 > 0$ the constant internal voltage modulus, $\alpha_i(t)$ the rotor angle and $\omega_i(t)$ the angular frequency at time $t \in \mathbb{R}$ of the i-th generator (modeled as a synchronous machine). Furthermore for $i, j = 1, \dots, n+m$ let $G_{ij} \geq 0$ and $B_{ij} \geq 0$ be the conductance and susceptance between bus i and j. Finally, let $V_i(t)$ be the voltage modulus and $\theta_i(t)$ be the voltage angle at the i-th bus at time t. The nonlinear differential equations representing the dynamic behavior of the generators for $i = 1, \dots, n$ are

$$\begin{aligned}
\dot{\alpha}_i(t) &= \omega_i(t),\\
M_i \dot{\omega}_i(t) &= P_{g,i}(t) - D_i \omega_i(t) - \frac{V_i^0 V_i(t)}{Z_i} \sin(\alpha_i(t) - \theta_i(t)),
\end{aligned}$$

where the input $P_{g,i}(t)$ is the mechanical power applied to the i-th generator. Let $P_i(t)$ and $Q_i(t)$ be the real and the reactive power infeed at the bus i representing time-dependent loads. Then we can write down the second part of the DAE consisting of the nonlinear equations for real and reactive power flow, which for $i = 1, \dots, n$ are given by

$$\begin{aligned}
P_i(t) = \sum_{j=1}^{m+n} & V_i(t) V_j(t) [B_{ij} \sin(\theta_i(t) - \theta_j(t))\\
&+ G_{ij} \cos(\theta_i(t) - \theta_j(t))]\\
&- \frac{V_i^0 V_i(t)}{Z_i} \sin(\alpha_i(t) - \theta_i(t)),\\
Q_i(t) = \sum_{j=1}^{m+n} & V_i(t) V_j(t) [G_{ij} \sin(\theta_i(t) - \theta_j(t))\\
&- B_{ij} \cos(\theta_i(t) - \theta_j(t))]\\
&- \frac{V_i(t)^2}{Z_i} + \frac{V_i^0 V_i(t)}{Z_i} \cos(\alpha_i(t) - \theta_i(t))
\end{aligned}$$

and for $i = n+1, \dots, n+m$ by

$$\begin{aligned}
P_i(t) = \sum_{j=1}^{m+n} & V_i V_j [B_{ij} \sin(\theta_i(t) - \theta_j(t))\\
&+ G_{ij} \cos(\theta_i(t) - \theta_j(t))],\\
Q_i(t) = \sum_{j=1}^{m+n} & V_i V_j [G_{ij} \sin(\theta_i(t) - \theta_j(t))\\
&- B_{ij} \cos(\theta_i(t) - \theta_j(t))].
\end{aligned}$$

Here the terms $\frac{V_i^0 V_i(t)}{Z_i} \sin(\alpha_i(t) - \theta_i(t))$ and $\frac{(V_i(t))^2}{Z_i} - \frac{V_i^0 V_i(t)}{Z_i} \cos(\alpha_i(t) - \theta_i(t))$ represent the real and reactive power infeed into the network by the i-th generator. Let $x_1 := [\alpha^T, \omega^T]^T$, $x_2 := [\theta^T, V^T]^T$, $u_1 = P_g$, $u_2 = [P^T, Q^T]^T$, where $\alpha(t), \omega(t), P_g(t) \in \mathbb{R}^n$ and $\theta(t), V(t), P(t), Q(t) \in \mathbb{R}^{n+m}$ are vectors with the corresponding entries as introduced above, then the above equations can be written compactly as a semi-explicit DAE of the form

$$\begin{aligned}
\dot{x}_1 &= f(x_1, x_2, u_1),\\
0 &= g(x_1, x_2, u_2),
\end{aligned} \tag{1}$$

with corresponding functions $f : \mathbb{R}^{2n} \times \mathbb{R}^{2(n+m)} \times \mathbb{R}^n \to \mathbb{R}^{2n}$, $g : \mathbb{R}^{2n} \times \mathbb{R}^{2(n+m)} \times \mathbb{R}^{n+m} \to \mathbb{R}^{2(n+m)}$.

