Funnel control for systems with relative degree two

Christoph Hackl* Norman Hopfe† Achim Ilchmann‡ Markus Mueller§
Stephan Trenn¶

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Abstract

Tracking of reference signals $y_{\text{ref}}(\cdot)$ by the output $y(\cdot)$ of linear (as well as a considerably large class of nonlinear) single-input, single-output system is considered. The system is assumed to have strict relative degree two with ("weak") stable zero dynamics. The control objective is tracking of the error $e = y - y_{\text{ref}}$ and its derivative $\dot{e}$ within two prespecified performance funnels, resp. This is achieved by the so called 'funnel controller': $u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t)$, where the simple proportional error feedback has gain functions k_0 and k_1 designed in such a way to preclude contact of e and $\dot{e}$ with the funnel boundaries, resp. The funnel controller also ensures boundedness of all signals.

We also show that the same funnel controller is (i) applicable to relative degree one systems, (ii) allows for input constraints provided a feasibility condition (formulated in terms of the system data, the saturation bounds, the funnel data, bounds on the reference signal and the initial state) holds, (iii) is robust in terms of the gap metric: if a system is sufficiently close to a system with relative degree two, stable zero dynamics and positive high-frequency gain, but does not necessarily have these properties, then for small initial values the funnel controller also achieves the control objective. Finally, we illustrate the theoretical results by experimental results: the funnel controller is applied to a rotatory mechanical system for position control.

Keywords. Output feedback, relative degree two, input saturation, robustness, gap metric, linear systems, nonlinear systems, functional differential equations, transient behaviour, tracking, funnel control.

1 Introduction

We study tracking of reference signals $y_{\text{ref}}(\cdot)$ by the output $y(\cdot)$ of single-input, single-output system with (strict) relative degree two and (weak) stable zero dynamics. For the purpose of illustration, we first explain our concept for the prototype of linear single-input, single-output systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \qquad x(0) = x^0, \\ y(t) &= cx(t), \end{aligned} \tag{1.1}$$

*Institute for Electrical Drive Systems and Power Electronics, Technische Universität München, Arcisstr. 21, 80333 München, Germany, christoph.hackl@tum.de

†FERCHAU Engineering GmbH, Elsenheimerstraße 57, 80687 München, Germany, supported by DFG grant IL 25/4, Norman.Hopfe@ferchau.de

‡Institute of Mathematics, Ilmenau University of Technology, Weimarer Straße 25, 98693 Ilmenau, Germany, supported by DFG grant IL 25/4, achim.ilchmann@tu-ilmenau.de

§College of Engineering, Computing & Mathematics, University of Exeter, Exeter EX4 4QF, UK, supported by DFG grant IL 25/4, M.Mueller@ex.ac.uk

¶Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1308 W. Main Street, Urbana, 61801, USA, supported by NFS grants CNS-0614993 and ECCS-0821153, trenn@illinois.edu

where $(A,b,c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}$, $x^0 \in \mathbb{R}^n$, has *relative degree two and positive high-frequency gain*, i.e.

$$cb = 0 \quad \text{and} \quad cAb > 0 \tag{1.2}$$

and *asymptotically stable zero dynamics* (equivalently called *minimum phase*), i.e.

$$\forall s \in \mathbb{C} \quad \text{with} \quad \operatorname{Re} s \geq 0 \ : \ \det \begin{bmatrix} sI_n - A & b \\ c & 0 \end{bmatrix} \neq 0. \tag{1.3}$$

1.1 Frequency domain: high-gain, zero dynamics, internal model, tracking

The zero dynamics of a system (1.1) (and also of nonlinear systems) play an essential role for the design of a controller, see the nice textbooks [20, 21]. We discuss it on an elementary level and write, in abuse of notation, $f(s)$ for the Laplace transform of $f(t)$. The transfer function of (1.1) may be written as

$$y(s) = c(sI - A)^{-1} b\, u(s) = \gamma \frac{q(s)}{d(s)}\, u(s) \quad \text{for coprime, monic } q, d \in \mathbb{R}[s] \text{ such that } \deg d > \deg q, \tag{1.4}$$

where the relative degree of (1.4) is $r := \deg d - \deg q$ and the high-frequency gain is $\gamma \neq 0$. By the Euclidean algorithm,

$$d(s) = a(s)\, q(s) + l(s) \qquad \text{for some } a, l \in \mathbb{R}[s] \text{ such that } \deg l < \deg q, \tag{1.5}$$

and a straightforward calculation shows that system (1.4) may be written as

$$y(s) = \frac{1}{a(s)} \left[\gamma\, u(s) - \frac{l(s)}{q(s)}\, y(s) \right] . \tag{1.6}$$

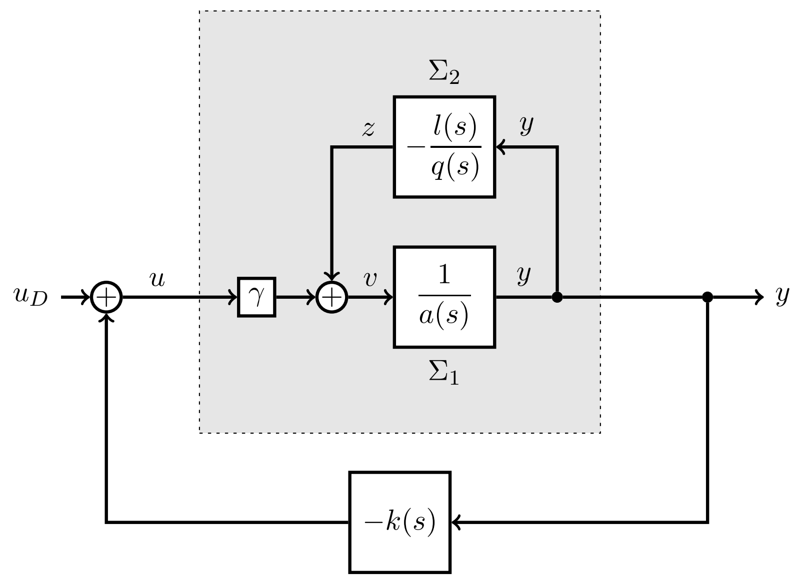


Figure 1.1: Time-invariant system decomposition

In the decomposition (1.6), see Figure 1.1, the subsystem $\Sigma_1 : v \mapsto y$ has the same relative degree r as (1.4); and the subsystem $\Sigma_2 : y \mapsto z$ is asymptotically stable if, and only if, $q(s)$ is Hurwitz. Recall that (provided (1.1) is stabilizable and detectable) (1.3) is equivalent to $q(s)$ being Hurwitz, see [10, Prop. 2.1.2], hence we see that Σ_2 captures the zero dynamics. Suppose we apply (time-invariant) derivative feedback of the form

$$u(s) = -k(s) y(s) + u_D(s), \qquad k(s) \in \mathbb{R}[s],$$

to (1.6), where u_D is a disturbance or a new input to check stability of the closed-loop system, then the transfer function of the closed-loop system is

$$y(s) = \frac{\gamma\, q(s)}{[a(s) + k(s)\gamma]\, q(s) + l(s)}\; u_D(s)\,. \tag{1.7}$$

We stress the following observations: (i) If the zero dynamics are asymptotically stable, i.e. $q(s)$ is Hurwitz, then a Hurwitz polynomial $k(s)$ may be chosen independently of the special structure of the zero dynamics to yield an asymptotically stable system (1.7). (ii) If the systems entries are unknown and only the structural assumptions of minimum phase and sign of the high-frequency gain γ of (1.4) are assumed, then we may choose $k(s) = \kappa\, \widetilde{k}(s)$ such that $\widetilde{k}(s)$ is Hurwitz and has coefficients of the same sign as γ, and for sufficiently large κ system (1.7) becomes asymptotically stable.
We illustrate (ii) for relative degree two systems (1.4) which are minimum phase and have high-frequency gain $\gamma > 0$. Let $a(s) = s^2 + a_1 s + a_0$ and choose $k(s) = k_1 s + k_0$ so that $k_0 = \gamma\,(k_1/2)^2$; then the zeros of $a(s) + \gamma\, k(s)$ are

$$s_{1;2} = -\frac{\gamma\, k_1 + a_1}{2} \pm \frac{1}{2}\sqrt{2\gamma\, a_1 k_1 + a_1^2 - 4a_0}\,,$$

and for large k_1 we have approximately $s_{1;2} \approx -(\gamma k_1 \pm \sqrt{2\gamma a_1 k_1})/2$ and so the denominator in (1.7) becomes stable for sufficiently large k_1.
These properties, and generalizations thereof, will be exploited to design adaptive controllers in the time domain in the following.
Note also that if we want to track asymptotically some reference signals $y_{\rm ref}(\cdot)$ then the internal model principle [37, Sec. 8.8] says, roughly speaking, that the feedback controller has to reduplicate the dynamics of the class of reference signal by an internal model. This internal model principle can be circumvented and the controller can be kept simple by weakening the control objective slightly: asymptotic tracking is replaced by practical tracking, i.e. the tracking error ultimately gets smaller than a prespecified error bound.

1.2 Classical adaptive control

We now explain the classical concept of high-gain adaptive control where the gain is determined adaptively. To illustrate the idea, we restrict to relative degree one systems (1.1) with positive high-frequency gain, i.e.

$$cb > 0\,, \tag{1.8}$$

and asymptotically stable zero dynamics, i.e. (1.3).
It is well known that proportional output feedback

$$u(t) = -k\, y(t) \tag{1.9}$$

applied to (1.1) yields a closed-loop system which is stable if $k > 0$ is sufficiently large: for a proof in the time domain see for example [10, Lem. 2.2.7], a proof in the frequency domain is straightforward by using the presentation in Section 1.1.
This inherent high-gain property of the system class is used in adaptive control (see the pioneering contributions by [3, 23, 24, 26, 36] and for more the survey [12]) as follows: Adaptive control means that the feedback law (1.9) becomes time-varying

$$u(t) = -k(t)\, y(t), \tag{1.10}$$

and the gain is adapted by the output, e.g.,

$$\dot{k}(t) = y(t)^2\,, \qquad k(0) = k^0. \tag{1.11}$$

If (1.10), (1.11) is applied to (1.1), then, for any initial data $x^0 \in \mathbb{R}^n$, $k^0 \in \mathbb{R}$, the closed-loop system satisfies $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$. The intuition of this adaptive control strategy is, roughly speaking, that as long as $|y(t)|$ is large, the gain $k(t)$ increases, until finally it is sufficiently large so that the closed-loop system is asymptotically stable. This control strategy, and all variants thereof, have the drawback that (i) the gain increases monotonically and, albeit bounded, may finally be very large and amplifies measurement noise and (ii) no transient behaviour is taken into account; an exception being the contribution [25] wherein the issue of prescribed transient behaviour is successfully addressed.

1.3 The funnel controller for systems with relative degree one

The fundamentally different approach of *funnel control*, introduced by [14], resolves these drawbacks. To explain the concept, we stick to the relative degree one case and consider first only output stabilization, i.e. $y_{\text{ref}} = 0$: The simplicity of the output feedback (1.10) is preserved, but the gain adaptation (1.11) is replaced by

$$k(t) = \frac{1}{\psi(t) - |y(t)|}, \tag{1.12}$$

where $\psi : \mathbb{R}_{\geq 0} \to [\lambda, \infty)$ is, for some $\lambda > 0$, a bounded differentiable function representing the funnel boundary, see Figure 2.1. Now if (1.10), (1.12) is applied to (1.1), then, for any initial data $x^0 \in \mathbb{R}^n$ such that the initial output is in the funnel: $|cx^0| < \psi(0)$, the closed-loop system has a unique solution on $\mathbb{R}_{\geq 0}$, the gain $k(\cdot)$ is bounded, and the output evolves within the funnel: $|y(t)| < \psi(t)$ for all $t \geq 0$. The intuition of funnel control is, roughly speaking, that the gain $k(t)$ is only "large" if $|y(t)|$ is "close" to the funnel boundary $\psi(t)$, and then the inherent high-gain property of the system class precludes boundary contact. Therefore, in contrast to the adaptive high-gain approach discussed above, the gain is no longer monotone, transient behavior within the funnel is guaranteed, the gain is not dynamically generated as in (1.11) and does not invoke any internal model. While in adaptive control the output (or the output error) tends to 0 as $t \to \infty$, in funnel control we may only guarantee that $\limsup_{t\to\infty} |y(t)| < \lambda$, however $\lambda > 0$ is prespecified and may be arbitrarily small.
Funnel control was introduced in [14] for systems described by functional differential equations including the class (1.1) of relative degree one systems, i.e. (1.8), with asymptotically stable zero dynamics, i.e. (1.3). It was generalized to wide classes of systems and has been successfully applied in experiments controlling the speed of electric devices [18]; see the survey [12] and references therein and further applications.
As in adaptive control, funnel control becomes a more difficult task if one has to cope with the obstacle of higher relative degree. Clearly, if (1.9) is applied to a relative-degree-two system, take for example the simple prototype $\ddot{y}(t) = u(t)$, then the closed-loop system is not asymptotically stable. In [15, 16] the concept of funnel controller has been extended to systems of higher relative degree. However, this controller involves a filter, the feedback strategy is dynamic and the gain occurs with $k(t)^6$, see [15, Rem. 4 (ii), (iii)].

1.4 Contributions of the present paper

We introduce a funnel controller with *derivative feedback* to achieve output tracking of relative-degree-two systems where a funnel for each output error and *its derivative* is prespecified to shape the transient behaviour. The funnel controller is simply

$$\boxed{u(t) = -k_0(t)^2\, e(t) - k_1(t)\, \dot{e}(t),} \tag{1.13}$$

where $k_0(\cdot), k_1(\cdot)$ are defined analogously as in (1.12) with funnel boundaries $\psi_0(\cdot)$ and $\psi_1(\cdot)$, resp., and $e(t) = y(t) - y_{\text{ref}}(t)$ is the error between the output and some desired reference signal. This simple controller applied to linear SISO systems (1.1) with stable zero dynamics and relative degree two, i.e. (1.3) and (1.2), ensures that the error and its derivative evolve within the funnels and all internal variables remain bounded. Based on the proof for the linear case, we can enlarge the system class to encompass also nonlinear systems described by functional differential equations. In addition, we are able to show that this controller also works for systems with relative degree one, i.e. we can apply this controller also in the case where only the upper bound two is known for the relative degree. Moreover, if input constraints are present, then the funnel controller is applicable provided the saturation is larger than a feasibility number. We also show that the funnel controller is robust in terms of the gap metric. Finally, our results are applied to position control with two stiff coupled machines; experimental results are shown.

These results are generalizations of results for relative degree one systems: standard funnel controller in [14], funnel control in the presence of input constraints in [19, 9], robustness of the funnel controller in [11]. It also presents a much simpler approach than in [15] to achieve tracking with prespecified bounds for the relative degree two case.

The structure of the paper is as follows. In Section 2, we introduce the funnel and state the main result for linear systems with relative degree two. Further results are presented in Section 3: In Section 3.1 nonlinear systems governed by functional differential equations are considered, Section 3.2 shows that the same funnel controller also works for relative degree one systems, funnel control in the presence of input saturation is studied in Section 3.3, and a robustness result is given in Section 3.4. The application of the proposed funnel controller to a laboratory setup of two stiff coupled machines is described in Section 4. To improve readability, all proofs are given in the Appendix; however, sketches of the proofs and intuitions are discussed in the corresponding sections.

We finalize this introduction with some nomenclature:

$\lvert x\rvert$	$=$	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\lvert M\rvert$	$=$	$\max\{\, \lvert M\,x\rvert \mid x\in\mathbb{R}^m,\ \lvert x\rvert = 1 \,\}$, induced matrix norm of $M \in \mathbb{R}^{n\times m}$
$\mathcal{L}^\infty(I\to M)$	:	the space of essentially bounded functions $y: I \to M \subseteq \mathbb{R}^\ell$, $I \subseteq \mathbb{R}$ some interval, with norm
$\lVert y\rVert_\infty := \lVert y\rVert_{\mathcal{L}^\infty}$	$=$	$\operatorname{ess\,sup}_{t\ge 0} \lvert y(t)\rvert$
$\mathcal{L}^\infty_{\text{loc}}(I\to M)$		the space of locally bounded functions $y : I \to M \subseteq \mathbb{R}^\ell$, with $\operatorname{ess\,sup}_{t\in K}\lvert y(t)\rvert < \infty$ for all compact $K \subseteq I$
$\mathcal{W}^{i,\infty}(I\to M)$	:	the Sobolev space of i-times weakly differentiable functions $y : I \to M \subseteq \mathbb{R}^\ell$ such that $y,\dots,y^{(i)} \in \mathcal{L}^\infty(I\to\mathbb{R}^\ell)$ and norm
$\lVert y\rVert_{\mathcal{W}^{i,\infty}}$	$=$	$\sum_{j=0}^{i} \lVert y^{(j)}\rVert_{\mathcal{L}^\infty}$, $i\in\mathbb{N}$,
$\mathcal{W}^{i,\infty}_{\text{loc}}(I\to M)$	:	the space of i-times weakly differentiable functions $y : I \to M \subseteq \mathbb{R}^\ell$ such that $y,\dots,y^{(i)} \in \mathcal{L}^\infty_{\text{loc}}(I\to\mathbb{R}^\ell)$
$\lVert\cdot\rVert_{\mathcal{L}^\infty\times\mathcal{W}^{i,\infty}}$	:	some product norm on the product space $\mathcal{L}^\infty(I\to M)\times\mathcal{W}^{i,\infty}(I\to M)$,
$\mathcal{C}^i(I\to M)$	:	the space of i-times continuously differentiable functions $y : I \to M \subseteq \mathbb{R}^\ell$

Note that $y \in \mathcal{W}^{i,\infty}_{(\text{loc})}(I\to M)$ implies that $y^{(i-1)}$ is absolutely continuous. Furthermore, we consider solutions of differential equations in the sense of Carathéodory, see e.g. [8, Sect. 2.1.2], and "a.a." stands for "almost all".

2 Funnel control for linear systems with relative degree two

2.1 The performance funnels

The central ingredient of our approach is the concept of two performance funnels within which the tracking error $e = y - y_{\text{ref}}$ and its derivative $\dot{e}$ are required to evolve; y_{ref} denotes a reference signal. A funnel

$$\mathcal{F}_\varphi := \{ \ (t,\eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \ \mid \varphi(t)|\eta| < 1 \ \}$$

is determined by a function φ belonging to the class

$$\mathcal{G}_1 := \left\{ \ \varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \ \left| \ \begin{array}{l} \varphi \text{ is absolutely continuous, } \forall t > 0 : \varphi(t) > 0 \text{ and} \\ \exists\, \lambda > 0 \ \forall\, \varepsilon > 0 : \ 1/\varphi\big|_{[\varepsilon,\infty)} \in \mathcal{W}^{1,\infty}\big([\varepsilon,\infty) \to [\lambda,\infty)\big) \end{array} \right. \right\}$$

Note that the funnel boundary is given by the *reciprocal* of φ. This formulation allows for $\varphi(0) = 0$ which, by $0 = \varphi(0)|e(0)| < 1$, puts no restriction on the initial value, hence we are able to prove global results. In the presence of input saturations we cannot allow for arbitrary initial values, hence we will later consider the class of *finite funnels*:

$$\mathcal{G}_1^{\text{fin}} := \left\{ \ \varphi \in \mathcal{G}_1 \ \left| \ 1/\varphi \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}) \ \right. \right\}.$$

From $\varphi \in \mathcal{G}_1^{\text{fin}}$ it follows that $\varphi(0) > 0$, however this is not sufficient as there exist funnels $\varphi \in \mathcal{G}_1$ with $\varphi(0) > 0$ which are not in $\mathcal{G}_1^{\text{fin}}$ because of an unbounded derivative $\frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi)$. Another important property of the funnel class $\mathcal{G}_1$ is that each funnel $\mathcal{F}_\varphi$ with $\varphi \in \mathcal{G}_1$ is *bounded away from zero*, i.e. there exists λ (depending on φ) such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. This condition is equivalent to the assumption that φ is bounded which should not be confused with the assumption that $1/\varphi$ is bounded corresponding to finite funnels in $\mathcal{G}_1^{\text{fin}}$. Two typical funnels are illustrated in Figure 2.1.