B. The linearized model

For simplification of (1) we assume that $V_i^0 = 1$ (per unit) for $i = 1, \dots, n$ and that the voltage moduli are regulated to be constantly equal to one: $V_i(t) = 1$ (per unit) for $i = 1, \dots, n+m$ and all $t \in \mathbb{R}$. Furthermore, assume that the lines are lossless, i.e. $G_{ij} = 0$ for all $i, j = 1, \dots, n+m$. Then we can linearize the equations around $\alpha_i - \theta_i = 0$ for $i = 1, \dots, n$ and $\theta_i - \theta_j = 0$ for $i, j = 1, \dots, n+m$. To obtain the linearized equations we define the diagonal matrices $M := \operatorname{diag}(M_1, \dots, M_n)$, $Z := \operatorname{diag}(Z_1, \dots, Z_n)$, $D := \operatorname{diag}(D_1, \dots, D_n)$, the admittance matrix $B = [B_{ij}]$ and the network matrix

$$\begin{aligned}
R &:= -\operatorname{diag}\left(\sum_{k=1}^{n+m} B_{1k}, \dots, \sum_{k=1}^{n+m} B_{n+m,k} \right) + B, \qquad (2)\\
R &=: \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix},
\end{aligned}$$

where $R_1 \in \mathbb{R}^{n \times n}, R_2, R_3^T \in \mathbb{R}^{n \times m}$ and $R_4 \in \mathbb{R}^{m \times m}$. Defining $\hat{\theta} := [\theta_1, \dots, \theta_n]^T$, $\tilde{\theta} := [\theta_{n+1}, \dots, \theta_{n+m}]^T$, $\hat{P} := [P_1, \dots, P_n]^T$, $\tilde{P} := [P_{n+1}, \dots, P_{n+m}]^T$,

$$x := \begin{bmatrix} \alpha \\ \omega \\ \hat{\theta} \\ \tilde{\theta} \end{bmatrix}, \quad u := \begin{bmatrix} P_g \\ \hat{P} \\ \tilde{P} \end{bmatrix}$$

and

$$\mathfrak{E} := \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{B} := \begin{bmatrix} 0 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix},$$

$$\mathfrak{A} := \begin{bmatrix} 0 & I_n & 0 & 0 \\ -Z^{-1} & -D & Z^{-1} & 0 \\ Z^{-1} & 0 & R_1 - Z^{-1} & R_2 \\ 0 & 0 & R_3 & R_4 \end{bmatrix},$$

the linearized equation is given by

$$\mathfrak{E}\dot{x} = \mathfrak{A}x + \mathfrak{B}u. \tag{3}$$

C. Preliminaries

Since B_{ij} is the susceptance between bus i and j, it holds $B_{ij} = B_{ji}$ and therefore R, as defined in (2), is a symmetric matrix. Furthermore the sum of all off-diagonal elements of every row of R is equal to the negative of the corresponding diagonal element. Matrices having this property are contained in the well-known set of diagonally dominant matrices.

Definition 1: A matrix $A \in \mathbb{R}^{n\times n}$ is called *diagonally dominant* if and only if

$$|A_{ii}| \geq |\sum_{\substack{j=1\\ j\neq i}}^{n} A_{ij}|, \, \forall i = 1, \dots, n. \tag{4}$$

It is called *diagonally balanced* if and only if equality holds in (4).

An important subset of the diagonally dominant matrices is the following:

Definition 2: Define $\mathcal{D}_-^n \subset \mathbb{R}^{n\times n}$ to be the set of matrices $A \in \mathbb{R}^{n\times n}$ for which holds

(i) A is symmetric,

(ii) $A_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{n} A_{ij}$ and

(iii) $A_{ij} \geq 0$ for all $i \neq j$.

The minus sign in the subscript in Definition 2 is used to denote that the diagonal elements are negative and obviously for the Matrix R, as defined in (2), holds $R \in \mathcal{D}_-^n$.

Since a power system has a network structure it is convenient to view it as a graph. This network structure is represented by the matrix R from (2), where bus i is directly connected to bus $j \neq i$ by a line if and only if $R_{ij} \neq 0$. Moreover we say that bus i is connected to bus j if and only if there exists an $1 \leq r \leq n-2$ and a subset $\{i_1, \dots, i_r\} \subset \{1, \dots, n\}$, such that $R_{ii_1} \cdot R_{i_1 i_2} \cdot \dots \cdot R_{i_n j} \neq 0$. Hence the matrix R has a structure similar to an adjacency matrix of an undirected graph. This motivates the following definition.

Definition 3: Let $H \in \mathcal{D}_-^n$. We define $G_H := (\mathcal{V}, \mathcal{E})$ to be the undirected *graph represented by* H where

- $\mathcal{V} := \{1, \dots, n\}$ is the set of all nodes of G_H,
- $\mathcal{E} := \{(i, j) \,|\, i \neq j \wedge H_{ij} \neq 0\}$ is the set containing all edges of G_H.

Every undirected graph G_H with n nodes can be divided into $1 \leq \xi \leq n$ nonempty *connected components* $C_\alpha := (\mathcal{V}_\alpha, \mathcal{E}_\alpha), 1 \leq \alpha \leq \xi$, where every connected component is a subgraph in which any two nodes are connected to each other by paths and connected to no additional nodes in the graph G_H.

Finally we provide the definitions of the properties we want to show for the presented model. These are regularity and the index.

Definition 4: A matrix pair (E, A) with $E, A \in \mathbb{R}^{n\times n}$ is called *regular* if and only if $\det(sE - A)$ is not the zero polynomial.

We will need the following well-known relationship between regularity and solvability of linear DAEs:

Lemma 1: The DAE $E\dot{x} = Ax + Bu$ is solvable for all sufficiently smooth u and each solution is uniquely determined by its initial value $x(0)$ if and only if the matrix pair (E, A) is regular.

Proof: The proof can be carried out analogously to the proof of [16, Theorem 6.3.2]. ■

Definition 5 ([11]): Let the general nonlinear DAE

$$f(\dot{x}(t), x(t), t) = 0$$

be solvable. Then it is said to have *differentiation index* (or *index*) μ if μ is the smallest number of differentiations

$$\frac{df(\dot{x}(t), x(t), t)}{dt} = 0, \dots, \frac{d^\mu f(\dot{x}(t), x(t), t)}{dt^\mu} = 0,$$

such that one can only by algebraic manipulations obtain an explicit expression for $\dot{x}$. In particular, the semi-explicit DAE (1) has index 1 if the Jacobian matrix $\frac{\partial g(x_1,x_2,u_2)}{\partial x_2}$ is invertible for all relevant x_1, x_2, u_2.

III. Solvability of the Linearized Model

We can now present our main result concerning the solvability of the linearized equation.

Theorem 1: The linearized equation (3) is solvable for all sufficiently smooth inputs u and its solution is uniquely determined by the initial value $x(0)$ if and only if every load node of the network graph is connected to a generator node.

In order to prove Theorem 1, we need two technical lemmas. Firstly we need to prove invertibility of an important class of matrices, closely related to the network graph.

Lemma 2: Let $R \in \mathcal{D}_-^{n+m}$ be given by (2) for given susceptances B_{ij} of a power network and let $G_R = (\mathcal{V}_R, \mathcal{E}_R)$ be the corresponding undirected graph of the power network. Then the following three statements are equivalent:

(i) $R - Q$ is invertible for any $Q = \operatorname{diag}(q_1, \dots, q_n, 0, \dots, 0) \in \mathbb{R}^{n+m\times n+m}$ with $q_i > 0$ for all $i = 1, \dots, n$.

(ii) There is no connected component of G_R with nodes only contained in $\{n+1, n+2, \dots, n+m\}$.