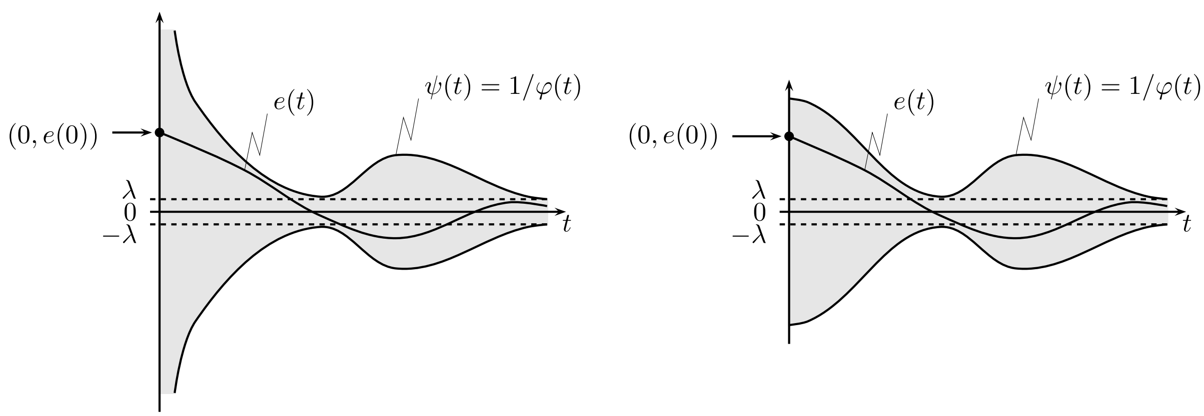


Figure 2.1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\psi(t) = 1/\varphi(t)$ for $t > 0$,
left: general funnel case $\varphi \in \mathcal{G}_1$, *right*: a finite funnel $\varphi \in \mathcal{G}_1^{\text{fin}}$.

As indicated in Figure 2.1, we do not assume that the funnel boundary decreases monotonically; whilst in most situation the control designer will choose a monotone funnel, there are situations where widening the funnel at some later time might be beneficial: e.g., when it is known that the reference signal changes strongly or the system is perturbed by some calibration so that a large error would enforces a large control action.

As mentioned above, we consider two funnels: one for the error and one for its derivative. The main control objective is to keep the error signal within prespecified error bounds, i.e. within some funnel. In order to achieve this control objective, we introduce a second funnel for the derivative of the error. This "derivative funnel" might originate in physical bounds on the derivative of the error or could be seen as a controller design parameter. If the error evolves within the funnel $\mathcal{F}_\varphi$ for some $\varphi \in \mathcal{G}_1$, then the derivative of the error eventually has to fulfill

$$\dot{e}(t) < \tfrac{\mathrm{d}}{\mathrm{d}t}(1/\varphi)(t) \quad \text{or} \quad \dot{e}(t) > -\tfrac{\mathrm{d}}{\mathrm{d}t}(1/\varphi)(t),$$

i.e. at some time the error must decrease faster than the upper funnel boundary gets smaller or the error must increase faster than the lower funnel boundary grows. This implies that the derivative funnel must be *large enough* to allow the error to follow the funnel boundaries. Therefore, we consider the following family of tuples (φ_0, φ_1):

$$\mathcal{G}_2 := \left\{ (\varphi_0, \varphi_1) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \varphi_0, \varphi_1 \in \mathcal{G}_1 \text{ and } \exists\, \delta \text{ such that for a.a. } t > 0 : \\ 1/\varphi_1(t) \geq \delta - \tfrac{\mathrm{d}}{\mathrm{d}t}(1/\varphi_0)(t) \end{array} \right\}$$

with corresponding funnel $\mathcal{F}_{\varphi_0}$ for the error and $\mathcal{F}_{\varphi_1}$ for the derivative of the error. The finite version $\mathcal{G}_2^{\text{fin}}$ is defined analogously as $\mathcal{G}_2$ by replacing $\mathcal{G}_1$ with $\mathcal{G}_1^{\text{fin}}$ in the definition.

2.2 Funnel control for linear systems with relative degree two

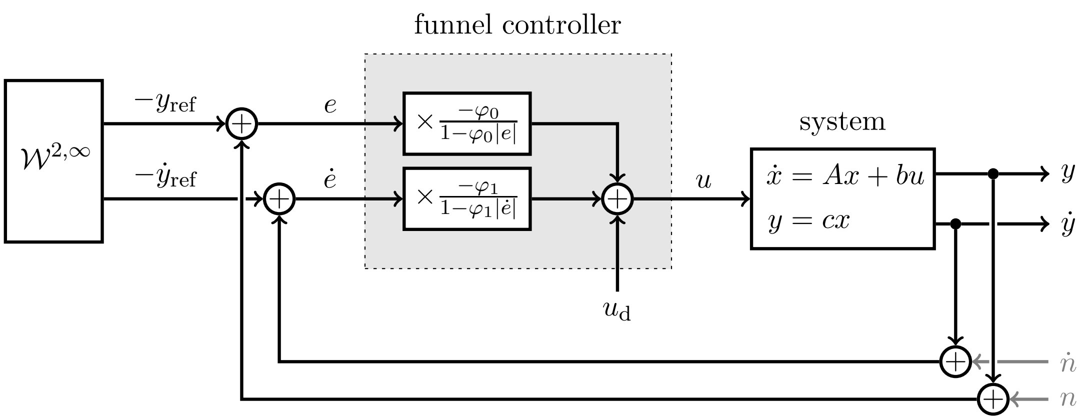


Figure 2.2: Closed-loop system (1.1), (2.2) subject to input disturbances u_d and measurement noise n, for the latter see Remark 2.2

In this section we show funnel control for linear systems with relative degree two and stable zero dynamics. This result is fundamental for various generalizations and aspects considered in Section 3.

Theorem 2.1 (Funnel control for linear systems with relative degree two)**.** *Consider linear systems* (1.1) *with relative degree two and positive high-frequency gain, i.e.* (1.2)*, and asymptotically stable zero dynamics, i.e.* (1.3)*. Let* $y_{\text{ref}} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *be a reference signal,* $u_{\text{d}} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *an input disturbance,* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *a pair of funnels for* $(\varphi_0, \varphi_1) \in \mathcal{G}_2$ *and* $x^0 \in \mathbb{R}^n$ *an initial value such that*

$$\varphi_0(0)\,|y_{\text{ref}}(0) - cx^0| < 1 \qquad \text{and} \qquad \varphi_1(0)\,|\dot{y}_{\text{ref}}(0) - cAx^0| < 1\,. \tag{2.1}$$

Then the funnel controller

$$\boxed{\begin{aligned} u(t) &= -k_0(t)^2\, e(t) - k_1(t)\,\dot{e}(t) + u_{\text{d}}(t), & e(t) &= y(t) - y_{\text{ref}}(t) \\ k_i(t) &= \frac{\varphi_i(t)}{1 - \varphi_i(t)|e^{(i)}(t)|}, & i &= 0, 1, \end{aligned}} \tag{2.2}$$

applied to (1.1) *yields a closed-loop system with the following properties:*

(i) *Precisely one maximal solution* $x : [0, \omega) \to \mathbb{R}^n$ *exists and this solution is global (i.e.* $\omega = \infty$*); in particular, the error and its derivative evolve within the corresponding funnels:*

$$\forall\, t \geq 0 : \; (t, e(t)) \in \mathcal{F}_{\varphi_0} \quad \textit{and} \quad (t, \dot{e}(t)) \in \mathcal{F}_{\varphi_1}.$$

(ii) *The input* $u(\cdot)$ *and the gain functions* $k_0(\cdot)$, $k_1(\cdot)$ *are uniformly bounded.*

(iii) *The solution* $x(\cdot)$ *and its derivative are uniformly bounded; furthermore, the signals* $e(\cdot)$, $\dot{e}(\cdot)$ *are uniformly bounded away from the funnel boundaries:*

$$\forall\, i \in \{0,1\} \; \exists\, \varepsilon_i > 0 \; \forall\, t > 0 \; : \; 1/\varphi_i(t) - |e^{(i)}(t)| \geq \varepsilon_i \,. \tag{2.3}$$

The proof is in the Appendix; however, we sketch its main ideas in the following.
First, assume without restriction of generality, that the funnels are finite: φ_0, $\varphi_1 \in \mathcal{G}_1^{\text{fin}}$; otherwise there will exist a local solution on $[0, \varepsilon)$ and we may consider the problem on the interval $[\varepsilon/2, \infty)$ instead of $[0, \infty)$. Therefore, $\psi_i := 1/\varphi_i$ denotes the finite funnel boundaries of $\mathcal{F}_{\varphi_i}$, $i = 0, 1$. Furthermore, to simplify the arguments, we assume that the derivatives of absolutely continuous functions are defined everywhere. Finally, we restrict our attention to positive errors $e(t)$, the negative case follows analogously.
In Section 1.1 we have, although in a time-invariant set up, motivated the gains: $k_0(t)^2$ for $e(t)$ (squared!) and $k_1(t)$ for $\dot{e}(t)$.
The standard theory of ordinary differential equations guarantees existence and uniqueness of a solution $x(\cdot)$ of (1.1) on $[0, \omega)$ for some maximal $\omega \in (0, \infty]$. Since e and $\dot{e}$ are bounded (they evolve within the bounded funnels), the minimum phase condition (1.3) yields that z is bounded and so there exists a constant $M > 0$ such that

$$\ddot{e}(t) < M + \gamma u(t) \quad \forall t \in [0, \omega). \tag{2.4}$$

In particular, if $u(t) \ll 0$ then $\ddot{e}(t) \ll 0$. If we knew that the product $k_0(\cdot)^2 e(\cdot)$ in the control law (2.2) is bounded, then it followed from (2.4) that $\dot{e}$ remains bounded away from the boundaries of the funnel $\mathcal{F}_1$ because we were able to choose $\varepsilon_1 > 0$ in such a way that the following implications hold, for all $t \in [0, \omega)$,

$$\begin{aligned} \dot{e}(t) = \psi_1(t) - \varepsilon_1 \qquad &\Longrightarrow \quad \ddot{e}(t) < \dot{\psi}_1(t), \\ \dot{e}(t) = -\psi_1(t) + \varepsilon_1 \qquad &\Longrightarrow \quad \ddot{e}(t) > -\dot{\psi}_1(t). \end{aligned}$$

Hence, it suffices to prove that k_0 is bounded or, equivalently, that e is uniformly bounded away from the funnel boundary, i.e. there exists $\varepsilon_0 > 0$ such that $|e(t)| \leq \psi(t) - \varepsilon_0$ for all $t \in [0, \omega)$. This is the key step of the proof, it is illustrated in Figure 2.3 and goes as follows.
Consider, for some "small" ε_0, $t_0 \geq 0$ such that $e(t_0) = \psi_0(t_0) - 2\varepsilon_0$ and $e(t) < \psi_0(t) - 2\varepsilon_0$ for some $t < t_0$. Then we show that there exists $\tau(\varepsilon_0) > 0$ such that $e(t) \leq \psi_0(t) - 2\varepsilon_0 + \tau(\varepsilon_0)$ for $t > t_0$ and that $\tau(\varepsilon_0)/\varepsilon_0 \to 0$ as $\varepsilon_0 \to 0$. This implies that, for sufficiently small $\varepsilon_0 > 0$ and all $t \geq 0$, it follows that $e(t) \leq \psi_0(t) - \varepsilon_0$. We show that the following three properties hold:

- *Parabolic phase* on $[t_0, t_1)$: $\boxed{\ddot{e}(t) < -\overline{M}(\varepsilon_0)}$ for some $\overline{M}(\varepsilon_0) > 0$ with $\overline{M}(\varepsilon_0) \to \infty$ as $\varepsilon_0 \to 0$.
- *Linear phase* on $[t_1, t_2)$: $\boxed{\dot{e}(t) < \dot{\psi}_0(t)}$.
- Once in the linear phase, we remain in it until $e(t) < \psi_0(t) - 2\varepsilon_0$.

The parabolic phase is characterized by

$$\textbf{(P)} : \; \dot{e}(t) \geq -\psi_1(t) + \delta/2,$$

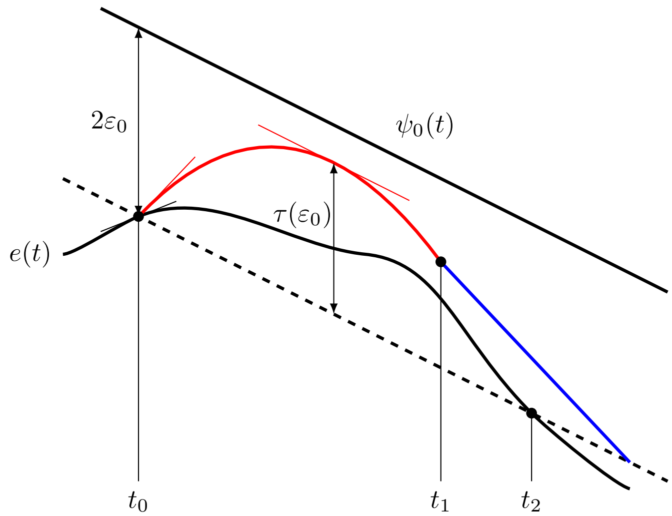


Figure 2.3: Illustration of the main idea of the proof of Theorem 2.1 showing the parabolic phase on $[t_0, t_1)$ and the linear phase on $[t_1, t_2)$

where $\delta > 0$ is given in the definition of $\mathcal{G}_2$, whilst the linear phase is characterized by

$$\textbf{(L1)}: \; e(t) \le \psi_0(t) - 2\varepsilon_0 + \tau(\varepsilon_0) \quad \text{and} \quad \textbf{(L2)}: \; \dot{e}(t) \le -\psi_1(t) + \delta/2,$$

additionally we may assume for both phases that

$$\textbf{(PL)}: \; e(t) \ge \psi_0(t) - 2\varepsilon_0.$$

Applying (PL) and (P) to the funnel controller (2.2) and for $2\varepsilon_0 \le \lambda_0/2$, we obtain

$$u(t) < -\frac{1}{(2\varepsilon_0)^2}\frac{\lambda_0}{2} + \frac{1}{\delta/2}\|\psi_1\|_\infty + \|u_{\mathrm{d}}\|_\infty,$$

which, together with (2.4), yields the proposed property $\ddot{e}(t) < -\overline{M}(\varepsilon_0)$ of the parabolic phase, where $\overline{M}(\varepsilon_0) \to \infty$ as $\varepsilon_0 \to 0$. Hence the error is bounded by a *parabola*:

$$\forall t \in [t_0, t_1): \quad e(t) < -\frac{\overline{M}(\varepsilon_0)}{2}(t - t_0)^2 + \underbrace{\dot{e}(t_0)}_{\le \|\psi_1\|_\infty}(t - t_0) + \underbrace{e(t_0)}_{\le \|\psi_0\|}.$$

In particular, there exists a maximal "overshoot" $\tau(\varepsilon_0)$ of the error starting at $\psi_0(t_0) - 2\varepsilon_0$ and we can show that $\tau(\varepsilon_0)/\varepsilon_0 \to 0$ as $\varepsilon_0 \to 0$ (here we exploit that the gain $k_0(\cdot)$ enters quadratically into the equation). The parabolic phase is only active as long as (P) holds, however if (P) does not hold, then the property of $\mathcal{G}_2$ yields

$$\dot{e}(t) \le -\psi_1(t) + \delta/2 < \dot{\psi}_0(t),$$

which ensures that the distance between the error e and the funnel boundary ψ_0 increases. Finally, it can be shown that first the parabolic phase is active for some time and either the distance of the error and the funnel boundary gets bigger than $2\varepsilon_0$ in this phase or it gets bigger than $2\varepsilon_0$ in the linear phase. Altogether, by choosing ε_0 small enough such that $\tau(\varepsilon_0) \le \varepsilon_0$, it follows that the error is uniformly bounded away from the funnel boundary with $e(t) \le \psi_0(t) - \varepsilon_0$ for all $t \ge 0$.

Remark 2.2 (Measurement noise)**.** *If system* (1.1) *is subject to measurement noise* $n(\cdot) \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$, *then the disturbed error signal is* $e = (y + n) - y_{\mathrm{ref}} = y - (y_{\mathrm{ref}} - n)$ *and the funnel controller tracks the disturbed reference signal* $y_{\mathrm{ref}} - n$. *Now Theorem 2.1 ensures that the disturbed error* e *and its derivative* $\dot{e}$ *remain within its funnels. Hence, the "real" error remains in the bigger funnel obtained by adding the corresponding bound of the noise to the funnel bounds used for the control.* ◇

3 Nonlinear systems, systems of relative degree one or two, input saturations and robustness

In this section, we show that the funnel controller (2.2) has far reaching consequences. We will show in Section 3.1 that it is also applicable to a fairly large class of nonlinear strict relative degree two systems with weakly stable zero dynamics and described by infinite-dimensional functional differential equations; in Section 3.2 it is shown that the funnel controller is applicable no matter whether the system is of relative degree one or two; in Section 3.3 we show that the funnel controller copes with input saturations if a feasibility condition is satisfied; and in Section 3.4 we show that the funnel controller is robust in terms of the gap metric.

3.1 Nonlinear and infinite-dimensional systems governed by functional differential equations

A careful inspection of the proof of Theorem 2.1 reveals that the essential property of the system (1.1) is the existence of constants $M > 0$ and $\gamma > 0$ such that

$$\forall\, t \geq 0 : \quad -M + \gamma u(t) < \ddot{e}(t) < M + \gamma u(t)\,, \tag{3.1}$$

i.e. the property that a large u implies a large value for $\ddot{e}$ with the same sign. In the following, see also Figure 3.1, we show that hence the funnel controller is also applicable to a large class of nonlinear systems described by functional differential equations as long as (i) the system has strict relative degree two with positive high-frequency gain, (ii) it is in a certain Byrnes-Isidori form, (iii) the zero dynamics map bounded signals to bounded signals, (iv) the operators involved are sufficiently smooth to guarantee local maximal existence of a solution of the close-loop system. We study the large class of infinite-dimensional nonlinear systems governed by functional differential equations with "memory" $h > 0$:

$$\ddot{y}(t) = f\big(p_f(t), T_f(y,\dot{y})(t)\big) + g\big(p_g(t), T_g(y,\dot{y})(t)\big)u(t), \quad y\big|_{[-h,0]} = y^0 \in \mathcal{W}^{1,\infty}([-h,0] \to \mathbb{R}) \tag{3.2}$$

where

- $p_f, p_g \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^P)$, $P \in \mathbb{N}$, are bounded disturbances,
- $f, g \in \mathcal{C}(\mathbb{R}^P \times \mathbb{R}^W \to \mathbb{R})$, $W \in \mathbb{N}$, sich that
$$\forall (p,w) \in \mathbb{R}^P \times \mathbb{R}^W : \quad g(p,w) > 0.$$
- $T_f, T_g : \mathcal{C}([-h,\infty) \to \mathbb{R}) \to \mathcal{L}^\infty_{\text{loc}}\big([0,\infty) \to \mathbb{R}^W\big)$ are operators with the following properties, where $T = T_f$ and $T = T_g$, resp.,
  - T maps bounded trajectories to bounded trajectories, i.e. there exists a function $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $y_0, y_1 \in \mathcal{L}^\infty([-h,\infty) \to \mathbb{R}) \cap \mathcal{C}([-h,\infty) \to \mathbb{R})$
$$\|T(y_0,y_1)\|_\infty \leq \alpha(\|y_0\|_\infty, \|y_1\|_\infty),$$
  - T is causal, i.e. for all $t \geq 0$ and all $\xi, \zeta \in \mathcal{C}([-h,\infty) \to \mathbb{R})^2$
$$\xi\big|_{[-h,t)} = \zeta\big|_{[-h,t)} \quad \Longrightarrow \quad T(\xi)\big|_{[0,t]} = T(\zeta)\big|_{[0,t]}\,,$$
  - T is "locally Lipschitz" continuous in the following sense: $\forall t \geq 0\ \exists \tau, \delta, c > 0$ such that for all $y_0, y_1, \Delta y_0, \Delta y_1 \in \mathcal{C}([-h,\infty) \to \mathbb{R})$ with $\Delta y_{0/1}\big|_{[-h,t]} \equiv 0$ and $\big\|(\Delta y_0, \Delta y_1)\big|_{[t,t+\tau]}\|_\infty < \delta$:
$$\Big\|\big(T(y_0 + \Delta y_0, y_1 + \Delta y_1) - T(y_0,y_1)\big)\big|_{[t,t+\tau]}\Big\|_\infty \leq \Big\|(\Delta y_0, \Delta y_1)\big|_{[t,t+\tau]}\Big\|_\infty .$$

For relative degree one systems, the operators T_f, T_g and similar systems as (3.2) are well studied, see [29, 13, 14, 17] and [16] for higher relative degree. In these references it is shown that: system (3.2) encompasses linear systems (1.1) with (1.2) and (1.3), and the generality of the operators T_f and T_g allows for infinite-dimensional linear systems, systems with hysteretic effects, systems with nonlinear delay elements, input-to-state stable (ISS) systems and combinations thereof.
We are now ready to state the nonlinear generalization of Theorem 2.1 for systems given by (3.2).

Theorem 3.1 (Funnel control for nonlinear functional differential equations with relative degree two)**.** *Consider systems given by* (3.2)*. Let* $y_{\rm ref} \in \mathcal{W}^{2,\infty}([0,\infty) \to \mathbb{R})$ *be a reference signal,* $u_{\rm d} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *be an input disturbance,* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *a pair of funnels for* $(\varphi_0, \varphi_1) \in \mathcal{G}_2$ *and* $y^0 \in \mathcal{W}^{1,\infty}([-h,0] \to \mathbb{R})$ *an initial trajectory such that*

$$\varphi_0(0)\,|y_{\rm ref}(0) - y^0(0)| < 1 \quad \text{and} \quad \varphi_1(0)\,|\dot y_{\rm ref}(0) - \dot y^0(0)| < 1\,. \tag{3.3}$$

Then the funnel controller (2.2) *applied to* (3.2) *yields a closed-loop system which also satisfies the properties (i)-(iii) of Theorem 2.1.*

The proof is in the Appendix.
Note that (3.2) may be written in block form as depicted in Figure 3.1.

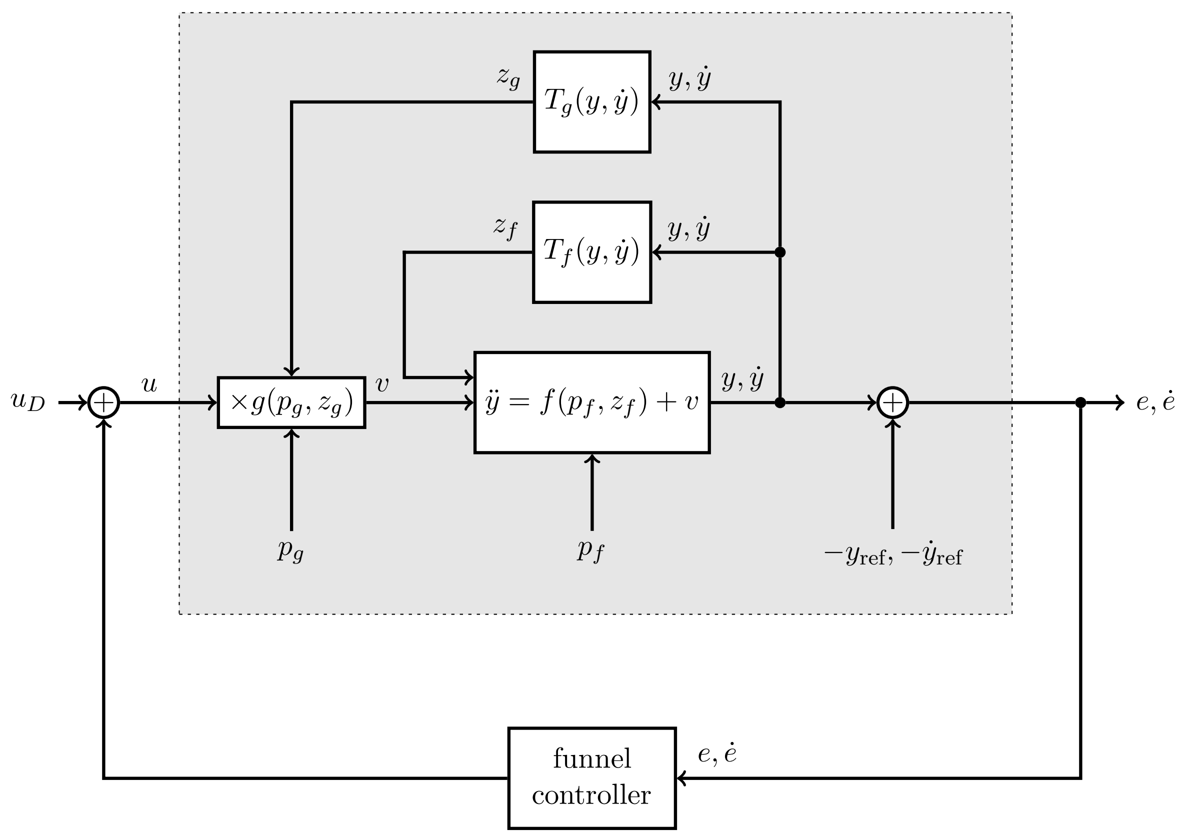


Figure 3.1: Nonlinear system decomposition

Comparing the linear and the nonlinear case, i.e. Figure 1.1 and Figure 3.1, the zero dynamics captured by Σ_2 are now captured by T_f. In [20, Sec 4.1] it is shown that for nonlinear (as opposed to linear) systems of a relative degree two, the zero dynamics in the Byrnes-Isidori form are driven by y and $\dot y$ (not only by y). Now the weak condition that T_f is a BIBO operator allows the same design of the controller as in the linear case. The function g stands for the high-frequency gain (see γ in Figure 1.1) and the assumptions on it ensures that it is uniformly bounded away from zero.

3.2 Linear systems with relative degree one

One may ask the question as to whether the funnel controller (2.2), which is designed for systems with relative degree two, also works for minimum-phase systems with relative degree one, i.e. (1.1) with (1.3) and (1.8). The answer is affirmative.

Theorem 3.2 (Relative degree one case)**.** *Consider linear systems* (1.1) *with relative degree one and positive high-frequency gain, i.e.* (1.8)*, and asymptotically stable zero dynamics, i.e.* (1.3)*. Let* $y_{\text{ref}} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *be reference signal,* $u_{\text{d}} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \cap \mathcal{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *an input disturbance,* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *a pair of funnels for* $(\varphi_0, \varphi_1) \in \mathcal{G}_2 \cap \mathcal{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R})^2$*,* $\varphi_1(0) = 0$*, and* $x^0 \in \mathbb{R}^n$ *an initial value such that* (2.1) *holds. Then the funnel controller* (2.1) *applied to* (1.1) *yields a closed-loop system which also satisfies the properties (i)-(iii) of Theorem 2.1.*

The proof is in the Appendix.

The mathematical difficulty for application of the relative degree two funnel controller to a relative degree one system is as follows: Due to the derivative feedback, the resulting closed-loop system yields an *implicit* differential equation. To utilize the Implicit Function Theorem to prove existence and uniqueness of solutions, we have to restrict slightly the class of allowed funnels and reference signals: φ_0, φ_1 and $\dot{y}_{\text{ref}}$ are assumed to be continuously differentiable instead of just being absolutely continuous. Additionally, we assume $\varphi_1(0) = 0$ for two reasons: (i) If $\varphi_1(0) > 0$, then $\dot{e}(0)$ has to fulfill $|\dot{e}(0)| < 1/\varphi_1(0)$ which might contradict the implicit differential equation. (ii) If $\varphi_1(0) = 0$, then $u(0)$ does not depend on $\dot{e}(0)$, hence the implicit ordinary differential equation is explicit for $\dot{e}$ at $t = 0$, which yields existence and uniqueness of at least a local solution starting at $t = 0$. For details see the Appendix.

3.3 Input saturation

In many practical applications, the input may be subject to certain bounds: say there is some maximal bound $\widehat{u} > 0$ such that $|u(t)| \leq \widehat{u}$ is required for all $t \geq 0$. In this case the funnel controller had to be replaced by

$$u(t) = \text{sat}_{\widehat{u}}\big(-k_0(t)^2 e(t) - k_1(t)\dot{e}(t) + u_{\text{d}}(t)\big)$$

with $e(\cdot)$, $k_0(\cdot)$, $k_1(\cdot)$ as in (2.2) and saturation function defined by

$$\text{sat}_{\widehat{u}} : \mathbb{R} \to \{\, w \in \mathbb{R} \mid |w| \leq \widehat{u} \,\}, \qquad v \mapsto \text{sat}_{\widehat{u}}(v) := \begin{cases} v, & |v| \leq \widehat{u} \\ \widehat{u} \operatorname{sgn} v, & |v| > \widehat{u}\,. \end{cases}$$

We will show that funnel control is also feasible in the presence of input constraints provided the saturation is larger than a certain feasibility number.

Theorem 3.3 (Funnel control with input saturation)**.** *Suppose the linear system* (1.1) *has relative degree two with positive high-frequency gain, i.e.* (1.2)*, and asymptotically stable zero dynamics, i.e.* (1.3)*. Let* $y_{\text{ref}} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *be a reference signal,* $u_{\text{d}} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *an input disturbance,* $(\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1})$ *a pair of finite funnels for* $(\varphi_0, \varphi_1) \in \mathcal{G}_2^{\text{fin}}$ *and* $x^0 \in \mathbb{R}^n$ *an initial value such that* (2.1) *holds. Then there exists a feasibility number* $f_{\text{feas}} > 0$ *such that, for any* $\widehat{u} \geq f_{\text{feas}}$*, the saturated funnel controller*

$$\boxed{\begin{aligned} u(t) &= \text{sat}_{\widehat{u}}\big(-k_0(t)^2 e(t_0) - k_1(t)\dot{e}(t) + u_{\text{d}}(t)\big), & e(t) &= y(t) - y_{\text{ref}}(t) \\ k_i(t) &= \frac{\varphi_i(t)}{1 - \varphi_i(t)|e^{(i)}(t)|}, & i &= 0, 1, \end{aligned}} \tag{3.4}$$

applied to (1.1) *yields a closed-loop system which also satisfies the properties (i)-(iii) of Theorem 2.1.*

The proof is in the Appendix.

As shown in Theorem 2.1, the input of the closed-loop system (1.1), (2.2) is bounded; however, in Theorem 3.3 we state that a saturated input yields the same result, provided this saturation bound is sufficiently large. In fact, we will show that the feasibility bound $f_{\text{feas}} > 0$ depends on all parameters involved in the closed-loop system. In most cases the calculated f_{feas} may be very conservative; in applications of small dimension, it may be useful. However, already for the position control problem considered in Section 4, f_{feas} is much larger than $\widehat{u}$ required in the experiments.

In the remainder of this section, we collect several bounds which in the end determine f_{feas}. This derivation has several consequences: (i) the bounds help to understand the interplay between the two different "players" $k_0(\cdot)$ and $k_1(\cdot)$; (ii) if the entries of (1.1) are known, it may be possible to determine a sharper number f_{feas}; (iii) for simplicity we have considered only symmetric funnels which is a rather hard assumption, this can be relaxed and the feasibility bound becomes smaller, see [22] for a more detailed analysis in a comparable context.

In the following, we consider the closed-loop system (1.1), (3.4). Existence and uniqueness of a solution is treated in the proof of Theorem 3.3. Here we assume that a solution exists on the whole of $\mathbb{R}_{\geq 0}$ and we may also assume, without restriction of generality, that the system (1.1) is in *Byrnes-Isidori form*; see [20] and, for an explicit calculation of the transformation, e.g. [16, Lem. 3.5]:

$$\begin{aligned} \ddot{y}(t) &= r_0 y(t) + r_1 \dot{y}(t) + s^\top z(t) + \gamma u(t), & \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix} &= \begin{pmatrix} cx^0 \\ cAx^0 \end{pmatrix}, \\ \dot{z}(t) &= p\, y + Q\, z, & z(0) &= z^0, \end{aligned} \tag{3.5}$$

where $r_0, r_1 \in \mathbb{R}$, $s, p \in \mathbb{R}^{n-2}$, $Q \in \mathbb{R}^{(n-2)\times n(n-2)}$, $z^0 \in \mathbb{R}^{n-2}$. By (1.2), the high-frequency gain is $\gamma = cAb > 0$.

3.3.1 A bound from the zero dynamics

Note that the minimum phase assumption (1.3) is equivalent to the matrix Q being *Hurwitz*, i.e.

$$\exists\, M_Q \geq 1\ \exists\, \lambda_Q > 0\ \forall\, t \geq 0\ :\ |e^{Qt}| \leq M_Q e^{-\lambda_Q t}\,. \tag{3.6}$$

Applying Variations of Constants to the second equation in (3.5) and taking norms yields

$$\forall\, t \geq 0\ :\ |z(t)| \leq M_Q e^{-\lambda_Q t} |z^0| + \int_0^t M_Q e^{-\lambda_Q (t-\tau)} |p|\, |y(\tau)|\, d\tau \leq M_Q |z^0| + \tfrac{M_Q}{\lambda_Q} \|p\|\ \left[\|y_{\text{ref}}\|_\infty + \|e|_{[0,t]}\|_\infty \right].$$

Writing

$$\begin{aligned} M_z &:= M_Q |z^0| + \tfrac{M_Q}{\lambda_Q} \|p\|\ [\|y_{\text{ref}}\|_\infty + \|\psi_0\|_\infty]\,, \\ M &:= |r_0| \|\psi_0\|_\infty + |r_1| \|\psi_1\|_\infty + |z^\top| M_z + \max\{|r_0|, |r_1|, 1\}\, \|y_{\text{ref}}\|_{W^{2,\infty}}\,, \end{aligned} \tag{3.7}$$

and observing that

$$\ddot{e}(t) = r_0 e(t) + r_1 \dot{e}(t) + s^\top z(t) + r_0 y_{\text{ref}}(t) + r_1 \dot{y}_{\text{ref}}(t) - \ddot{y}_{\text{ref}}(t) + \gamma u(t)$$

together with the fact that both e and $\dot{e}$ are bounded since they evolve within the bounded funnels, we conclude that the key inequality (3.1) holds.

3.3.2 Bounds from the parabolic phase

We consider the parabolic and linear phases as described in Section 2.2 separately to determine a sufficient large $\widehat{u}$. In the following we will only consider the case that the error e is positive, by symmetry the obtained bound will also be valid for negative errors. Choose $\varepsilon_0 > 0$ such that

$$2\varepsilon_0 \leq \frac{\lambda_0}{2}, \text{ where } \lambda_0 := \inf_{t\geq 0} \psi_0(t) > 0 \quad \text{ and } \quad 2\varepsilon_0 \leq \psi(0) - |e(0)|\,, \text{ the latter is positive by (2.1),}$$

and assume the parabolic phase is active on the interval $[t_0, t_1)$. Then, by (**P**) and (**PL**),

$$\forall\, t \in [t_0, t_1) \;:\; e(t) \geq \psi_0(t) - 2\varepsilon_0 \geq \lambda_0/2, \quad \psi_1(t) > \dot{e}(t) \geq -\psi_1(t) + \delta/2,$$

where $\delta > 0$ exists by definition of $\mathcal{G}_2$. Hence, for all $t \in [t_0, t_1)$

$$-k_0(t)^2 e(t) - k_1(t)\dot{e}(t) + u_d(t) < -\frac{\lambda_0}{8\varepsilon_0^2} + \frac{2\|\psi_1\|_\infty}{\delta} + \|u_\mathrm{d}\|_\infty =: -U_{2\varepsilon_0} \tag{3.8}$$

and if $\widehat{u} \geq U_{2\varepsilon_0}$ we obtain by (3.1), which is proved in Section 3.3.1,

$$e(t) < \frac{1}{2}(M - \gamma U_{2\varepsilon_0})(t - t_0)^2 + \dot{e}(t_0)(t - t_0) + e(t_0).$$

Since $e(t_0) = \psi(t_0) - 2\varepsilon_0$, we can easily obtain the following sufficient condition which ensures that $e(t) \leq \psi_0(t) - \varepsilon_0$ for all $t \in [t_0, t_1)$:

$$\frac{1}{2}(M - \gamma U_{2\varepsilon_0})(t - t_0)^2 + (\|\psi_1\|_\infty + \|\dot{\psi}_0\|_\infty)(t - t_0) - \varepsilon_0 \leq 0. \tag{3.9}$$

Under the assumption

$$M - \gamma U_{2\varepsilon_0} < 0 \quad \text{ or, equivalently, } \quad \varepsilon_0 < \sqrt{\frac{\gamma\lambda_0}{M_0}}\,,$$

where

$$M_0 := 8(M + 2\gamma\|\psi_1\|_\infty/\delta + \gamma\|u_\mathrm{d}\|_\infty), \tag{3.10}$$

the parabola (3.9) obtains its maximum at $t_{\max} > t_0$ which is the solution of

$$(M - \gamma U_{2\varepsilon_0})(t_{\max} - t_0) + (\|\psi_1\|_\infty + \|\dot{\psi}_0\|_\infty) = 0.$$

Some basic calculations reveal that, with M_0 as in (3.10),

$$0 < \varepsilon_0 \leq \overline{\varepsilon}_0 := \tfrac{-2(\|\psi_1\|_\infty + \|\dot{\psi}_0\|_\infty)^2}{M_0} + \sqrt{\tfrac{\gamma\lambda_0}{M_0} + \tfrac{4(\|\psi_1\|_\infty + \|\dot{\psi}_0\|_\infty)^4}{M_0^2}} < \sqrt{\frac{\gamma\lambda_0}{M_0}} \tag{3.11}$$

together with $\widehat{u} \geq U_{2\varepsilon_0}$ ensures that $e(t) \leq \psi_0(t) - \varepsilon_0$ for all $t \in [t_0, t_1)$.