(iii) Any load bus of the power network is connected to at least one generator bus.

Proof: The equivalence of (ii) and (iii) is obvious, hence it remains to be shown that (i) is equivalent to (ii). For any $x \in \mathbb{R}^{n+m}$ it holds

$$\begin{aligned}
x^T Rx &= \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} R_{ij}x_i x_j \\
&\overset{\text{Def.2,(i)}}{=} \sum_{i=1}^{n+m} R_{ii}x_i^2 + 2\sum_{i=2}^{n+m}\sum_{j=1}^{i-1} R_{ij}x_i x_j \\
&\overset{\text{Def.2,(ii)}}{=} -\sum_{i=1}^{n+m}\sum_{\substack{j=1\\ j\neq i}}^{n+m} R_{ij}x_i^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} R_{ij}x_i x_j \\
&= -\sum_{i=1}^{n+m}\sum_{j=1}^{i-1} R_{ij}x_i^2 - \sum_{i=1}^{n+m}\sum_{j=i+1}^{n+m} R_{ij}x_i^2 \\
&\quad + 2\sum_{i=1}^{n+m}\sum_{j=1}^{i-1} R_{ij}x_i x_j \\
&\overset{\text{Def.2,(i)}}{=} -\sum_{i=2}^{n+m}\sum_{j=1}^{i-1} R_{ij}x_i^2 - \sum_{i=2}^{n+m}\sum_{j=1}^{i-1} R_{ij}x_j^2
\end{aligned}$$

$$+2\sum_{i=2}^{n+m}\sum_{j=1}^{i-1} R_{ij}x_ix_j$$
$$= \quad -\sum_{i=2}^{n+m}\sum_{j=1}^{i-1} R_{ij}(x_i-x_j)^2 \overset{\text{Def.2,(iii)}}{\leq} 0.$$

Thus R is negative semi-definite. Now, defining

$$\Psi_1(x) := \sum_{i=2}^{n+m}\sum_{j=1}^{i-1} R_{ij}(x_i-x_j)^2 \text{ and}$$
$$\Psi_2(x) := \sum_{i=1}^{n} q_i x_i^2,$$

we obtain

$$x^T(R-Q)x = -(\Psi_1(x)+\Psi_2(x)), \tag{5}$$

where $\Psi_1(x) \geq 0$ and $\Psi_2(x) \geq 0$ for arbitrary $x \in \mathbb{R}^{n+m}$. Therefore also $R-Q$ is negative semi-definite.
Furthermore we have

$$\Psi_1(x)+\Psi_2(x) = 0 \quad \Leftrightarrow \quad \Psi_1(x) = 0 \ \wedge \ \Psi_2(x) = 0.$$

The summands of the right-hand side of (5) have the properties

$$\Psi_1(x) = 0 \quad \Leftrightarrow \quad \forall (i,j) \in \mathcal{E}_R : \ x_i = x_j, \tag{6}$$
$$\Psi_2(x) = 0 \quad \Leftrightarrow \quad \forall i \in \{1,\dots,n\} : \ x_i = 0. \tag{7}$$

Let G_R be divided into $1 \leq \xi \leq n$ connected components $C_\alpha := (\mathcal{V}_\alpha, \mathcal{E}_\alpha), 1 \leq \alpha \leq \xi$. It remains to be shown that

$$\begin{aligned} &\forall x \in \mathbb{R} : [\Psi_1(x) = 0 \wedge \Psi_1(x) = 0 \ \Rightarrow \ x = 0] \\ &\Leftrightarrow \forall \alpha \in \{1,\dots,\xi\} : \ \mathcal{V}_\alpha \cap \{1,2,\dots,n\} \neq \emptyset. \end{aligned} \tag{8}$$