3.3.3 Bounds from the linear phase

It remains to consider the linear phase on $[t_1, t_2)$ characterized by

$$\forall t \in [t_1, t_2): \quad \psi_0(t) - 2\varepsilon_0 \leq e(t) \leq \psi_0(t) - \varepsilon_0 \quad \text{ and } \quad \dot{e}(t) \leq -\psi_1(t) + \delta/2.$$

Since $-\psi_1(t) + \delta/2 \leq \dot{\psi}_0(t) - \delta/2$ for almost all $t \geq 0$, the linear phase ensures $e(t) \leq \psi_0(t) - \varepsilon_0$ for all $t \in [t_1, t_2)$. Thus we have to find a sufficient large $\widehat{u}$ which ensures that we remain in the linear phase until the distance of the error e and the funnel boundary ψ_0 is bigger than $2\varepsilon_0$. First observe that, for $2\varepsilon_0 \le \lambda_0/2$,

$$\forall t \in [t_1, t_2): \quad \frac{\lambda_0}{8\varepsilon_0^2} - \|u_{\rm d}\|_\infty \;\le\; k_0(t)^2 e(t) - u_{\rm d}(t) \;<\; \frac{\|\psi_0\|_\infty}{\varepsilon_0^2} + \|u_{\rm d}\|_\infty,$$

hence the following implications hold for all $t \in [t_1, t_2)$ and all $\varepsilon_1 \in \big(0, \max\{\lambda_1/2, \delta/2\}\big]$, where $\lambda_1 := \inf_{t\ge 0} \psi_1(t)$:

$$\begin{aligned}
\dot e(t) = -\psi_1(t) + \delta/2 \quad &\Rightarrow \quad k_0(t)^2 e(t) + k_1(t)\dot e(t) - u_{\rm d} > \tfrac{\lambda_0}{8\varepsilon_0^2} - \|u_{\rm d}\|_\infty - \tfrac{2\|\psi_1\|_\infty}{\delta} = U_{2\varepsilon_0},\\
\dot e(t) = -\psi_1(t) + \varepsilon_1 \quad &\Rightarrow \quad k_0(t)^2 e(t) + k_1(t)\dot e(t) - u_{\rm d} < \frac{\|\psi_0\|_\infty}{\varepsilon_0^2} + \|u_{\rm d}\|_\infty - \tfrac{\lambda_1/2}{\varepsilon_1} =: -U_{\varepsilon_0,\varepsilon_1}.
\end{aligned}$$

Clearly, by (3.1), for small enough ε_0 and ε_1 (and corresponding large enough $\widehat{u} \ge \max\{U_{2\varepsilon_0}, U_{\varepsilon_0,\varepsilon_1}\}$) we can ensure that the set

$$\{\ (t, \dot e) \in \mathbb{R}_{\ge 0} \times \mathbb{R} \ \mid\ -\psi_1(t) + \varepsilon_1 \le \dot e \le -\psi_1(t) + \delta/2\ \} \subseteq \mathcal{F}_{\varphi_1}$$

is positively invariant, i.e. once in the linear phase we remain there and $\dot e$ is bounded away from the funnel boundary $-\psi_1$. In fact, with M_0 as in (3.10),

$$\varepsilon_0 \le \overline{\varepsilon_0}^* := \sqrt{\frac{\gamma\lambda_0}{M_0 + 8\|\dot\psi_1\|_\infty}} \tag{3.12}$$

and, with M from (3.7),

$$\varepsilon_1 \le \overline{\varepsilon}_1(\varepsilon_0) := \frac{\gamma\lambda_1/2}{\|\dot\psi_1\|_\infty} + M + \frac{\gamma\|\psi_0\|_\infty}{\varepsilon_0^2} + \gamma\|u_{\rm d}\|_\infty \tag{3.13}$$

together with sufficiently large $\widehat{u}$ and (3.1) ensure that

$$\begin{aligned}
\dot e(t) = -\psi_1(t) + \delta/2 \quad &\Rightarrow \quad \ddot e(t) < -\|\dot\psi_1\|_\infty,\\
\dot e(t) = -\psi_1(t) + \varepsilon_1 \quad &\Rightarrow \quad \ddot e(t) > \|\dot\psi_1\|_\infty.
\end{aligned}$$

3.3.4 Feasibility number

Summarizing, if we set

$$\begin{aligned}
\varepsilon_0^{\max} &:= \min\left\{\frac{\lambda_0}{4}, \frac{\psi(0) - |cx^0 - y_{\rm ref}(0)|}{2}, \bar\varepsilon_0, \bar\varepsilon_0^*\right\}\\
\varepsilon_1^{\max} &:= \min\left\{\frac{\lambda_1}{2}, \frac{\delta}{2}, \overline{\varepsilon}_1(\varepsilon_0^{\max})\right\}
\end{aligned}$$

and

$$f_{\rm feas} := \max\left\{\frac{\lambda_0}{8(\varepsilon_0^{\max})^2} - \frac{2\|\psi_1\|_\infty}{\delta} - \|u_{\rm d}\|_\infty, \frac{\lambda_1}{2\varepsilon_1^{\max}} - \frac{\|\psi_0\|_\infty}{(\varepsilon_0^{\max})^2} - \|u_{\rm d}\|_\infty\right\},$$

then funnel control (3.4) with saturation is applicable if the saturation is larger than the feasibility number: $\widehat{u} \ge f_{\rm feas}$.

3.4 Robustness in the sense of the gap metric

We now study robustness of the funnel controller (2.2) in terms of the gap metric [38], see also [27] and the references therein.
Define the class of nominal systems (1.1) with asymptotically stable zero dynamics and relative degree two with positive high-frequency gain:

$$\mathcal{P} := \left\{ (A,b,c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n} \;\middle|\; n\in\mathbb{N}, (A,b,c) \text{ satisfies (1.2) and (1.3)} \right\}.$$

Clearly, the funnel controller, as a universal controller, is already robust for disturbed systems within the class $\mathcal{P}$. However, the aim of this section is to study robustness also for disturbances of a nominal plant $\theta = (A,b,c) \in \mathcal{P}$ which yield a disturbed plant $\widetilde{\theta} = (\widetilde{A},\widetilde{b},\widetilde{c}) \notin \mathcal{P}$. We will give sufficient conditions in terms of the gap metric for the funnel controller (2.2) to achieve the control objective if applied to a disturbed system belonging to the more general systems class

$$\widetilde{\mathcal{P}} := \left\{ (\widetilde{A},\widetilde{b},\widetilde{c}) \in \mathbb{R}^{q\times q} \times \mathbb{R}^q \times \mathbb{R}^{1\times q} \;\middle|\; q\in\mathbb{N},\ (\widetilde{A},\widetilde{b},\widetilde{c}) \text{ is stabilizable and detectable, } \widetilde{c}\widetilde{b}=0 \right\} \supsetneq \mathcal{P}.$$

In particular, the disturbance of the nominal plant can yield a plant which has a different state space dimension, has a higher relative degree than two, does not have a positive high-frequency gain and/or is not minimum phase. Note that we do not consider disturbances which yield a relative-degree-one system, the reason for this is twofold: (i) due to the implicit nature of the resulting closed-loop system, we were not able to prove the general robustness result for $cb < 0$, (ii) we have already shown in Section 3.2 that the funnel controller works for any minimum-phase, relative-degree-one system with positive high-frequency gain.
In order to define the gap metric between plants in $\widetilde{\mathcal{P}}$ we first have to introduce the *plant operator* associated to $\theta = (A,b,c) \in \widetilde{\mathcal{P}}$ as follows:

$$P_{\theta,x^0} : \mathcal{L}^\infty(\mathbb{R}_{\ge 0}\to\mathbb{R}) \to \mathcal{W}^{2,\infty}_{\text{loc}}(\mathbb{R}_{\ge 0}\to\mathbb{R}), \quad u\mapsto y, \tag{3.14}$$

where $x^0 \in \mathbb{R}^{\dim\theta}$, $\dim\theta$ is such that $A\in\mathbb{R}^{\dim\theta\times\dim\theta}$, and y is the unique output of the intial value problem:

$$\dot{x} = Ax + bu, \quad x(0)=x^0, \quad y = cx.$$

Since $cb = 0$, it is easy to see, that P_{θ,x^0} is well defined and causal, i.e. for all $u\in\mathcal{L}^\infty(\mathbb{R}_{\ge 0}\to\mathbb{R})$ it follows that the corresponding output fulfills $y\in\mathcal{W}^{2,\infty}_{\text{loc}}(\mathbb{R}_{\ge 0}\to\mathbb{R})$ and $y\big|_{[0,\tau)}$ does not depend on $u\big|_{[\tau,\sup\operatorname{dom}u)}$ for all $\tau\in\operatorname{dom}u$. With abuse of notation, we write $P\in\widetilde{\mathcal{P}}$ if there exists $\theta\in\widetilde{\mathcal{P}}$ and $x^0\in\mathbb{R}^{\dim\theta}$ such that $P = P_{\theta,x^0}$. For $P\in\widetilde{\mathcal{P}}$ define the *graph of* P as

$$\mathcal{G}_P := \left\{ (u,P(u)) \;\middle|\; u\in\mathcal{L}^\infty(\mathbb{R}_{\ge 0}\to\mathbb{R}),\ P(u)\in\mathcal{W}^{2,\infty}(\mathbb{R}_{\ge 0}\to\mathbb{R}) \right\}.$$

We are now able to define the gap metric of two systems in $\widetilde{\mathcal{P}}$:

Definition 3.4 (Directed gap metric, [6])**.** *For $P_1,P_2\in\widetilde{\mathcal{P}}$ define the (possibly empty) set*

$$\mathcal{O}_{P_1,P_2} := \left\{ \Phi : \mathcal{G}_{P_1}\to\mathcal{G}_{P_2} \;\middle|\; \Phi \textit{ is causal, surjective and } \Phi(0)=0 \right\}.$$

The directed gap *is given by*

$$\vec{\delta}(P_1,P_2) := \inf_{\Phi\in\mathcal{O}_{P_1,P_2}} \sup\left\{ \frac{\|(\Phi-I)(x)\|_{\mathcal{L}^\infty\times\mathcal{W}^{2,\infty}}}{\|x\|_{\mathcal{L}^\infty\times\mathcal{W}^{2,\infty}}} \;\middle|\; x\in\mathcal{G}_{P_1},\ \|x\|_{\mathcal{L}^\infty\times\mathcal{W}^{2,\infty}} > 0 \right\},$$

with the convention that $\vec{\delta}(P_1,P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$.

Note that this definition generalizes the gap metric for linear operators on Hilbert spaces introduced in [38], see also [34]. Note also, that we here define the system graphs and the gap metric in the signal space setting of Theorem 2.1, i.e. $\mathcal{G}_P \subset \mathcal{L}^\infty \times \mathcal{W}^{2,\infty}$. It is also possible to define the system graphs and the gap metric, resp., in different signal space settings. This may simplify the calculation of upper bounds for the gap metric. For purpose of illustration when systems are "close" in the gap metric, we consider the following example.

Example 3.5. *Consider the linear system $P \in \mathcal{P}$, $a > 0$, given by*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a^2 & 2a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [1,0]x$$

and the "disturbed system" $\widetilde{P} \in \widetilde{\mathcal{P}}$, $M > 0$, given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2a^2M^2 & 4aM^2 - 3a^2M & 6aM - 2M^2 - a^2 & 2a - 3M \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2M \end{bmatrix} u, \quad y = [-M, 1, 0, 0]\, x.$$

Their transfer functions are given by $g(s) = \frac{1}{(s-a)^2}$ and $\widetilde{g}(s) = \frac{-2M(s-M)}{(s-a)^2(s+2M)(s+M)}$, resp., hence $\widetilde{P}$ is a system with relative degree three, with negative high-frequency gain $-2M$ and with a zero M in the right-half plane, in particular, the system is not minimum phase. Both system arise from the example in [27, 6.3.1] by multiplication with $\frac{1}{s-a}$. Note that the line of arguments in [27, 6.3.1] is incomplete and we were not able to prove the following estimation for the gap metric defined in $\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}$. However, if we replace $\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}$ by $\mathcal{W}_0^{1,\infty} \times \mathcal{W}^{2,\infty}$ in the definition of graphs and gap metric, one can adopt the idea from [27, 6.3.1] to show that

$$\lim_{M\to\infty} \vec{\delta}(P, \widetilde{P}) = 0,$$

i.e. in an arbitrary small neighbourhood of the nominal plant $P \in \mathcal{P}$, we find a plant $\widetilde{P}$ which has relative degree three with negative high-frequency gain and is non-minimum phase.

We are now ready to state the main robustness result. Note that we have to assume that the funnels are *not finite*, the reason being that in the analysis we study the plant and controller as operators on certain signal spaces separately. In particular, the (bounded) signals can have arbitrary big bounds, and if the funnels are finite, we could in general not guarantee existence of a local solution for large "inputs" to the funnel controller operator because the values at $t = 0$ might not be contained within the funnels.

Theorem 3.6 (Robustness of the funnel controller)**.** *Consider funnel controller* (2.2) *with infinite funnels $\mathcal{F}_{\varphi_0}, \mathcal{F}_{\varphi_1}$, $(\varphi_0, \varphi_1) \in \mathcal{G}_2 \backslash \mathcal{G}_2^{\text{fin}}$, input disturbance $u_{\text{d}} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and reference signal $y_{\text{ref}} \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Let $\theta \in \mathcal{P}$ be a nominal system with associated zero-initial-value plant operator $P_{\theta,0}$ given by* (3.14)*. Then there exist functions $\eta : (0,\infty) \to (0,\infty)$ and $\alpha : \widetilde{\mathcal{P}} \to (0,\infty)$ such that, for $\widetilde{\theta} \in \widetilde{\mathcal{P}}$, $\widetilde{x}^0 \in \mathbb{R}^{\dim \widetilde{\theta}}$ and $r > 0$,*

$$\alpha(\widetilde{\theta})|\widetilde{x}^0| + \|(u_{\text{d}}, y_{\text{ref}})\|_{\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}} \leq r \quad \wedge \quad \vec{\delta}(P_{\theta,0}, P_{\widetilde{\theta},0}) \leq \eta(r) \tag{3.15}$$

implies that the closed loop of disturbed plant $P_{\widetilde{\theta},\widetilde{x}^0}$ and funnel controller (2.2) *works, that means the properties (i)-(iii) of Theorem 2.1 hold.*

The proof is in the Appendix.

Theorem 3.6 also holds true for $u_{\text{d}} \in \mathcal{W}_0^{1,\infty}$ and $y_{\text{ref}} \in \mathcal{W}^{2,\infty}$, which allows a simpler calculation of upper bounds for the gap metric, see Example 3.5.

Remark 3.7. *Given an input disturbance* u_{d} *and a reference signal* y_{ref} *with* $\|(u_{\mathrm{d}}, y_{\mathrm{ref}})\|_{\mathcal{L}^\infty \times \mathcal{W}^{2,\infty}} \le C$ *for some* $C > 0$*, and choose* $r > C$*. Then properties (i)-(iii) of Theorem 3.6 ensure that for any disturbed plant* $\widetilde{\theta} \in \widetilde{\mathcal{P}}$ *which is "close enough" to the nominal plant* $\theta \in \mathcal{P}$*, i.e. for which the directed gap metric is smaller than* $\eta(r)$*, the funnel controller will also work for the disturbed plant* $P_{\widetilde{\theta},\widetilde{x}^0} \in \widetilde{\mathcal{P}}$*, whenever the initial value* $\widetilde{x}^0$ *of* $P_{\widetilde{\theta},\widetilde{x}^0}$ *is "small enough", i.e.* $|\widetilde{x}^0| < (r - C)/\alpha(\widetilde{\theta})$*.*

4 Experimental results

In this section we consider a simple rotatory model for the standard position control problem and will apply the funnel controller to a laboratory setup of two stiff coupled machines, see Figure 4.1.

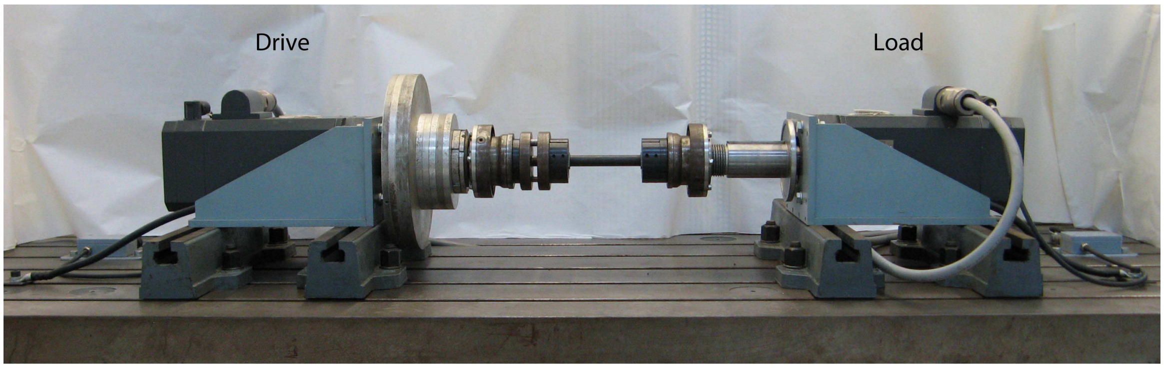


Figure 4.1: Laboratory setup of rotatory system: stiff coupled machines (drive and load)

4.1 Standard position control problem

The mathematical model of a rotatory system (translational is similar) with actuator for position control is given by

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{\Theta} \end{pmatrix} \big(\mathrm{sat}_{\widehat{u}_A}(u(t) + u_A(t)) - u_L(t) - (T_{\vartheta^0} x_2)(t)\big), \quad x(0) = \begin{pmatrix} \phi^0 \\ \Omega^0 \end{pmatrix} \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)\,, \end{aligned} \tag{4.1}$$

where the state variable $x(t) = (\phi(t), \Omega(t))^\top$ represents angle $\phi(t)$ and angular velocity $\Omega(t) = \dot{\phi}(t)$ at time $t \ge 0$ in $[rad]$ and $[rad/s]$, resp.
In the "real world", the drive (or load) torque is generated by a saturated actuator comprising inverter and machine (with current/torque control-loop), that is a nonlinear dynamical system. Since its dynamics are very fast, e.g. $u(t) \approx \mathrm{sat}_{\widehat{u}_A}(u(t) + u_A(t))$ for $|u(t) + u_A(t)| \le \hat{u}_A$ (see e.g. [30, pp. 775–779]), we model the actuator by the (small) disturbance $u_A \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})\,[Nm]$ and the saturation $\mathrm{sat}_{\widehat{u}_A}(\cdot)$ with $\widehat{u}_A > 0\,[Nm]$. The input $u(\cdot)$ $[Nm]$ represents the 'desired' drive torque. It is additionally corrupted by an external load disturbance $u_L \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$ $[Nm]$ and friction modelled by an operator $T_{\vartheta^0} : \Omega(\cdot) \mapsto (T_{\vartheta^0}\Omega)(\cdot)$ $[Nm]$ explained in the next section.
The moment of inertia $\Theta > 0\,[kg\,m^2]$ is a constant, the reciprocal of which is the high-frequency gain $\gamma := (1, 0)\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\left(\begin{smallmatrix} 0 \\ 1/\Theta \end{smallmatrix}\right) = 1/\Theta > 0$. The influence of gears and elasticity in the shaft is neglected. Note that if a gear is applied and yields a negative high-frequency gain, then the gains $k_0(t)^2$ and $k_1(t)$ in funnel controller (2.2) have to be modified to $-k_0(t)^2$ and $-k_1(t)$, resp., and the same results hold true.
The output $y(\cdot) = \phi(\cdot)$ and its derivative $\dot{y}(\cdot) = \Omega(\cdot)$ are available for feedback and corrupted by measurement noise $n \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$. The control objective is tracking of a reference signal $y_{\mathrm{ref}}(\cdot)$

and its derivative in the presence of input constraints. Although in many applications derivative feedback is a problem, in the present setup of stiff coupled machines, or more general in joint position control of robotics it may be justified, see e.g. [31, pp. 210-213 and 290-292].
We will show that (4.1), without saturation, is in the system class (3.2), hence Theorem 3.1 ensures existence and uniqueness of a global solution. Furthermore, we are able to establish inequality (3.1) and can therefore derive feasibility bounds for the controller with input saturation via Theorem 3.3.

4.2 Friction model

Friction counteracts the acceleration of the body in motion. The popular (nonlinear and dynamic) Lund-Grenoble (LuGre) friction model introduced in [4] cannot reproduce hysteretic behaviour with nonlocal memory (see [33]) and nonphysical drift phenomena may occur for small vibrational forces (see [5]). However, it is adequate for the position control problem since most of the friction effects observed in "reality" are covered: e.g. sticking, break-away (and varying break away forces), pre-sliding displacement and frictional lag; moreover, stick-slip and hunting for controllers with integrational part can be reproduced (see e.g. [4, 28]) and it can be rendered passive [2].
To explain the Lund-Grenoble friction model, we first introduce, following [28], the Stribeck function: For Coulomb friction torque u_C and static friction (stiction) torque u_S such that $0 < u_C \le u_S$, Stribeck velocity $\Omega_S > 0$, stiffness $\sigma_0 > 0$ of the bristles and $\delta_S \in [1/2, 2]$, let the Stribeck function be given by

$$\beta : \mathbb{R} \to [u_C/\sigma_0, u_S/\sigma_0], \quad \Omega \mapsto \sigma_0^{-1} \left(u_C + (u_S - u_C) \exp^{-(|\Omega|/\Omega_S)^{\delta_S}} \right). \tag{4.2}$$

The function $\beta(\cdot)$ covers the Stribeck-effect (Stribeck curve): a 'rapid' decrease in friction for increasing but very low speeds close to standstill [32].
Next, the dynamics of the average bristle deflection $\vartheta(\cdot)$ of the asperity junctions is modeled, for some angular velocity $\Omega \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R})$ and initial average bristle deflection $\vartheta^0 \in \mathbb{R}$, by

$$\dot{\vartheta}(t) = \Omega(t) - \frac{|\Omega(t)|}{\beta}(\Omega(t))\ \vartheta(t), \qquad \vartheta(0) = \vartheta^0. \tag{4.3}$$

The damping (of the deflection rate $\dot{\vartheta}(\cdot)$) and the viscous friction is modeled, for $\sigma_1, \sigma_2, \Omega_D > 0$ and $\delta_D, \delta_V \ge 1$, by

$$\sigma_D : \mathbb{R} \to [0, \sigma_1], \quad \Omega \mapsto \sigma_1 \exp^{-(|\Omega|/\Omega_D)^{\delta_D}} \qquad \text{and} \qquad \sigma_V : \mathbb{R} \to \mathbb{R}, \quad \Omega \mapsto \sigma_2 |\Omega|^{\delta_V} \operatorname{sgn}(\Omega).$$

We are now ready to define the friction operator mapping the angular velocity to the friction torque and which is parameterized by ϑ^0

$$\begin{aligned} T_{\vartheta^0} :\ & \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}) \to \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}) \\ & \Omega(\cdot) \mapsto \sigma_0 \vartheta_{\Omega(\cdot)} + \sigma_D(\Omega) \left(\Omega - \frac{|\Omega|}{\beta}(\Omega) \vartheta_{\Omega(\cdot)} \right) + \sigma_V(\Omega), \quad \text{where } \vartheta_{\Omega(\cdot)} \text{ solves (4.3).} \end{aligned} \tag{4.4}$$

Some care must be exercised to show that (4.4) is well-defined: We first show that the initial value problem (4.3) has a unique solution for each $\Omega \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$

$$\vartheta_{\Omega(\cdot)} : \mathbb{R}_{\ge 0} \to \left[-\max\left\{ u_S/\sigma_0, |\vartheta^0| \right\},\ \max\left\{ u_S/\sigma_0, |\vartheta^0| \right\} \right].$$

Existence, uniqueness and extension on $\mathbb{R}_{\ge 0}$ follows from the standard theory of linear, time-varying differential equations; furthermore it is easy to see that if $|\vartheta_{\Omega(\cdot)}(t)| \ge u_S/\sigma_0$, then

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\vartheta_{\Omega(\cdot)}(t)^2 \right) = -2 |\vartheta_{\Omega(\cdot)}(t) \Omega(t)| \Bigg(\underbrace{-\operatorname{sgn}\left(\vartheta_{\Omega(\cdot)}(t) \Omega(t) \right)}_{\in\{-1,0,1\}} + \frac{|\vartheta_{\Omega(\cdot)}}{(} t)| \beta(\Omega(t)) \Bigg) \le 0,$$

and hence $|\vartheta_{\Omega(\cdot)}(t)| \leq \max\{u_S/\sigma_0, |\vartheta^0|\}$ for all $t \geq 0$. Therefore, we have, for all $\Omega \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$,

$$\|T_{\vartheta^0}(\Omega)\|_\infty \leq \sigma_0 \max\left\{\frac{u_S}{\sigma_0}, |\vartheta^0|\right\} + \sigma_1\|\Omega\|_\infty \left(1 + \frac{\sigma_0}{u_C} \max\left\{\frac{u_S}{\sigma_0}, |\vartheta^0|\right\}\right) + \sigma_2\|\Omega\|_\infty^{\delta_V} \tag{4.5}$$

and so T_{ϑ^0} is well-defined.

Let

$$f(p,w) := \tfrac{1}{\Theta}(p+w), \quad p(t) := -u_L(t) + u_A(t), \quad g(p,w) := \tfrac{1}{\Theta}, \quad T_f(y,\dot{y}) := -T_{\vartheta_0}(\dot{y}), \quad h := 0\,.$$

Then (4.1) without saturation reads as (3.2) and fulfills all its properties. Hence, for any funnels $\mathcal{F}_{\varphi_0}$ and $\mathcal{F}_{\varphi_1}$ with $(\varphi_0,\varphi_1) \in \mathcal{G}_2$, Theorem 3.1 ensures existence of a global solution $x : [0,\infty) \to \mathbb{R}^2$ of the closed-loop of the unsaturated system (4.1) and funnel controller (2.2); in particular, y and its derivative $\dot{y}$ evolve within the funnels $\mathcal{F}_{\varphi_0}$ and $\mathcal{F}_{\varphi_1}$ around the reference signal y_{ref} and its derivative $\dot{y}_{\text{ref}}$, resp.
Furthermore, if $(\varphi_0,\varphi_1) \in \mathcal{G}_2^{\text{fin}}$, we can also show (3.1) for $\gamma = 1/\Theta$ and

$$M = \|\ddot{y}_{\text{ref}}\|_\infty + \gamma\Big(\|u_L\|_\infty + \sup\{\ \|T_{\vartheta_0}(\Omega)\|_\infty\ \mid \Omega \in \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \text{ with } \|\Omega\|_\infty \leq \|\psi_1\|_\infty\ \}\Big). \tag{4.6}$$

Note that Theorem 3.3 also holds for nonlinear systems with relative degree two if (1.1) is replaced by (3.2) (and the corresponding properties described in Sec. 3.1) and (2.1) by (3.3). Therefore, properties (i)-(iii) of Theorem 2.1 hold in the presence of input saturations for (4.1).

4.3 Controller and funnel design

We are now ready to apply the saturated funnel controller (3.4) to the stiff coupled machines in the laboratory, see Figure 4.1. We have introduced a saturation with $\widehat{u} > 0$ to prevent destruction of the actuator and for safety reasons. The considered reference signal $y_{\text{ref}} : [0,T] \to \mathbb{R}$ with $T = 40\,[s]$ for the experiment is shown in Figure 4.2.

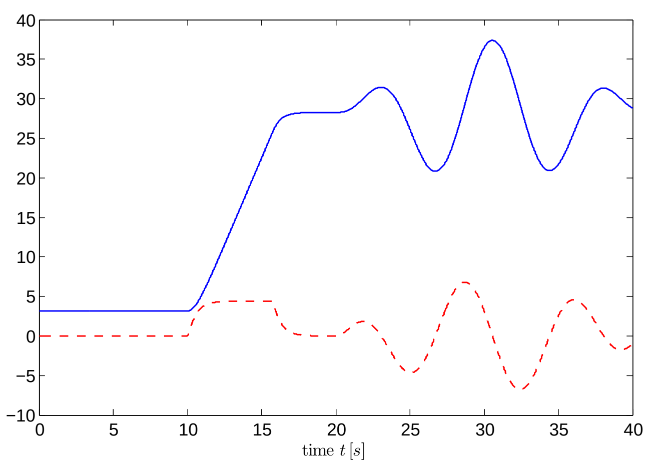


Figure 4.2: Reference signal used for experiment: —— $y_{\text{ref}}(\cdot)\,[rad]$, - - - $\dot{y}_{\text{ref}}(\cdot)\,[rad/s]$

The functions $(\varphi_0,\varphi_1) \in \mathcal{G}_2^{\text{fin}}$ determine the funnels $\mathcal{F}_{\varphi_0}$, $\mathcal{F}_{\varphi_1}$ and their reciprocals by

$$\psi_0(t) := (\Lambda_0 - \lambda_0)\exp(-t/T_E) + \lambda_0, \qquad \psi_1(t) := -\dot{\psi}_0(t) + \lambda_1, \qquad \Lambda_0 \geq \lambda_0 > 0,\ \lambda_1, T_E > 0, \tag{4.7}$$

resp. Note that $\psi_0,\psi_1 \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0})$ with $\|\psi_0\|_\infty = \Lambda_0$, $\|\dot{\psi}_0\|_\infty = (\Lambda_0 - \lambda_0)/T_E$, $\|\psi_1\|_\infty = (\Lambda_0 - \lambda_0)/T_E + \lambda_1$, $\|\dot{\psi}_1\|_\infty = (\Lambda_0 - \lambda_0)/T_E^2$ and, furthermore, $\inf_{t\geq 0}\psi_0(t) = \lambda_0$ and $\inf_{t\geq 0}\psi_1(t) = \lambda_1$.

Description	Symbol(s) & Value(s)	Dimension(s)
Moment of inertia	$\Theta = \Theta_D + \Theta_L$, $\Theta_D = 0.3333$, $\Theta_L = 0.009$	$[kg\, m^2]$
Gain (assumed bounds)	$\gamma = 1/\Theta = 2.924$ ($\gamma_{min} = \gamma/2$, $\gamma_{max} = 3\gamma$)	$\left[rad/(Nms^2)\right]$
Initial values	$\phi^0 = 0, \quad \Omega^0 = 0$	$[rad], \ [rad/s]$
Initial reference values	$y_{\rm ref}(0) = \pi, \quad \dot{y}_{\rm ref}(0) = 0$	$[rad], \ [rad/s]$
Input saturation	$\widehat{u} = 7.0$ (chosen), $\widehat{u}_A = 22.0$ (specified)	$[Nm]$
Disturbance bounds	$\|u_A\|_\infty \le 0.56$ (measured), $\|u_L\|_\infty \le 4.0$	$[Nm]$
Measured noise bounds	$\|n\|_\infty \le 5.8 \cdot 10^{-5}, \quad \|\dot{n}\|_\infty \le 0.024$	$[rad], \ [rad/s]$
Reference bounds	$\|y_{\rm ref}\|_\infty = 37.37, \ \|\dot{y}_{\rm ref}\|_\infty = 6.81, \ \|\ddot{y}_{\rm ref}\|_\infty = 6.05$	$[rad], \ [\frac{rad}{s}], \ [\frac{rad}{s^2}]$
Initial boundary values	$\psi_0(0) = \Lambda_0 = 2\pi, \quad \psi_1(0) = 8.853$	$[rad], \ [rad/s]$
Time constant	$T_E = 0.8189$	$[s]$
Asymptotic accuracies	$\lambda_0 = 0.2618, \quad \lambda_1 = 1.5$	$[rad], \ [rad/s]$
Sampling time (xPC)	$h = 1 \cdot 10^{-3}$	$[s]$

Table 1: Systems, implementation and controller design data

To check the feasibility condition in Theorem 3.3, we collect the implementation, design and system data in Table 1.
By Theorem 3.3 and neglecting (unknown) friction $T_{\vartheta^0}\Omega$ in (4.6), we conclude:

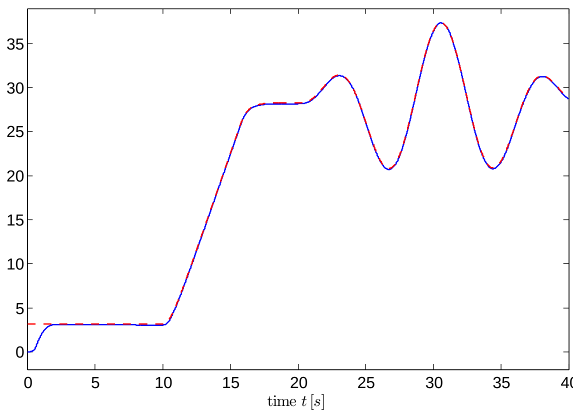
$$M = 41.14, \ M_0 = 473.72, \ \overline{\varepsilon}_0^* = 0.0265, \ \varepsilon_0^{\max} = \overline{\varepsilon}_0 = 3.64 \cdot 10^{-4}, \ \overline{\varepsilon}_1(\varepsilon_0^{\max}) = \varepsilon_1^{\max} = 1.58 \cdot 10^{-8},$$

where we used, based on worst case analysis, $\gamma = \gamma_{\max}$ for calculating M and $\gamma = \gamma_{\min}$ in (3.1) and hence in the rest of the calculation. Finally, the feasibility numbers are

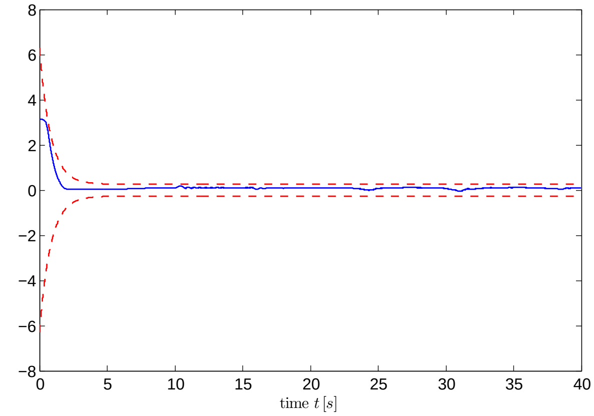
$$\widehat{u}_A \ge 2.466 \cdot 10^5 \, [Nm] \qquad \text{and} \qquad \widehat{u} \ge 2.466 \cdot 10^5 + \|u_A\|_\infty \approx 2.467 \cdot 10^5 \, [Nm] \, .$$

This computed lower bound of $\widehat{u}$ is very large and unrealistic compared to the actually required maximal torque of approximately $7.0\,[Nm]$ (see Figure 4.3e); it demonstrates how conservative the feasibility bound of Theorem 3.3 can be. A more careful derivation of the feasibility bound, following a related approach in [22], reveals the following: A main reason for the conservative bound is that the time-varying nature of the funnels is not taken into account. In fact, for constant funnels $1/\varphi_0 \equiv \Lambda_0$ and $1/\varphi_1 \equiv \lambda_1$ and known γ, one obtains the much more realistic bounds $\widehat{u}_A \ge 7.45\,[Nm]$ and $\widehat{u} \ge 8.01\,[Nm]$.

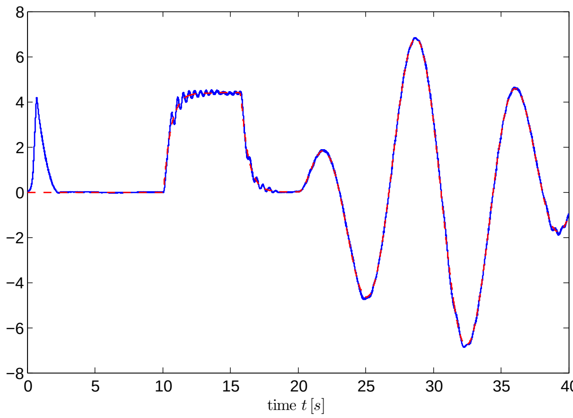
Finally, we illustrate the application of the funnel controller to the laboratory setup of two permanent magnetic synchronous machines, two power inverters, a remote host for monitoring and a real-time xPC target rapid-prototyping system. Figure 4.1 depicts the coupled machines – drive and load. Both machines and inverters are identical in construction. Each machine is driven by its own power inverter. The actuators generate the torques $u(\cdot) + u_A(\cdot)$ and $u_L(\cdot)$, resp. The build-in encoders of the machines provide position (and velocity) information and allow field-oriented control of each machine. The motor drive accelerates or decelerates $\Theta = \Theta_D + \Theta_L$ (the sum of drive Θ_D and load Θ_L inertia), whereas the load drive emulates external loads u_L. Due to the stiff shaft and an appropriate ratio $\Theta_L/\Theta_D = 2.7\%$ (see Tab. 1), the coupling can be considered stiff regarding the operation bandwidth $< 10\,[Hz]$. The dynamics (faster than $1 \cdot 10^{-3}\,[s]$) of each actuator are negligible compared to those of the mechanical system (4.1) (see also the experiments in e.g. [7, 18]).
Figure 4.3 depicts the measurements for the funnel controller (3.4) at the laboratory setup. The control error and its derivative remain within the prescribed funnel (see Figure 4.3 b,d). The control gains are adjusted "instantaneously" (see Figure 4.3 f) so that boundary contact is excluded. The



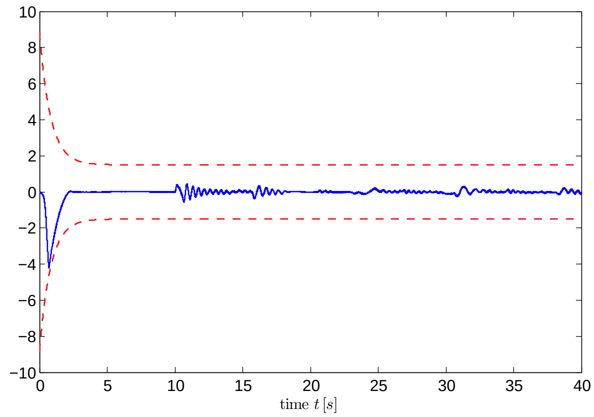
(a) $y(\cdot) + n(\cdot)$ [rad] and $y_{\text{ref}}(\cdot)$ [rad]



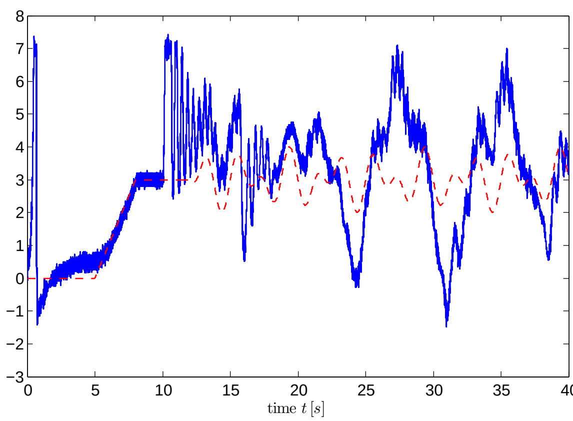
(b) $e(\cdot)$ [rad] and $\pm\psi_0(\cdot)$ [rad]



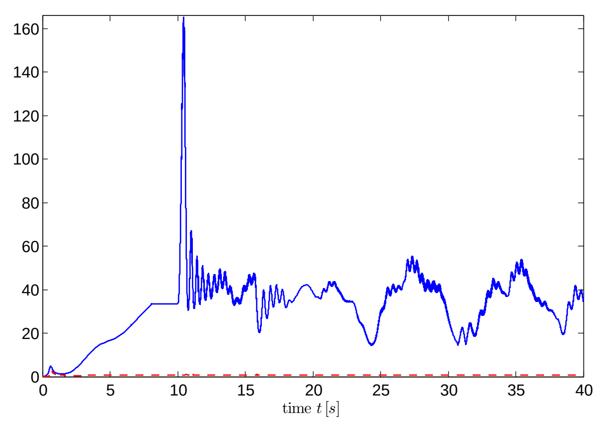
(c) $\dot{y}(\cdot) + \dot{n}(\cdot)$ [$\frac{rad}{s}$] and $\dot{y}_{\text{ref}}(\cdot)$ [$\frac{rad}{s}$]



(d) $\dot{e}(\cdot)$ [$\frac{rad}{s}$] and $\pm\psi_1(\cdot)$ [$\frac{rad}{s}$]



(e) $u(\cdot) + u_A(\cdot)$ [Nm] and $u_L(\cdot)$ [Nm]



(f) $k_0(\cdot)^2$ [$\frac{Nm}{rad}$] and $k_1(\cdot)$ [$\frac{Nms}{rad}$]

Figure 4.3: Experimental results of the controller design at the laboratory setup

funnel controller is capable of tracking the time-varying reference (see Figure 4.3 a,c) with prescribed accuracy (see Figure 4.3 b,d) also when load torques $u_L(\cdot) \neq 0$ are induced (see Figure 4.3 e). Noise amplification (see Figure 4.3 e) and "oscillations" in gains, speed and torque (see Figure 4.3 d,e,f) are acceptable.

A Appendix: Proofs

To simplify the notation we introduce, for $(\varphi_0, \varphi_1) \in \mathcal{G}_2$, the funnel boundaries

$$\psi_i : (0,\infty) \to (0,\infty), \quad t \mapsto 1/\varphi_i(t), \quad i = 0,1. \tag{A.1}$$

A.1 Proof of Theorem 2.1: Funnel control for linear systems with relative degree two.