"$\Leftarrow$": According to (7) from $\Psi_2(x) = 0$ it follows that $x_i = 0$ for all $i = 1,\dots,n$. Consider now x_i with $i \in \{n+1,\dots,n+m\}$ and the connected component C_α containing node i. Since $\mathcal{V}_\alpha \cap \{1,\dots,n\}$ there exists $j \in \{1,2,\dots,n\}$ and a path in the graph C_α connecting node i with node j, hence invoking $\Psi_1(x) = 0$ and (6) we conclude $x_i = x_j = 0$.
"$\Rightarrow$" (by contradiction): Assume there exists a connected component $C_{\alpha_*} = (\mathcal{V}_{\alpha_*}, \mathcal{E}_{\alpha_*})$ with $\mathcal{V}_{\alpha_*} \subseteq \{n+1,\dots,n+m\}$. Define x_*, such that

$$x_{*,i} = \begin{cases} 1, & i \in \mathcal{V}_{\alpha_*}, \\ 0, & i \in \mathcal{V} \setminus \mathcal{V}_{\alpha_*}. \end{cases}$$

Then from (6) and (7) it follows that $\Psi_1(x_*) = 0 = \Psi_2(x_*)$, but $x_* \neq 0$, which is the sought contradiction. ■

Remark 1: Lemma 2 can also be applied to $-R+Q$ instead of $R-Q$ and there are two theorems in literature, which are directly related to it. They both deal with slightly more general matrices.

- A well-known theorem (see e.g. [14]) says that "an irreducible, diagonally dominant matrix A is invertible if and only if at least one diagonal element is strictly larger than the sum of the off-diagonal elements of the corresponding row". Applying this to our setting we can see that the matrix in Lemma 2 is irreducible if and only if the graph is connected i.e. consists of only one connected component.
- In [13] it is shown that the equivalent invertibility condition of Lemma 2 is also a sufficient invertibility condition for matrices which are more general than the matrices considered in Lemma 2. It says: "Let $A \in \mathbb{C}^{n\times n}$ be diagonally dominant and let for the rows i with $i \in J \subset \{1,\dots,n\}$ hold strict inequality in (4). Then R is invertible if for each $i \notin J$ there exists a sequence of nonzero elements of A of the form $A_{ii_1}, A_{i_1i_2}, \dots, A_{i_rj}$ with $j \in J$." In contrast to Lemma 2 the proof in [13] is made by permutations and with so-called M-matrices. Assuming symmetry, real valued matrix entries and the condition on the signs of the entries simplifies not only the proof for the sufficient condition, but also makes it possible to prove the condition to be necessary.

We need a second lemma which provides a rather technical sufficient and necessary condition for the regularity of (3).

Lemma 3: Let (3) be given and define $L(s) \in (\mathbb{R}(s))^{n\times n}$

$$L(s) := -R + \begin{bmatrix} Z^{-1} - (s^2MZ^2 + sDZ^2 + Z)^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

then the matrix pair $(\mathfrak{E}, \mathfrak{A})$ from (3) is regular if and only if $\det(L(s))$ is not the zero polynomial.

Proof: First note, that $\det(s^2MZ^2 + sDZ^2 + Z) = \det(s^2MZ + sDZ + I)\det(Z)$ is not the zero polynomial because Z is invertible, hence $L(s)$ is well-defined. We define the rational matrices $Q_1(s) \in (\mathbb{R}[s])^{3n+m\times 3n+m}$ and $Q_2(s) \in (\mathbb{R}(s))^{3n+m\times 3n+m}$ as

$$Q_1(s) := \begin{bmatrix} I_n & 0 & 0 & 0 \\ Ms+D & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \text{ and}$$
$$Q_2(s) := \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & (s^2MZ+sDZ+I)^{-1} & I_n & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix},$$

and obtain

$$Q_2(s)Q_1(s)(s\mathfrak{E} - \mathfrak{A}) =: \begin{bmatrix} T_1(s) & \begin{bmatrix} 0 & 0 \\ -Z^{-1} & 0 \end{bmatrix} \\ 0 & T_2(s) \end{bmatrix},$$

where $T_1(s) \in (\mathbb{R}[s])^{2n\times 2n}$ and $T_2 \in (\mathbb{R}(s))^{n+m\times n+m}$, with