Without restriction of generality, we may assume that system (1.1) is in Byrnes-Isidori form (3.5). The main difficulties in proving Theorem 2.1 is first that the closed-loop initial-value problem (1.1), (2.2) has a potential singularity (a pole) on the right hand side of the differential equation and, secondly, to show that the solution does not have a finite escape time, i.e. exists globally on $[0,\infty)$.

Step 1: We show existence and uniqueness of a maximal solution.
Define, for $(\varphi_0, \varphi_1) \in \mathcal{G}_2$,

$$\mathcal{D} := \left\{ (t, \mu_0, \mu_1, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \;\middle|\; (t,\mu_0) \in \mathcal{F}_{\varphi_0},\ (t,\mu_1) \in \mathcal{F}_{\varphi_1} \right\} \tag{A.2}$$

and $f : \mathcal{D} \to \mathbb{R}^n$ by

$$f(t,\mu_0,\mu_1,\xi) = \begin{pmatrix} \left[\begin{smallmatrix} 0, & 1 \\ r_0, & r_1 \end{smallmatrix}\right] \left(\begin{smallmatrix} y_{\mathrm{ref}}(t)+\mu_0 \\ \dot y_{\mathrm{ref}}(t)+\mu_1 \end{smallmatrix}\right) - \left[\begin{smallmatrix} 0 \\ s^\top \end{smallmatrix}\right]\xi - \left(\begin{smallmatrix} \dot y_{\mathrm{ref}}(t) \\ \ddot y_{\mathrm{ref}}(t) \end{smallmatrix}\right) + \gamma \left(\begin{smallmatrix} 0 \\ u_{\mathrm{d}}(t) - \frac{\varphi_0(t)^2 \mu_0}{(1-\varphi_0(t)|\mu_0|)^2} - \frac{\varphi_1(t)\mu_1}{1-\varphi_1(t)|\mu_1|} \end{smallmatrix}\right) \\ \left[\begin{smallmatrix} p, & 0 \end{smallmatrix}\right] \left(\begin{smallmatrix} y_{\mathrm{ref}}(t)+\mu_0 \\ \dot y_{\mathrm{ref}}(t)+\mu_1 \end{smallmatrix}\right) + Q\xi \end{pmatrix}.$$

The relative degree two condition (1.2) implies $\gamma = cAb > 0$, and the minimum-phase condition (1.3) is equivalent to Q being Hurwitz, i.e. (3.6). Then the initial-value problem (3.5), (2.2) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} e(t) \\ \dot e(t) \\ z(t) \end{pmatrix} = f(t, e(t), \dot e(t), z(t)), \qquad \begin{pmatrix} e(0) \\ \dot e(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} cx^0 - y_{\mathrm{ref}}(0) \\ cAx^0 - \dot y_{\mathrm{ref}}(0) \\ z^0 \end{pmatrix}. \tag{A.3}$$

Clearly, f is locally Lipschitz in μ_0, μ_1 and ξ and measurable in t, hence the theory of ordinary differential equations, see e.g. [35, Thm. III.§10.XX], ensures existence of a unique absolutely continuous solution $(e, \dot e, z) : [0,\omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$, $0 < \omega \leq \infty$, which is maximally extended, i.e. the graph of the solution is not completely contained in any compact subset of $\mathcal{D}$.
In the following, let $(e, \dot e, z) : [0,\omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ be the unique and maximally extended solution of the closed-loop initial-value problem (A.3).

Step 2: We show that there exists $M > 0$ such that

$$\text{for a.a. } t \in [0,\infty) : \quad \boxed{-M + \gamma u(t) < \ddot e(t) < M + \gamma u(t)}. \tag{A.4}$$

By continuity of $e(\cdot)$, $\dot e(\cdot)$, $z(\cdot)$ and the corresponding $k_0(\cdot)$, $k_1(\cdot)$, there exists $\varepsilon \in (0,\omega)$ such that

$$\forall\, i = 0,1\ \forall\, t \in [0,\varepsilon] \;:\quad |e^{(i)}(t)| \leq |e^{(i)}(0)| + 1, \quad |z(t)| \leq |z^0| + 1, \quad k_i(t) \leq k_i(0) + 1\,. \tag{A.5}$$

Hence it suffices to consider the interval $[\varepsilon,\omega)$ and we may adopt the notation (A.1). By definition of $\mathcal{G}_1$ we have that $\|\psi_i\|_{\varepsilon,\infty} := \left\| \psi_i|_{[\varepsilon,\infty)} \right\|_\infty$, $i = 0,1$, and hence

$$\forall t \in [\varepsilon,\omega) : \quad |e(t)| < \psi_0(t) \leq \|\psi_0\|_{\varepsilon,\infty} \quad \text{and} \quad |\dot e(t)| < \psi_1(t) \leq \|\psi_1\|_{\varepsilon,\infty}. \tag{A.6}$$

Applying Variation of Constants to the third subsystem in (A.3) yields

$$\forall\, t \in [\varepsilon, \omega): \quad z(t) = \mathrm{e}^{Q(t-\varepsilon)} z(\varepsilon) + \int_\varepsilon^t \mathrm{e}^{Q(t-\tau)} p \left[y_{\mathrm{ref}}(\tau) - e(\tau) \right] \mathrm{d}\tau,$$

and thus, in view of (3.6), (A.5) and (A.6), it follows that

$$\forall t \in [0, \omega): \ \|z(t)\| \le M_z, \tag{A.7}$$

where, with $M_Q \ge 1$ and $\lambda_Q > 0$ as in (3.6),

$$M_z := M_Q[|z^0| + 1] + \frac{M_Q}{\lambda_Q} \|p\| \left[\|y_{\mathrm{ref}}\|_\infty + \|\psi_0\|_\infty \right].$$

Since, for almost all $t \in [0, \omega)$,

$$\ddot{e}(t) = r_0 \left[e(t) + y_{\mathrm{ref}}(t) \right] + r_1 \left[\dot{e}(t) + \dot{y}_{\mathrm{ref}}(t) \right] + s^\top z(t) - \ddot{y}_{\mathrm{ref}}(t) + \gamma u(t),$$

we obtain, by invoking (A.5) and (A.6), the claimed inequality (A.4) for

$$\begin{aligned} M :=& |r_0| \left[\max\{\|e(0)\| + 1, \|\psi_0\|_{\varepsilon,\infty}\} + \|y_{\mathrm{ref}}\|_\infty \right] + |r_1| \left[\max\{|\dot{e}(0)| + 1, \|\psi_1\|_{\varepsilon,\infty}\} + \|\dot{y}_{\mathrm{ref}}\|_\infty \right] \\ &+ |s^\top| M_z + \|\ddot{y}_{\mathrm{ref}}\|_\infty. \end{aligned}$$

Step 3: We show that $|e(\cdot)|$ is uniformly bounded away from the funnel boundary $\psi_0(\cdot)$, more precisely:

$$\exists\, \varepsilon_0 > 0 \ \forall\, t \in [\varepsilon, \omega): \quad \boxed{\psi_0(t) - |e(t)| \ge \varepsilon_0}. \tag{A.8}$$

Consider two phases: a *parabolic phase* and a *linear phase*. In the parabolic phase the distance of the error $e(\cdot)$ to the funnel boundary $\psi(\cdot)$ is bounded by a parabola as formalized by Steps 3a. Step 3b ensures that the "overshoot" of this parabola can be made sufficiently small. In the linear phase, the distance of the error and the funnel boundary grows linearly as formalized in Step 3c, and Step 3d ensures that the linear phase remains active as long as the error is close to the boundary.

Step 3a: We show that for $\varepsilon_0 \in (0, \lambda_0/2)$ the following implication holds on any interval $[t_0, t_1] \subseteq [\varepsilon, \omega)$:

$$\begin{aligned} &\left[\psi_0(t_0) - |e(t_0)| = 2\varepsilon_0 \ \wedge \ \text{for a.a. } t \in [t_0, t_1] \ : \ \ddot{e}(t) \operatorname{sgn} e(t) \le -(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2 / (2\varepsilon_0) \right] \\ &\qquad\qquad\qquad\qquad \Longrightarrow \quad \forall\, t \in [t_0, t_1]: \ \psi_0(t) - |e(t)| \ge \varepsilon_0. \end{aligned} \tag{A.9}$$

First note that there exists a countable family of pairwise disjoint intervals $T_i = [\underline{\tau}_i, \overline{\tau}_i]$, $i \in \mathcal{I}$, and $S_j = (\underline{\sigma}_i, \overline{\sigma}_i)$, $j \in \mathcal{J}$, with $[t_0, t_1] \subseteq \bigcup_{i \in \mathcal{I}} T_i \cup \bigcup_{j \in \mathcal{I}} S_j$ such that

$$\begin{aligned} &\forall i \in \mathcal{I}: \quad \psi_0(\underline{\tau}_i) - |e(\underline{\tau}_i)| = 2\varepsilon_0 \quad \wedge \quad && \forall t \in T_i: \ \psi_0(t) - |e(t)| \le 2\varepsilon_0, \\ &\forall j \in \mathcal{J}: && \forall t \in S_j: \ \psi_0(t) - |e(t)| > 2\varepsilon_0. \end{aligned}$$

On the intervals S_j, $j \in \mathcal{J}$, the conclusion of (A.9) is trivially true, hence we only have to consider the intervals T_i, $i \in \mathcal{I}$, i.e. to show (A.9) under the additional assumption

$$\forall\, t \in [t_0, t_1] \ : \ \psi_0(t) - |e(t)| \le 2\varepsilon_0. \tag{A.10}$$

From $\lambda_0 > 2\varepsilon_0$ it follows that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. We only consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, the case $\operatorname{sgn} e(\cdot) \equiv -1$ follows analogously.

Integrating the inequality $\ddot{e}(\cdot) \le -(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2/(2\varepsilon_0)$ twice over $[t_0, t]$ yields

$$\forall\, t \in [t_0, t_1] \;:\; e(t) \le e(t_0) - \tfrac{(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2}{4\varepsilon_0}(t-t_0)^2 + \underbrace{\dot{e}(t_0)}_{\le \|\psi_1\|_{\varepsilon,\infty}}\,(t-t_0),$$

and in combination with the inequality $\psi_0(t) \ge \psi_0(t_0) - \|\dot{\psi}_0\|_{\varepsilon,\infty}(t-t_0)$ we conclude, for all $t \in [t_0, t_1]$,

$$\psi_0(t) - e(t) \ge \underbrace{\psi_0(t_0) - e(t_0)}_{=2\varepsilon_0} - \Big((\|\dot{\psi}_0\|_{\varepsilon,\infty} + \|\psi_1\|_{\varepsilon,\infty})(t-t_0) - \tfrac{(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2}{4\varepsilon_0}(t-t_0)^2 \Big).$$

The parabola $t \mapsto (\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})(t-t_0) - \frac{(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2}{4\varepsilon_0}(t-t_0)^2$ attains its maximum at $t - t_0 = \frac{2\varepsilon_0}{\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty}}$ with the maximum value ε_0, hence

$$\forall\, t \in [t_0, t_1] \;:\quad \psi_0(t) - e(t) \ge \varepsilon_0.$$

This proves Step 3a.

Step 3b: We show that there exists $\bar{\varepsilon}_0 \in (0, \lambda_0/4]$ such that the following implication holds on any interval $[t_0, t_1] \subseteq [\varepsilon, \omega)$ and for all $\varepsilon_0 \in (0, \bar{\varepsilon}_0]$:

$$\begin{aligned} &\big[\forall\, t \in [t_0, t_1] \;:\; \dot{e}(t)\operatorname{sgn} e(t) \ge -\psi_1(t) + \delta/2 \;\wedge\; \psi_0(t) - \big|e(t)\big| \le 2\varepsilon_0\,\big] \\ &\qquad\Longrightarrow\quad \text{for a.a. } t \in [t_0, t_1] \;:\; \ddot{e}(t)\operatorname{sgn} e(t) \le -(\|\psi_1\|_{\varepsilon,\infty} + \|\dot{\psi}_0\|_{\varepsilon,\infty})^2/(2\varepsilon_0)\,. \end{aligned} \tag{A.11}$$

The condition $2\varepsilon_0 \le \lambda_0/2$ together with $\psi_0(t) - \big|e(t)\big| \le 2\varepsilon_0$ on $[t_0, t_1]$ implies that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. We only consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, $\operatorname{sgn} e(\cdot) \equiv -1$ follows analogously.
The condition $\dot{e}(t) \ge \delta/2 - \psi_1(t)$ on $[t_0, t_1]$ implies that

$$\forall\, t \in [t_0, t_1] \;:\quad -k_1(t)\dot{e}(t) = \frac{-\dot{e}(t)}{\psi_1(t) - |\dot{e}(t)|} \le \frac{2|\dot{e}(t)|}{\delta} < \frac{2\|\psi_1\|_{\varepsilon,\infty}}{\delta}.$$

From $\psi_0(t) - e(t) \le 2\varepsilon_0$ and $2\varepsilon_0 \le \lambda_0/2$, it follows that $e(t) \ge \lambda_0/2$ on $[t_0, t_1]$ and hence

$$\forall\, t \in [t_0, t_1] :\quad -k_0(t)^2 e(t) \le -\frac{\lambda_0}{8\varepsilon_0^2}.$$

Inserting these inequalities into (A.4) and invoking (2.2) yields

$$\text{for a.a. } t \in [t_0, t_1] \;:\; \ddot{e}(t) < M - \gamma\,\frac{\lambda_0}{8\varepsilon_0^2} + \gamma\,\frac{2\|\psi_1\|_{\varepsilon,\infty}}{\delta} + \gamma\|u_{\mathrm{d}}\|_\infty, \tag{A.12}$$

whence (A.11) for sufficiently small $\varepsilon_0 > 0$.

Step 3c: We show the following implication:

$$\begin{aligned} &\big[\forall\, t \in [t_1, t_2] \;:\; \dot{e}(t)\operatorname{sgn} e(t) \le -\psi_1(t) + \delta/2 \qquad\wedge\qquad \psi_0(t) - |e(t)| \le 2\varepsilon_0\,\big] \\ &\qquad\qquad\Longrightarrow\quad t \mapsto \psi_0(t) - |e(t)| \text{ is monotonically increasing on } [t_1, t_2]. \end{aligned} \tag{A.13}$$

Note that the presupposition in (A.13) precludes a sign change of $e(\cdot)$ on $[t_1, t_2]$. We only consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, the other case follows analogously. Invoking the definition of $\mathcal{G}_2$,

$$\text{for a.a. } t \in [t_1, t_2] \;:\; \dot{\psi}_0(t) - \dot{e}(t) \ge \dot{\psi}_0(t) + \psi_1(t) - \delta/2 \ge \delta - \delta/2 = \delta/2$$

yields (A.13).

Step 3d: We show that there exists $\bar{\varepsilon}_0^* \in (0, \lambda_0/4]$ such that the following implication holds for any $[t_1, t_2] \subseteq [\varepsilon, \omega)$ and any $\varepsilon_0 \in (0, \bar{\varepsilon}_0^*]$:

$$\left[\, \dot{e}(t_1) \operatorname{sgn} e(t_1) = -\psi_1(t_1) + \delta/2 \;\wedge\; \forall t \in [t_1, t_2] :\; \psi_0(t) - |e(t)| \le 2\varepsilon_0 \,\right] \implies \quad \forall t \in [t_1, t_2] :\; \dot{e}(t) \operatorname{sgn} e(t) \le -\psi_1(t) + \delta/2. \quad \text{(A.14)}$$

From $2\varepsilon_0 \le \lambda_0/2$ and $\psi_0(t) - |e(t)| \le 2\varepsilon_0$ it follows that $\operatorname{sgn} e(\cdot)$ is constant on $[t_1, t_2]$, we only consider $\operatorname{sgn} e(\cdot) \equiv 1$ here, the negative case follows analogously.
We show that the existence of $\hat{t} \in (t_1, t_2]$ with $\dot{e}(\hat{t}) > -\psi_1(\hat{t}) + \delta/2$ yields a contradiction to the assumptions of the implication (A.14). Therefore, choose $\hat{t}_1 \in [t_1, \hat{t})$ with $\dot{e}(\hat{t}_1) = -\psi_1(\hat{t}_1) + \delta/2$ and $\dot{e}(t) \ge -\psi_1(t) + \delta/2$ for all $t \in [\hat{t}_1, \hat{t}]$. Together with $\psi_0(t) - e(t) \le 2\varepsilon_0$ we can conclude as in Step 3b that (A.12) holds for the interval $[\hat{t}_1, \hat{t}]$, hence for small enough $\bar{\varepsilon}_0^*$ and all $\varepsilon_0 \in (0, \bar{\varepsilon}_0^*]$:

$$\forall t \in [\hat{t}_1, \hat{t}] :\; \ddot{e}(t) < -\|\dot{\psi}_1\|_{\varepsilon,\infty}.$$

Now,

$$\delta/2 < \dot{e}(\hat{t}) + \psi_1(\hat{t}) = \dot{e}(\hat{t}_1) + \psi_1(\hat{t}_1) + \int_{\hat{t}_1}^{\hat{t}} \ddot{e}(\tau) + \dot{\psi}_1(\tau)\, \mathrm{d}\tau < \dot{e}(\hat{t}_1) + \psi_1(\hat{t}_1) = \delta/2,$$

whence a contradiction to the choice of $\hat{t}$.

Step 3e: We show that for sufficiently small $\varepsilon_0 > 0$ the claim of Step 3 holds.

Choose $\varepsilon_0 > 0$ such that (A.11), (A.14) and $\psi_0(\varepsilon) - |e(\varepsilon)| \ge 2\varepsilon_0$ hold. Seeking a contradiction, assume that there exists $t_2 \in (\varepsilon, \omega)$ such that $\psi_0(t_2) - |e(t_2)| < \varepsilon_0$. Choose $t_0 \in [\varepsilon, t_2)$ such that

$$\psi_0(t_0) - |e(t_0)| = 2\varepsilon_0 \quad \text{and} \quad \forall t \in [t_0, t_2] :\; \psi_0(t) - |e(t)| \le 2\varepsilon_0. \quad \text{(A.15)}$$

Since $2\varepsilon_0 < \lambda_0$, it follows that $e(\cdot)$ has a constant sign on $[t_0, t_2]$, we consider here only the positive case, the negative follows analogously. It follows from (A.15) that there exists $\nu > 0$ such that

$$\text{for a.a. } t \in (t_0, t_0 + \nu] : \quad \dot{\psi}_0(t) - \dot{e}(t) \le 0,$$

hence, by the property of $\mathcal{G}_2$:

$$\text{for a.a. } t \in (t_0, t_0 + \nu] : \quad \dot{e}(t) \ge -\psi_1(t) + \delta > -\psi_1(t) + \delta/2$$

and by continuity of $\dot{e}$ and ψ_1 it follows that $\dot{e}(t_0) > -\psi_1(t_0) + \delta/2$, hence there exists a maximal $t_1 \in (t_0, t_2]$ such that

$$\forall t \in [t_0, t_1] :\; \dot{e}(t) \ge -\psi_1(t) + \delta/2. \quad \text{(A.16)}$$

Now the implications (A.11) and (A.9) from Step 3b and 3a, resp., together with (A.15) and (A.16) show that

$$\forall\, t \in [t_0, t_1] \;:\; \quad \psi_0(t) - e(t) \ge \varepsilon_0.$$

Hence $t_1 < t_2$, which implies $\dot{e}(t_1) = -\psi_1(t_1) + \delta/2$. Combining this with (A.15) and implication (A.14) from Step 3d yields

$$\forall t \in [t_1, t_2] :\; \dot{e}(t) \le -\psi_1(t) + \delta/2.$$

Implication (A.13) from Step 3c now gives

$$\psi_0(t_2) - e(t_2) \ge \psi_0(t_1) - e(t_1) \ge \varepsilon_0,$$

which contradicts the choice of t_2. Hence Step 3 is shown.