$$T_1(s) := \begin{bmatrix} sI_n & -I_n \\ s^2M + sD + Z^{-1} & 0 \end{bmatrix} \text{ and}$$
$$T_2(s) := -R + \begin{bmatrix} Z^{-1} - (s^2MZ^2 + sDZ^2 + Z)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\det(Q_1(s)) = \det(Q_2(s)) = 1$, it holds

$$\det(s\mathfrak{E} - \mathfrak{A}) = \det(T_1(s))\det(T_2(s)),$$

with $\det(T_1(s)) = \det(s^2M + sD + Z^{-1})$, which is not the zero polynomial. Hence it follows that $(\mathfrak{E}, \mathfrak{A})$ is regular if and only if $T_2(s)$ is not the zero polynomial. ■

With Lemma 2 and Lemma 3 we now have the tools to finally prove Theorem 1.

Proof: [Proof of Theorem 1] Due to Lemma 1 unique solvability of the DAE (3) is equivalent to regularity of the matrix pair $(\mathfrak{E}, \mathfrak{A})$ which in view of Lemma 3 is equivalent to the existence of $\lambda \in \mathbb{C}$ such that $L(\lambda)$ given there is invertible. Hence we have to show the following equivalence:

$$\text{Any load node is connected to a generator node} \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{C} : L(\lambda) \text{ is invertible.}$$

For that, let $Q(s) \in (\mathbb{C}(s))^{n+m \times m+m}$ with

$$Q(s) := \begin{bmatrix} Z^{-1} - (s^2 MZ^2 + sDZ^2 + Z)^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

which is a diagonal rational matrix with

$$Q_{ii}(s) = \frac{1}{Z_i} - \frac{1}{s^2 M_i Z_i^2 + sD_i Z_i^2 + Z_i}, \forall i = 1, \ldots, n.$$

"$\Rightarrow$": Due to positivity of Z_i, M_i, D_i it follows that $Q_{ii}(\lambda) > 0$ for all real $\lambda > 0$. Hence, noting that $L(s) = R - Q(s)$, we can conclude from Lemma 2 that from the property that every load bus is connected to some generator bus it follows that $L(\lambda)$ is invertible for all real $\lambda > 0$.
"$\Leftarrow$": Assume that there exists a connected component of the network graph that doesn't contain a generator. Then, by Lemma 2, $L(s) = R - Q(\lambda)$ is not invertible for any real $\lambda > 0$. Since $\det(L(s))$ is a rational function vanishing on the positive real line it follows that $\det(L(s))$ has infinitely many zeros, hence $L(s)$ is the zero polynomial and we cannot find any $\lambda \in \mathbb{C}$ such that $L(\lambda)$ is invertible. ∎

IV. Differentiation index

When dealing with solvability also in the numerical sense, it is important to analyze the index of the considered DAE (see e.g. [2]). Roughly speaking, the higher the index is, the more involved it is to solve the equations. Therefore it is preferable to have an index one system, which we show in the following theorem to be the case for (3) if regularity is provided.

Theorem 2: Let the matrix pair $(\mathfrak{E}, \mathfrak{A})$ from (3) be regular, then the DAE has index one. In particular, unique solvability of the linear DAE model (3) of a power network already implies index one.

Proof: Since (3) is a linear semi-explicit DAE, the index equals one if $R - \begin{bmatrix} Z^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is invertible. By Theorem 1 and Lemma 2, this is the case if regularity is given. ∎

It is desirable to have a similar result as Theorem 2 not only for the linearized, but also for the nonlinear model presented in Section II-A. The general case is still work in progress, but we are able to characterize the index-one-property for a simplified nonlinear version of the nonlinear DAE.

For the simplified nonlinear model of the power network we assume analogously as for the linearized model that the lines are lossless and all bus voltages are constantly one. However, we do not assume anymore that the angle differences are close to zero; we relax this assumption and assume instead that they are bounded by $\pi/2$. Since in the nonlinear framework there is no such easy regularity condition, we have to assume that the DAE is uniquely solvable, such that the index is defined. Under these assumptions we can state the following Lemma.