Step 4: We show that $\dot e(\cdot)$ is uniformly bounded away from the funnel boundary $\psi_1(\cdot)$, i.e.

$$\exists \varepsilon_1 > 0 \ \forall t \in [\varepsilon, \omega) : \quad \boxed{\psi_1(t) - |\dot e(t)| \geq \varepsilon_1}. \tag{A.17}$$

We have, for $\varepsilon_0 > 0$ as in Step 3, that $k_0(t)^2 \leq 1/\varepsilon_0^2$ for all $t \in [\varepsilon, \omega)$ which together with (A.6) yields

$$\forall\, t \in [\varepsilon, \omega) : k_0(t)^2 |e(t)| < \|\psi_0\|_{\varepsilon,\infty}/\varepsilon_0^2.$$

Assume $\varepsilon_1 \leq \min\{\lambda_1/2,\ \psi_1(\varepsilon) - |\dot e(\varepsilon)|\}$. Then, in view of (A.4) and (2.2), the following implication holds for almost all $t \in [\varepsilon, \omega)$

$$\psi_1(t) - |\dot e(t)| \leq \varepsilon_1 \quad \Longrightarrow \quad \ddot e(t) \operatorname{sgn} \dot e(t) < M + \gamma \tfrac{\|\psi_0\|_{\varepsilon,\infty}}{\varepsilon_0^2} - \gamma \frac{\lambda_1/2}{\varepsilon_1} + \gamma \|u_{\rm d}\|_\infty,$$

hence, for sufficiently small enough $\varepsilon_1 > 0$ and a.a. $t \in [\varepsilon, \infty)$,

$$\psi_1(t) - |\dot e(t)| \leq \varepsilon_1 \quad \Longrightarrow \quad \ddot e(t) \operatorname{sgn} \dot e(t) < -\|\dot\psi_1\|_{\varepsilon,\infty}.$$

The above implication ensures that the set $\{\ (t, \xi) \in [\varepsilon,\, \omega) \times \mathbb{R} \ |\ \psi_1(t) - |\xi| \geq \varepsilon_1\ \}$ is positively invariant for $\dot e(\cdot)$. Hence Step 4 is proved.

Step 5: We show Assertions (i)–(iii).
Boundedness of $e(\cdot)$, $\dot e(\cdot)$, $z(\cdot)$, $k_0(\cdot)$, $k_1(\cdot)$ on $[0, \omega)$ follows from (A.5), (A.6), (A.7), (A.8) and (A.17). The inequality (2.3) holds on $[0, \omega)$ because $k_i(\cdot)$ is bounded, $i = 0, 1$. Therefore, Assertion (i)–(iii) hold if $\omega = \infty$. Let, for ε_0 and ε_1 as in (2.3) and M_z as in (A.7),

$$\mathfrak{C} := \left\{ (t, e_0, e_1, z) \in [0, \omega] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \ \middle|\ \begin{array}{ll} \forall\, i \in \{0, 1\} : & |e_i| \leq |e^{(i)}(0)| + 1 \text{ if } t \in [0, \varepsilon], \\ & |e_i| \leq \psi_i(t) - \varepsilon_i \text{ otherwise,} \\ \|z\| \leq M_z & \end{array} \right\} \tag{A.18}$$

Let $\mathcal{D}$ be as in Step 1. If $\omega < \infty$ then $\mathfrak{C} \subsetneq \mathcal{D}$ is a compact subset of $\mathcal{D}$ which contains the whole graph of the solution $t \mapsto (e(t), \dot e(t), z(t))$, which contradicts the maximality of the solution. Hence $\omega = \infty$. This completes the proof. □

A.2 Proof of Theorem 3.1: nonlinear systems described by functional differential equations

It suffices to show that there exists a maximal solution $y : [-h, \omega)$ and each solution fulfills the minor modification of (A.4):

$$\exists\, M > 0\ \exists\, \gamma > 0 \text{ for a.a. } t \in [0, \omega) \ : \quad \ddot e(t) \operatorname{sgn} u(t) \geq -M \operatorname{sgn} u(t) + \gamma u(t)\,. \tag{A.19}$$

Then Steps 3-5 of the proof of Theorem 2.1 can then be repeated identically to prove Assertions (i)–(iii) of Theorem 3.1.

Step 1: We show existence of maximally extended solutions $y : [-h, \omega)$ with $\omega \in (0, \infty]$.
Define

$$F : [-h, \infty) \times \mathcal{D} \times \mathbb{R}^{2W} \to \mathbb{R}^3, \quad (t, (\tau, e_0, e_1), (w_1, w_2)) \mapsto \\ \Big(1, e_1, f(p_f(t), w_1) + g(p_g(t), w_2) \Big(-\tfrac{\varphi_0(\tau)^2}{(1 - \varphi_0(\tau)|e_0|)^2} e_0 - \tfrac{\varphi_1(\tau)}{1 - \varphi_1(\tau)|e_1|} e_1 + u_{\rm d}(t) \Big) - \ddot y_{\rm ref}(t) \Big)$$

where

$$\mathcal{D} := \{\ (\tau, e_0, e_1) \in [-h,\infty) \times \mathbb{R} \times \mathbb{R} \ \mid (|\tau|, e_0) \in \mathcal{F}_{\varphi_0},\ (|\tau|, e_1) \in \mathcal{F}_{\varphi_1}\ \}$$

and define the operator $\widehat{T} : [-h,\infty) \times \mathcal{C}([-h,\infty) \to \mathbb{R})^2 \to \mathcal{L}^\infty_{\text{loc}}\left([0,\infty) \to \mathbb{R}^W\right)^2$ by

$$\widehat{T}(\tau, e_0, e_1) := (T_f(e_0 + y_{\text{ref}}, e_1 + \dot{y}_{\text{ref}}), T_g(e_0 + y_{\text{ref}}, e_1 + \dot{y}_{\text{ref}}))$$

where y_{ref} is extended to $[-h, 0)$ in such a way that $y_{\text{ref}} \in \mathcal{W}^{2,\infty}([-h,\infty) \to \mathbb{R})$ and

$$\forall\, t \in [-h, 0]\ :\quad \varphi_0(|t|)\,|y^0(t) - y_{\text{ref}}(t)| < 1 \quad \text{and} \quad \varphi_1(|t|)\,|\dot{y}^0(t) - \dot{y}_{\text{ref}}(t)| < 1.$$

This is possible since (3.3) and $(\varphi_0, \varphi_1) \in \mathcal{G}_2$ hold.
Writing $\tau^0 : [-h, 0] \to \mathbb{R}$, $t \mapsto t$, it follows that $x = (\tau, e, \dot{e})$ is a solution of

$$\dot{x} = F(t, x, \widehat{T}(x)), \quad x\big|_{[-h,0]} = (\tau^0, y^0 - y_{\text{ref}}|_{[-h,0]}, \dot{y}^0 - \dot{y}_{\text{ref}}|_{[-h,0]}),$$

if, and only if, $y = y_{\text{ref}} + e$ solves the closed-loop system (3.2), (2.2). Finally, [14, Thm. 5] ensures the existence of a maximally extended solution $y : [-h, \omega) \to \mathbb{R}$, $\omega \in (0, \infty]$.

Step 2: We show (A.19)*.*
Consider a fixed solution $y : [-h, \omega) \to \mathbb{R}$ of (3.2), (2.2), i.e.

$$\text{for a.a. } t \in [0, \omega) : \quad \ddot{e}(t) = f(p_f(t), T(y, \dot{y})(t)) + g(p_g(t), T_g(y, \dot{y})(t))u(t) - \ddot{y}_{\text{ref}}(t).$$

Choose $\varepsilon > 0$ such that $|y(t)| \le \|y^0\|_\infty + 1$ and $|y(t)| \le \|\dot{y}^0\|_\infty + 1$ for all $t \in [0, \varepsilon]$ and since

$$\forall\, t \in [\varepsilon, \omega) : \quad |y(t)| < \|y_{\text{ref}}\|_\infty + \|1/\varphi_0\big|_{[\varepsilon,\infty)}\|_\infty \quad \text{and} \quad |\dot{y}(t)| < \|\dot{y}_{\text{ref}}\|_\infty + \|1/\varphi_1\big|_{[\varepsilon,\infty)}\|_\infty,$$

the trajectories y and $\dot{y}$ are bounded on $[0, \omega)$, hence $T_f(y, \dot{y})\big|_{[0,\omega)}$ and $T_g(y, \dot{y})\big|_{[0,\omega)}$ are well defined and bounded, say by M_{T_f} and M_{T_g}. Let M_{p_f} and M_{p_g} be the corresponding bounds of $p_f(\cdot)$ and $p_g(\cdot)$, then by continuity of f and g

$$\max_{|p| \le M_{p_f}, |w| \le M_{T_f}} |f(p, w)| =: M_f < \infty \quad \text{and} \quad \min_{|p| \le M_{p_g}, |w| \le M_{T_g}} g(p, w) =: \gamma > 0,$$

and so (A.19) holds for $M := M_f + \|\ddot{y}_{\text{ref}}\|_\infty$. ☐

A.3 Proof of Theorem 3.2: systems with relative degree one

The proof is based on the following existence and uniqueness of the solution of an implicit ordinary differential equation.

Lemma A.1 (Existence and uniqueness of the solution of an implicit ODE)**.** *Let $\mathcal{D} \subseteq \mathbb{R}_{\ge 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ be a non-empty and relatively open set and $(t_0, e_0^0, e_1^0, z^0) \in \mathcal{D}$. Let $F \in \mathcal{C}^1(\mathcal{D} \to \mathbb{R})$ be such that*

$$F(t_0, e_0^0, e_1^0, z^0) = 0 \tag{A.20}$$

and

$$\forall (t, e_0, e_1, z) \in \mathcal{D} : \quad \frac{\partial F}{\partial e_1}(t, e_0, e_1, z) \neq 0. \tag{A.21}$$

Consider, for $p \in \mathbb{R}^{n-1}$, $Q \in \mathbb{R}^{(n-1)\times(n-1)}$ and $g \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R})$, the implicit initial-value problem

$$\begin{aligned} 0 &= F(t, e, \dot{e}, z), & e(t_0) &= e^0 \\ \dot{z} &= pe + Qz + g(t), & z(t_0) &= z^0. \end{aligned} \tag{A.22}$$

Then there exists a unique maximal solution $(e, z) : [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega \in (t_0, \infty]$, of (A.22) *such that*

$$\text{graph}(e, \dot{e}, z) = \{\ (t, e(t), \dot{e}(t), z(t)) \ \mid t \in [0, \omega)\ \} \subseteq \mathcal{D},$$

and maximality implies that graph$(e, \dot{e}, z)$ *is not completely contained in any compact subset of $\mathcal{D}$.*

Proof. Step 1: We show existence and uniqueness of a local solution of the initial-value problem (A.22).
Differentiability of $F(\cdot)$ together with (A.20) and (A.21) allow us to apply the Implicit Function Theorem (see, for example, [1, Th. VII.8.2]) to conclude the following: there exist a relatively open neighbourhood $\mathcal{U} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^{n-1}$ of (t_0, e_0^0, z^0), an open neighbourhood $\mathcal{V} \subseteq \mathbb{R}$ of e_1^0, a unique function $f \in \mathcal{C}^1(\mathcal{U} \to \mathcal{V})$ such that $f(t_0, e_0^0, z^0) = e_1^0$ and $F(t, e_0, f(t, e_0, z), z) = 0$ for all $(t, e_0, z) \in \mathcal{U}$; moreover,

$$\forall (t, e_0, z) \in \mathcal{U}: \quad [F(t, e_0, e_1, z) = 0 \ \wedge \ e_1 \in \mathcal{V}] \quad \Longleftrightarrow \quad e_1 = f(t, e_0, z). \tag{A.23}$$

Consider next the initial-value problem

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} e_0 \\ z \end{pmatrix} = \begin{pmatrix} f(t, e_0, z) \\ pe_0 + Qz + g(t) \end{pmatrix}, \qquad \begin{pmatrix} e_0(t_0) \\ z(t_0) \end{pmatrix} = \begin{pmatrix} e_0^0 \\ z^0 \end{pmatrix}. \tag{A.24}$$

The right hand side of (A.24) is continuous on the relatively open set $\mathcal{U}$ and locally Lipschitz in e_0 and z, hence standard theory of ordinary differential equations (see e.g. [35, Thm. III.§11.III]) yields existence of a unique solution $(e, z) : [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega \in (t_0, \infty]$, of the initial value problem (A.24). From (A.23) it follows that this solution is also a unique (local) solution of (A.22).

Step 2: We show that every solution of (A.22) *can be maximally extended.*
Let $(e, z) : [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega \in (t_0, \infty]$, be a solution of (A.22). If $\omega = \infty$ nothing is to show, hence assume $\omega < \infty$. Define

$$\mathcal{A}_{(\omega,e,z)} := \left\{ (\sigma, \xi(\cdot)) \;\middle|\; \begin{array}{l} \sigma \in [\omega, \infty], \ \xi \in \mathcal{C}^1([t_0, \sigma) \to \mathbb{R}^n), \xi\big|_{[t_0,\omega)} = (e, z), \\ (\xi_1, \dot{\xi}_1, \xi_2, \ldots, \xi_n) \text{ solves (A.22) on } [t_0, \sigma) \end{array} \right\},$$

that is, the set comprising the solution (e, z) and all proper right extensions of (e, z) that are also solutions. Define on this non-empty set a partial order $\preceq$ by

$$(\sigma_1, \xi_1(\cdot)) \preceq (\sigma_2, \xi_2(\cdot)) \quad :\Longleftrightarrow \quad \sigma_1 \leq \sigma_2 \quad \text{and} \quad \xi_1(\cdot) = \xi_2|_{[t_0,\sigma_1]}.$$

Let $\mathcal{A}_1$ be a totally ordered subset of $\mathcal{A}_{(\omega,e,z)}$. Set

$$\sigma^* := \sup \{ \ \sigma \in [\omega, \infty] \ \mid \exists \ (\sigma, \xi(\cdot)) \in \mathcal{A}_1 \ \}$$

and let $\xi^* : [t_0, \sigma^*) \to \mathbb{R}^n$ be defined by the property that, for every $(\sigma, \xi) \in \mathcal{A}_1$, $\xi^*|_{[t_0,\sigma)} = \xi(\cdot)$. Then $(\sigma^*; \xi^*) \in \mathcal{A}_{(\omega,e,z)}$ and it is an upper bound for $\mathcal{A}_1$. By Zorn's Lemma, see e.g. [35, II.§7.XIII], it follows that $\mathcal{A}_{(\omega,e,z)}$ contains at least one maximal element. Hence there exists a maximal solution $(e, z) : [t_0, \omega^*) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $\omega^* \in (t_0, \infty]$, of the initial value problem (A.22).

Step 3: We show uniqueness of the solution of the initial value problem (A.22).
Let $(e, z) : [t_0, \omega) \to \mathbb{R}^n$, $\omega \in (t_0, \infty]$, and $(\widetilde{e}, \widetilde{z}) : [t_0, \widetilde{\omega}) \to \mathbb{R}^n$, $\widetilde{\omega} \in (t_0, \infty]$, be two solutions of the initial value problem (A.22). Seeking a contradiction, suppose that there exists a first time $t_1 \in [t_0, \infty)$ where the two solutions separate, more precisely:

$$t_1 := \max \left\{ \ t \in [t_0, \min\{\omega, \widetilde{\omega}\}) \ \middle| \ (e, z)\big|_{[t_0,t]} = (\widetilde{e}, \widetilde{z})\big|_{[t_0,t]} \ \right\} \in \mathbb{R}.$$

According to Step 1, the corresponding initial value problem (A.22) at t_1 with initial value $(e^1, z^1) := (e(t_1), z(t_1)) = (\widetilde{e}(t_1), \widetilde{z}(t_1))$ has a unique local solution on $[t_1, t_1 + \delta) \subseteq [t_0, \min\{\omega, \widetilde{\omega}\})$ for some $\delta > 0$, hence $(e, z)\big|_{[t_1,t_1+\delta)} = (\widetilde{e}, \widetilde{z})\big|_{[t_1,t_1+\delta)}$. This contradicts the definition of t_1 and proves the claim of Step 3.

Step 4: We show that the graph of the maximal solution $(e, \dot{e}, z)$ is not contained in any compact subset of $\mathcal{D}$.

Let $(e,\dot e,z):[t_0,\omega)\to\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-1}$, $\omega\in(t_0,\infty]$, be the unique maximal solution of (A.22). An equivalent formulation of the claim of Step 4 is that the closure of $\operatorname{graph}(e,\dot e,z)$ is not a compact subset of $\mathcal{D}$. Denote the closure of $\operatorname{graph}(e,\dot e,z)$ by $\mathfrak{C}\subseteq\mathbb{R}_{\geq 0}\times\mathbb{R}\times\mathbb{R}^{n-1}$ and, seeking a contradiction, assume that $\mathfrak{C}$ is a compact subset of $\mathcal{D}$. Then, first of all, $\omega<\infty$.
By continuity of F and by construction of $\mathfrak{C}$ we have

$$\forall (t,e_0,e_1,z)\in\mathfrak{C}: F(t,e_0,e_1,z)=0.$$

Hence the Implicit Function Theorem ensures, for each $(t^*,e_0^*,e_1^*,z^*)\in\mathfrak{C}$, existence of (relatively) open neighbourhoods $\mathcal{U}_{(t^*,e_0^*,z^*)}$ of (t^*,e_0^*,z^*) and $\mathcal{V}_{e_1^*}$ of e_1^* as well as a function $f_{(t^*,e_0^*,e_1^*,z^*)}\in\mathcal{C}^1(\mathcal{U}_{(t^*,e_0^*,z^*)}\to\mathcal{V}_{e_1^*})$ such that (A.23) holds. Let $\mathcal{W}_{(t^*,e_0^*,z^*)}:=\left\{\,(t,e_0,e_1,z)\;\middle|\;(t,e_0,z)\in\mathcal{U}_{(t^*,e_0^*,z^*)},\ e_1\in\mathcal{V}_{e_1^*}\,\right\}$ which is an (relatively) open neighbourhood of (t^*,e_0^*,e_1^*,z^*) and $\bigcup_{(t^*,e_0^*,e_1^*,z^*)\in\mathfrak{C}}\mathcal{W}_{(t^*,e_0^*,e_1^*,z^*)}$ is an open covering of $\mathfrak{C}$.
By compactness of $\mathfrak{C}$ we may choose a finite subcovering of $\mathfrak{C}$, in particular, there exist $\varepsilon>0$ and $(t^\omega,e_0^\omega,e_1^\omega,z^\omega)\in\mathfrak{C}$ such that $\operatorname{graph}\left((e,\dot e,z)\big|_{[\omega-\varepsilon,\omega)}\right)\subseteq\mathcal{W}_{(t^\omega,e_0^\omega,e_1^\omega,z^\omega)}$. Hence, by (A.23), $\dot e(t)=f_{(t^\omega,e_0^\omega,e_1^\omega,z^\omega)}(t,e(t),z(t))$ on $[\omega-\varepsilon,\omega)$, i.e. $(e,z)\big|_{[\omega-\varepsilon,\omega)}$ is a solution of an (explicit) ordinary differential equation whose graph is contained in the compact set $\mathfrak{C}$. Now an application of [35, Lem. II.§6.VI] ensures that this solution can be extended to the closed interval $[\omega-\varepsilon,\omega]$, in particular $(e(\omega),\dot e(\omega),z(\omega))=\lim_{t\nearrow\omega}(e(t),\dot e(t),z(t))\in\mathfrak{C}\subseteq\mathcal{D}$ is well defined and $F(e(\omega),\dot e(\omega),z(\omega))=0$. Hence, by Step 1 with initial time ω and corresponding initial value, the solution can be extended locally to the interval $[\omega,\omega^*)$ for some $\omega^*>\omega$ which contradicts maximality of the solution. This shows the assertion of Step 4 and the proof of the lemma is complete. □

Proof of Theorem 3.2:
Step 1: We show existence of a maximal solution.
Without restriction of generality, we may assume that the system (1.1) is in Byrnes-Isidori form, i.e.

$$\begin{aligned}\dot y &= ry+s^\top z+\gamma u(t), \qquad & y(0)&=cx^0,\\ \dot z &= py+Qz, & z(0)&=z^0,\end{aligned}$$

where $r\in\mathbb{R}$, $s,p\in\mathbb{R}^{n-1}$, $Q\in\mathbb{R}^{(n-1\times n-1)}$ is Hurwitz by (1.3), $z^0\in\mathbb{R}^{n-1}$ and $\gamma:=cb>0$ by (1.8). The closed-loop system (1.1), (2.2) may be written as the following implicit differential equation:

$$\begin{aligned}\dot e(t)&=r(e(t)+y_{\rm ref}(t))+s^\top z(t)-\dot y_{\rm ref}(t)+\gamma\left(-\tfrac{\varphi_0(t)^2 e(t)}{(1-\varphi_0(t)|e(t)|)^2}-\tfrac{\varphi_1(t)\dot e(t)}{1-\varphi_1(t)|\dot e(t)|}+u_d(t)\right),\\ \dot z(t)&=p(e(t)+y_{\rm ref}(t))+Qz(t)\end{aligned}\tag{A.25}$$

with initial values $e(0)=e^0:=cx^0-y_{\rm ref}(0)$ and $z(0)=z^0$. For $\mathcal{D}\subseteq\mathbb{R}_{\geq0}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-1}$ defined analogously as in (A.2), the implicit ordinary differential equation (A.25) can be written as

$$\begin{aligned}0&=F(t,e(t),\dot e(t),z(t)),\\ \dot z(t)&=pe(t)+Qz(t)+py_{\rm ref}(t),\end{aligned}$$

with corresponding $F:\mathcal{D}\to\mathbb{R}$. Some simple observations, taking also into account the absolute value function $|\cdot|$, reveal that F is differentiable and

$$\forall\,(t,e_0,e_1,z)\in\mathcal{D}:\qquad \frac{\partial F}{\partial e_1}(t,e_0,e_1,z)=1+\gamma\frac{\varphi_1(t)}{(1-\varphi_0(t)|e_1|)^2}\geq 1.$$

Since $\varphi_1(0)=0$, it follows that (A.25) is *explicit* in $\dot e$ at $t=0$, hence

$$F(0,e_0^0,e_1^0,z^0)=0\quad\Longleftrightarrow\quad e_1^0=r(e_0^0+y_{\rm ref}(0))+s^\top z^0-\dot y_{\rm ref}(0)-\gamma\tfrac{\varphi_0(0)e_0^0}{(1-\varphi_0(0)e_0^0)^2}+\gamma u_{\rm d}(0).$$

Lemma A.1 now yields that there exists a unique and maximally extended solution $(e,z):[0,\omega)\to\mathbb{R}\times\mathbb{R}^{n-1}$ of (A.25) with $(t,e(t),\dot e(t),z(t))\in\mathcal{D}$ for all $t\in[0,\omega)$.