Lemma 4: Consider the nonlinear semi-explicit DAE (1) with $V_i(t) = 1$ for all $t \in \mathbb{R}$, $i = 1, \ldots, n$, $G_{ij} = 0$ for all $i, j = 1, \ldots, n+m$ and neglected reactive power flow, that is we consider the following DAE:

$$\dot{\alpha}_i = \omega_i, \tag{9a}$$

$$M_i \dot{\omega}_i = P_{g,i} - D_i \omega_i - \frac{V_i^0}{Z_i} \sin(\alpha_i - \theta_i), \tag{9b}$$

$$0 = \sum_{j=1}^{m+n} B_{ij} \sin(\theta_i - \theta_j) - \frac{V_i^0}{Z_i} \sin(\alpha_i - \theta_i) - P_i, \tag{9c}$$

for $i = 1, \ldots, n$ and

$$0 = \sum_{j=1}^{m+n} B_{ij} \sin(\theta_i - \theta_j) - P_i, \tag{9d}$$

for $i = n+1, \ldots, n+m$.

Assume the DAE (9) is uniquely solvable and

$$|\alpha_i - \theta_i| < \pi/2 \qquad \text{for } i = 1, \ldots, n, \text{ and} \tag{10}$$

$$|\theta_i - \theta_j| < \pi/2 \qquad \text{for } i, j = 1, \ldots, n+m. \tag{11}$$

Then the DAE (9) has index one if from every load node of the network graph there exists at least one path to a generator node.

Proof: We write the algebraic equations (9c) and (9d) compactly as $0 = g(\alpha, \omega, \theta, P)$ with corresponding function $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$. Then by Definition 5 the semi-explicit nonlinear DAE has index one, if the Jacobian matrix $\frac{\partial g}{\partial \theta}$ is non-singular, which we show in the following. To do so, we partition g in two additive terms

$$g(\alpha, \omega, \theta, P) = \tilde{h}(\alpha, \theta) + \hat{h}(\theta, P),$$

with $\tilde{h} : \mathbb{R}^n \times \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, where for $i = 1, \ldots, n+m$

$$\tilde{h}_i(\alpha, \theta) := \begin{cases} -\frac{V_i^0}{Z_i} \sin(\alpha_i - \theta_i), & i \leq n, \\ 0, & \text{otherwise} \end{cases}$$

and $\hat{h} : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, where for $i = 1, \ldots, n+m$

$$\hat{h}_i(\theta, P) := \sum_{k=1}^{n+m} B_{ik} \sin(\theta_i - \theta_k) - P_i.$$

Then it holds for the Jacobian matrix $\frac{\partial g}{\partial \theta} = \frac{\partial \tilde{h}}{\partial \theta} + \frac{\partial \hat{h}}{\partial \theta}$ that

$$\frac{\partial \tilde{h}_i}{\partial \theta_j} = \begin{cases} \frac{V_i^0}{Z_i} \cos(\alpha_i - \theta_i), & i = j \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

$$\frac{\partial \hat{h}_i}{\partial \theta_j} = \begin{cases} \sum_{k \neq j} B_{ik} \cos(\theta_i - \theta_k), & i = j, \\ -B_{ij} \cos(\theta_i - \theta_j), & j \neq i. \end{cases}$$

For any $\theta \in \mathbb{R}^{n+m}$ and $\alpha \in \mathbb{R}^n$, fulfilling (10) and (11) it holds $\cos(\theta_i - \theta_j) > 0, \forall i, j = 1, \ldots, n+m$ and $\cos(\alpha_i - \theta_i) > 0, \forall i = 1, \ldots, n$ and therefore

- $-\frac{\partial \tilde{h}}{\partial \theta}$ is a diagonal matrix with $-(\frac{\partial \tilde{h}}{\partial \theta})_{ii} < 0$ for $i = 1, \ldots, n$ as well as $(\frac{\partial \tilde{h}}{\partial \theta})_{ii} = 0$ for $i = n+1, \ldots, n+m$,
- $-\frac{\partial \hat{h}}{\partial \theta} \in \mathcal{D}_-^{n+m}$.