Step 2: We show existence of $M>0$ such that

$$\forall t\in[0,\omega):\quad -M+\gamma u(t)<\dot e(t)<M+\gamma u(t). \tag{A.26}$$

This follows analogously as in Step 2 of the proof of Theorem 2.1.

Step 3: We show existence of $\varepsilon_0\in(0,\lambda_0/2)$ such that

$$\forall t\in(0,\omega):\quad 1/\varphi_0(t)-|e(t)|\geq\varepsilon_0.$$

Adopting the notation (A.1) and choosing $\varepsilon>0$ as in (A.5), it suffices to show that the set

$$\{\ (t,e_0)\in[\varepsilon,\omega)\times\mathbb{R}\ \mid\ \psi_0(t)-|e_0|\geq\varepsilon_0\ \}$$

is positively invariant for sufficiently small $\varepsilon_0>0$ and $\psi_0(\varepsilon)-|e(\varepsilon)|\geq\varepsilon_0$. The former clearly follows if the following implication holds for all $t\in[\varepsilon,\omega)$:

$$\psi_0(t)-|e(t)|=\varepsilon_0\qquad\Longrightarrow\qquad\dot e(t)\operatorname{sgn}e(t)\leq-\psi_1(t)+\delta/2,$$

because, by definition of $\mathcal{G}_2$, $-\psi_1(t)+\delta/2\leq\dot\psi_0(t)-\delta/2$. Seeking a contradiction, assume there exists $t\in[\varepsilon,\omega)$ with $\psi_0(t)-|e(t)|=\varepsilon_0$ and $\dot e(t)\operatorname{sgn}e(t)>-\psi_1(t)+\delta/2$. From $\varepsilon_0<\lambda_0/2$ together with (2.2) and (A.26) it then follows that

$$\dot e(t)\operatorname{sgn}e(t)<M-\gamma\frac{\lambda_0/2}{\varepsilon_0^2}+\gamma\frac{\|\psi_1|_{[\varepsilon,\infty)}\|_\infty}{\delta/2}+\gamma\|u_{\rm d}\|_\infty,$$

hence, for sufficiently small ε_0, a contradiction to the assumption $\dot e(t)\operatorname{sgn}e(t)>-\psi_1(t)+\delta/2$.

Step 4: We show existence of $\varepsilon_1\in(0,\lambda_1/2]$ such that

$$\forall t\in(0,\omega):\quad 1/\varphi_1(t)-|\dot e(t)|\geq\varepsilon_1.$$

Adopting the notation (A.1) and choosing $\varepsilon>0$ as in (A.5) it suffices to show that

$$\forall t\in[\varepsilon,\omega):\quad |\dot e(t)|\leq\psi_1(t)-\varepsilon_1$$

for sufficiently small ε_1. Seeking a contradiction, assume $|\dot e(t)|>\psi_1(t)-\varepsilon_1$ for some $t\in[\varepsilon,\omega)$ and arbitrary small $\varepsilon_1>0$. We only consider the case $\dot e(t)>0$, the other case follows analogously. Choose $\varepsilon_0>0$ accordingly to Step 3. From $\varepsilon_1\leq\lambda_1/2$ together with (2.2) and (A.26) it follows that

$$\lambda_1/2\leq\psi_1(t)-\varepsilon_1<\dot e(t)<M+\gamma\frac{\|\psi_0|_{[\varepsilon,\omega)}\|_\infty}{\varepsilon_0^2}-\gamma\frac{\lambda_1/2}{\varepsilon_1}+\gamma\|u_{\rm d}\|_\infty;$$

a contradiction for sufficiently small ε_1.

Step 5: We show that the maximal solution is global.
Assume $\omega<\infty$ then $\mathfrak{C}\subseteq\mathbb{R}_{\geq0}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-1}$ defined as in (A.18) is a compact subset containing graph$(e,\dot e,z)$ which according to Lemma A.1 contradicts maximality of the solution, hence $\omega=\infty$ and the proof of Theorem 3.2 is complete. □

A.4 Proof of Theorem 3.3: input saturations

Existence and uniqueness of a maximal solution $(e,\dot e,z):[0,\omega)\to\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-2}$ follows similarly to Step 1 in the proof of Theorem 2.1. Now all the inequalities derived in Sections 3.3.1-3.3.4 hold true on $[0,\omega)$ instead of $\mathbb{R}_{\geq 0}$ and with minor modifications the steps of the proof of Theorem 2.1 go through. We omit the details. □

A.5 Proof of Theorem 3.6: robustness in the gap metric

A.5.1 Prerequisites

To match the notation of the gap metric, see e.g. [27], we rename the signals from Theorem 3.6 as follows:

$$u_0 := u_{\mathrm{d}},\qquad u_1 := u,\qquad u_2 := u_0 - u_1 = k_0^2 e + k_1\dot e$$
$$y_0 := y_{\mathrm{ref}},\qquad y_1 := y,\qquad y_2 := y_0 - y_1 = -e\,.$$

Corresponding to this notation, we consider the plant operator and the operator representing the funnel controller

$$P_{\theta,x^0}:\quad u_1\mapsto y_1,\qquad C_{\varphi_0,\varphi_0}:\quad y_2\mapsto u_2,\quad \text{resp.}$$

Due to possible finite escape time, we introduce the *ambient signal spaces* [27, Sec. 6.1]

$$\begin{aligned}\mathcal{L}_a^\infty &:= \left\{\, u:[0,\omega)\to\mathbb{R} \;\middle|\; \forall\tau\in(0,\omega), u\big|_{[0,\tau)}\in\mathcal{L}^\infty([0,\tau)\to\mathbb{R})\,\right\}\\ \mathcal{W}_a^{2,\infty} &:= \left\{\, y:[0,\omega)\to\mathbb{R} \;\middle|\; \forall\tau\in(0,\omega), u\big|_{[0,\tau)}\in\mathcal{W}^{2,\infty}([0,\tau)\to\mathbb{R})\,\right\},\end{aligned}$$

so that the plant and the controller can be considered as the maps

$$P_{\theta,x^0}:\quad \mathcal{L}_a^\infty\to\mathcal{W}_a^{2,\infty},\qquad C_{\varphi_0,\varphi_1}:\quad \mathcal{W}_a^{2,\infty}\to\mathcal{L}_a^\infty.$$

Finally, let the closed loop equations be given by

$$[P_{\theta,x^0},C_{\varphi_0,\varphi_1}]:\quad y_1=P_{\theta,x^0}(u_1),\; u_2=C_{\varphi_0,\varphi_1}(y_2),\; u_0=u_1+u_2,\; y_0=y_1+y_2.$$

Theorem 2.1 ensures that for all $(u_0,y_0)\in\mathcal{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})\times\mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R})$ there exists unique $(u_1,y_1),(u_2,y_2)\in\mathcal{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})\times\mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R})$ which solves the closed loop $[P_{\theta,x^0},C_{\varphi_0,\varphi_1}]$. This implies, in the terminology of [27], that the closed loop $[P_{\theta,x^0},C_{\varphi_0,\varphi_1}]$ is globally well posed and $\big(\mathcal{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})\times\mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R})\big)$-stable.
We now study the closed loop $[\widetilde P,C_{\varphi_0,\varphi_1}]$ of the disturbed plant $\widetilde P\in\widetilde{\mathcal{P}}$ and the (unchanged) funnel controller C_{φ_0,φ_1}. In general, this closed loop will not generate globally defined solutions, however we can show the following properties.

Lemma A.2. *Let $(\varphi_0,\varphi_1)\in\mathcal{G}_2\setminus\mathcal{G}_2^{\mathrm{fin}}$, $\widetilde P\in\widetilde{\mathcal{P}}$ and $(u_0,y_0)\in\mathcal{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})\times\mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R})$. Then the closed-loop $[\widetilde P,C_{\varphi_0,\varphi_1}]$ has the following properties:*

(i) *There exist unique, maximally extended solutions $(u_1,y_1),(u_2,y_2):[0,\omega)\to\mathbb{R}^2$, for some $\omega\in(0,\infty]$.*

(ii) *If $(u_2,y_2)\in\mathcal{L}^\infty([0,\omega),\mathbb{R}^m)\times\mathcal{W}^{2,\infty}([0,\omega),\mathbb{R}^m)$, then $\omega=\infty$ and y_2 and $\dot y_2$ are uniformly bounded away from the funnel boundaries $\varphi_i(\cdot)^{-1}$, $i=0,1$ resp.;*

(iii) *$[\widetilde P,C_{\varphi_0,\varphi_1}]$ is regularly well posed [27], i.e. it is locally well-posed and*

$$\omega<\infty\quad\Longrightarrow\quad \|(u_2,y_2)\big|_{[0,\tau)}\|_{\mathcal{L}^\infty\times\mathcal{W}^2}\to\infty \text{ as } \tau\nearrow\omega.$$

Proof. (i): Let $\widetilde{\theta} = (\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \widetilde{\mathcal{P}}$ and $\widetilde{x}^0 \in \mathbb{R}^{\dim \widetilde{\theta}}$ be such that $\widetilde{P} = P_{\widetilde{\theta}, \widetilde{x}^0}$. The closed loop can then be rewritten as

$$\dot{x} = f(t, x), \quad x(0) = \widetilde{x}^0,$$

where

$$f : \mathcal{D} \to \mathbb{R}^n, (t, x) \mapsto \\ \widetilde{A}x + \widetilde{b}u_0(t) + \widetilde{b}\tfrac{\varphi_0(t)^2}{(1-\varphi_0(t)|y_0(t)-cx|)^2}(y_0(t) - \widetilde{c}x) + \widetilde{b}\tfrac{\varphi_1(t)}{1-\varphi_1(t)|\dot{y}_0(t)-\widetilde{c}\widetilde{A}x|}(\dot{y}_0(t) - \widetilde{c}\widetilde{A}x),$$

for

$$\mathcal{D} := \{\ (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \ \mid (t, y_0(t) - cx) \in \mathcal{F}_{\varphi_0},\ (t, \dot{y}_0(t) - cAx) \in \mathcal{F}_{\varphi_1}\ \},$$

and

$$\begin{aligned} y_1 &= \widetilde{c}x, & \dot{y}_1 &= \widetilde{c}\widetilde{A}x, \\ y_2 &= y_0 - y_1, & \dot{y}_2 &= \dot{y}_0 - \dot{y}_1, \\ u_2 &= -\left(\tfrac{\varphi_0}{1-\varphi_0|y_2|}\right)^2 y_2 - \tfrac{\varphi_1}{1-\varphi_0|\dot{y}_2|}\dot{y}_2, & u_1 &= u_0 - u_2. \end{aligned}$$

Now, as in the proof of Theorem 2.1, the theory of ordinary differential equations, see e.g. [35, Thm. III.§10.XX], ensures existence and uniqueness of a maximally extended solution.
(ii): For $t \in [0, \omega)$, let $k_i(t) = \frac{\varphi_i(t)}{1-\varphi_i(t)|y_2^{(i)}(t)|}$, $i = 0, 1$.

By construction we have $\varphi_i(t)|y_2^{(i)}(t)| < 1$ for all $t \in [0, \omega)$. Note that we may choose $\varepsilon \in (0, \omega)$ such that

$$\forall\, t \in [0, \varepsilon]\ \forall\, i \in \{0, 1\}\ :\ |y_2^{(i)}(t)| \leq |y_2^{(i)}(0)| + 1 \quad \wedge \quad k_i(t) \leq k_i(0) + 1.$$

In the following we adopt the notation (A.1), i.e. use $\psi_{0/1}(\cdot)$ to denote the funnel boundaries. We will show that boundedness of u_2 implies boundedness of $k_0(\cdot)$ and $k_1(\cdot)$ on the interval $[0, \omega)$. Then the same line of arguments as in the proof of Theorem 2.1 shows that $\omega = \infty$ and that $y_2, \dot{y}_2$ are uniformly bounded away from their corresponding funnel boundaries.
Seeking a contradiction, assume (a) k_0 is unbounded and k_1 is bounded, (b) k_0 is bounded and k_1 is unbounded or (c) both k_0 and k_1 are unbounded. The cases (a) and (b) can be treated analogously, therefore consider only case (a) first. Boundedness of u_2 implies that the product $k_0^2 y_2$ is bounded, hence unboundedness of k_0 implies that we may choose a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow \omega$ and $k_0(t_n) \to \infty$ and $y_2(t_n) \to 0$. This is a contradiction because $|y_2(t_n)| < \lambda_0/2$ implies $\psi_0(t_0) - |y_2(t_n)| > \lambda_0/2$, hence $k_0(t_n) < 2/\lambda_0$.
It remains to consider (c). Assume that k_0 and k_1 are both unbounded. Since the (weak) derivative of ψ_i is essentially bounded on $[\varepsilon, \omega)$ and the (weak) derivative of $y_2^{(i)}$, $i = 0, 1$, is essentially bounded on $[0, \omega)$ by assumption, it follows that

$$\forall\, i \in \{0, 1\}\ \forall\, s, t \in [\varepsilon, \omega) \text{ with } t > s : \\ \psi_i(t) - |y_2^{(i)}(t)| \leq \underbrace{\psi_i(s) - |y_2^{(i)}(s)|}_{=1/k_i(s)} + \underbrace{\left(\|\dot{\psi}_i\|_{\varepsilon,\infty} + \|y_2^{(i+1)}\|_\infty\right)}_{=:M_i}(t - s)$$

Hence, by choosing s such that $0 < t - s \leq \omega - s$ is small enough and $k_i(s)$ is big enough it holds that

$$\forall M > 0\ \forall i \in \{0, 1\}\ \exists s_i \in [\varepsilon, \omega)\ \forall t \in [s, \omega) : \\ k_i(t) = 1/(\psi_i(t) - |y_2^{(i)}|) \geq \frac{1}{1/k_i(s_i) + M_i(\omega - s_i)} \geq M \quad \text{(A.27)}$$

This implies that $k_i(t) \to \infty$ as $t \nearrow \omega$ and therefore, by positivity and continuity of ψ_i, we have $\lim_{t\nearrow\omega} |y_2^{(i)}(t)| \to \psi_i(\omega)$ and close to ω no sign change occurs for $y_2^{(i)}$, $i = 0, 1$. Assume first that $\dot y_2$ is positive near ω, then choose $t^* \in [\varepsilon, \omega)$ such that

$$\text{for a.a. } t \in [t^*, \omega) : \dot y_2(t) \overset{\text{(A.27)}}{\geq} \psi_1(t) - \delta \overset{\mathcal{G}_2}{>} -\dot\psi_0(t).$$

Hence $t \mapsto \psi_0(t) + y_2(t)$ is strictly increasing on $[t^*, \omega)$ which, in view of $\lim_{t\nearrow\omega} \psi_0(t) - |y_2(t)| = 0$, is only possible if $y_2(t)$ is positive on $[t^*, \omega)$. With the analogue argument we can show that a negative sign of $\dot y_2$ near ω implies a negative sign of y_2 near ω. Altogether this shows that y_2 and $\dot y_2$ have the same sign near ω. In particular, boundedness of u_2 implies that both products $k_0^2 y_2$ and $k_1 \dot y_2$ must be bounded, which yields a contradiction in the same way as in cases (a) and (b).
(iii): This follows directly from (i) and (ii). □

A.5.2 Proof of Theorem 3.6

Since the perturbed closed loop $[P_{\widetilde\theta,\widetilde x^0}, C_{\varphi_0,\varphi_1}]$ is, according to Lemma A.2, regularly well posed we can repeat the proofs of [11, Props. 4.3, 4.4], see also [27, Thms. 6.5.3, 6.5.4] for signal spaces in the present setting, to show existence of functions η and α such that (3.15) implies that the closed loop $[P_{\widetilde\theta,\widetilde x^0}, C_{\varphi_0,\varphi_1}]$ maps $(u_0, y_0) = (u_{\rm d}, y_{\rm ref}) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ to $(u_1, y_1), (y_2, u_2) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. In particular, there exists a unique global and uniformly bounded solution. As shown in the proof of Lemma A.2 (ii), boundedness of (u_2, y_2) implies that the gain functions k_0 and k_1 of the funnel controller are bounded, which in turn shows that the the error and its derivative, i.e. y_2 and $\dot y_2$ are uniformly bounded away from the funnel boundaries.
It remains to show that the state variable x of the linear system corresponding to $\widetilde\theta = (\widetilde A, \widetilde b, \widetilde c)$ and its derivative are bounded. Detectability of $(\widetilde A, \widetilde b, \widetilde c)$ yields the existence of $F \in \mathbb{R}^q$, $q := \dim \widetilde\theta$, such that spec $(\widetilde A + F\widetilde c) \subseteq \mathbb{C}_-$. Setting $g := -\big[F - k_0^2 \widetilde b\big] y_2 + k_1 \widetilde b \dot y_2 + \widetilde b\, u_0$ gives

$$\dot x = \big[\widetilde A - k_0^2 \widetilde b \widetilde c\big] x - k_1 \widetilde b \widetilde c\, \dot x + \widetilde b\, u_0 + k_0^2 \widetilde b\, y_0 + k_1 \widetilde b\, \dot y_0 = \big[\widetilde A + F\widetilde c\big] x + g\,. \tag{A.28}$$

Since $y_2 \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $k_i \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, $i \in \{0, 1\}$, and since $w_0 = (u_0, y_0) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ it follows that $g \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Hence, by (A.28) and Variation of Constants we obtain $x \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ and, by (A.28), also $\dot x \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. □

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