Hence we can apply Lemma 2 and the invertibility of the Jacobian matrix follows if any load bus is connected to a generator bus. ∎

V. EXAMPLE

To illustrate the results on solvability and index for the linearized model, we consider the 6-bus power system with two different topologies, as drawn in Figure 1. The parameters for

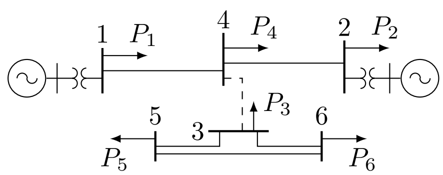

Fig. 1. A power system with 2 generator buses and 4 load buses. For topology 1 the dashed line is assumed to be existing, for topology 2 not.

the example are similar to those used in [9] and [8]. The generator parameters are given as $Z^{-1} := \text{diag}(0.0576, 0.625)$, $M := \text{diag}(0.1254, 0.034)$ and $D := \text{diag}(0.1254, 0.068)$. We consider two topologies:

1. For the first topology we assume that the dashed connection in Figure 1 exists, i.e. all buses are interconnected by paths and the solvability condition of Theorem 1 is fulfilled. To verify the regularity also numerically, we take

$$R := \left[\begin{array}{cc|cccc} -0.08 & 0 & 0 & 0.08 & 0 & 0 \\ 0 & -0.16 & 0 & 0.16 & 0 & 0 \\ \hline 0 & 0 & -0.34 & \boxed{0.07} & 0.17 & 0.1 \\ 0.08 & 0.16 & \boxed{0.07} & -0.31 & 0 & 0 \\ 0 & 0 & 0.17 & 0 & -0.26 & 0.09 \\ 0 & 0 & 0.1 & 0 & 0.09 & -0.19 \end{array}\right].$$

Defining the block matrices $\mathfrak{E}$ and $\mathfrak{A}$ as in (3), we obtain the determinant of the pencil as

$$\det(s\mathfrak{E} - \mathfrak{A}) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s,$$

with coefficients $a_4 := 0.1834 \cdot 10^{-6}, a_3 := 0.5501 \cdot 10^{-6}, a_2 := 0.5533 \cdot 10^{-6}$ and $a_1 := 0.2215 \cdot 10^{-6}$. Hence the Matrix pair $(\mathfrak{E}, \mathfrak{A})$ is regular. Moreover from Theorem 2 follows that the index equals one, which indeed is true, since

$$\det\left(\begin{bmatrix} R_1 - Z^{-1} & R_2 \\ R_3 & R_4 \end{bmatrix}\right) = 4.7019 \cdot 10^{-5}.$$

2. For the second topology we assume that the dashed connection in Figure 1 does not exist, i.e. the load buses 3, 5 and 6 are not connected to any generator bus. Thus the solvability condition of Theorem 1 is not fulfilled. We verify the singularity numerically by setting $R_{34} := 0$, $R_{43} := 0$ (consequently $R_{33} = 0.27$, $R_{44} = 0.24$) and obtain

$$(s\mathfrak{E} - \mathfrak{A}) \cdot [0, 0, 0, 0, 0, 0, 1, 0, 1, 1]^T = 0 \text{ for all } s \in \mathbb{C},$$

since $[R_2^T, R_4^T]^T [1, 0, 1, 1]^T = 0$.

VI. CONCLUSIONS

We have presented a topological characterization of the solvability of the linearized power network equation. We also showed that solvability implies index one which is an important and satisfying property: in theory, this excludes Dirac impulses in the solution and in praxis, numerical simulations do not run into trouble. We were able to extend this index-one characterization also to a simplified nonlinear model of the power network; however, a characterization of the index for the general nonlinear model is still work in progress.

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