Stability of Piecewise Affine Systems through Discontinuous Piecewise Quadratic Lyapunov Functions

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Abstract— State-dependent switched systems characterized by piecewise affine (PWA) dynamics in a polyhedral partition of the state space are considered. Sufficient conditions on the vectors fields such that the solution crosses the common boundaries of the polyhedra are expressed in terms of quadratic inequalities constrained to the polyhedra intersections. A piecewise quadratic (PWQ) function, not necessarily continuous, is proposed as a candidate Lyapunov function (LF). The sign conditions and the negative jumps at the boundaries are expressed in terms of linear matrix inequalities (LMIs) via cone-copositivity. A sufficient condition for the asymptotic stability of the PWA system is then obtained by finding a PWQ-LF through the solution of a set LMIs. Numerical results with a conewise linear system and an opinion dynamics model show the effectiveness of the proposed approach.

I. INTRODUCTION

The asymptotic stability of continuous-time systems which are piecewise affine (PWA) over a polyhedral partition of the state space can be investigated by using Lyapunov approaches. A common quadratic Lyapunov function does not always exists and to find a more general Lyapunov function is a nontrivial issue [12], [10]. In order to get less conservative conditions a typical approach consists in using piecewise quadratic (PWQ) Lyapunov functions (LFs) [8]. A seminal work on the use of PWQ-LFs for PWA systems is [6] where the $\mathcal{S}$-procedure is employed to determine stability conditions expressed in terms of linear matrix inequalities (LMIs). A typical assumption for this technique is the continuity of the PWQ-LF on the boundaries shared by different polyhedra [7], [16].

A stability analysis with *discontinuous* PWQ-LFs for planar PWA systems with continuous vector fields is proposed in [1]. The stability conditions proposed in [11] for the PWQ-LF allows discontinuities but the apriori knowledge of the sequence of modes and the $\mathcal{S}$-procedure are required. By using the copositive programming approach [13], in this paper we translate polyhedra-constrained conditions on the PWQ-LF into corresponding linear matrix inequalities (LMIs). The proposed approach can be considered as a generalization of the stability analysis presented in [4], [5] where the continuity of the PWQ-LF was required and a more restrictive class of PWA systems was considered. We adopt the cone-copositive approach in order to formulate a crossing

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condition of the system trajectory through the polyhedra boundaries. This condition allows to analyze existence and uniqueness of solutions for a quite general class of PWA systems and to express a non-increasing condition for the candidate PWQ-LF in terms of LMIs.

The paper is organized as follows. In Sec. II some definitions on polyhedra and cones are recalled and some preliminary results on the sign analysis of a PWQ function on a polyhedron is presented. The class of PWA systems with the relative solution analysis is presented in Sec. III. The main stability result with the conditions for the existence of a possibly discontinuous PWQ-LF is presented in Sec. IV. The numerical examples illustrated in Sec. V confirm the effectiveness of the approach. Sec. VI concludes the paper.

II. PRELIMINARIES

In this section some preliminary definitions and concepts on (polyhedral) cones, polyhedra, homogenization procedure and copositivity are recalled.

A. Cones and homogenization procedure

Definition 1: Given a finite number ρ of points $\{r_\ell\}_{\ell=1}^\rho$, $r_\ell \in \mathbb{R}^n$, $\rho \in \mathbb{N}$, a conical hull

$$\mathcal{C} = \text{cone}\,\{r_\ell\}_{\ell=1}^\rho \tag{1}$$

is the set of points $v \in \mathbb{R}^n$ such that $v = \sum_{\ell=1}^\rho \theta_\ell r_\ell$, with $\theta_\ell \in \mathbb{R}_+$, $\mathbb{R}_+$ being the set of nonnegative real numbers. The set $\mathcal{C}$ is also called (polyhedral) cone and the points $\{r_\ell\}_{\ell=1}^\rho$ are called rays of the cone. The matrix $R \in \mathbb{R}^{n \times \rho}$ whose columns are the points $\{r_\ell\}_{\ell=1}^\rho$ in an arbitrary order is called ray matrix. Any $v \in \mathcal{C}$ can be written as $v = R\theta$ where $\theta \in \mathbb{R}_+^\rho$.

Definition 2: Given a finite number λ of points $\{v_\ell\}_{\ell=1}^\lambda$, $v_\ell \in \mathbb{R}^n$, $\lambda \in \mathbb{N}$, a convex hull, say $\text{conv}\{v_\ell\}_{\ell=1}^\lambda$, is the subset of points in a conical hull for which $\sum_{\ell=1}^\lambda \theta_\ell = 1$.

Definition 3: Given a finite number λ of vertices $\{v_\ell\}_{\ell=1}^\lambda$ and a finite number ρ of rays $\{r_\ell\}_{\ell=1}^\rho$, $v_\ell, r_\ell \in \mathbb{R}^n$, $\lambda, \rho \in \mathbb{N}$, the (convex) set

$$X = \text{conv}\{v_\ell\}_{\ell=1}^\lambda + \text{cone}\{r_\ell\}_{\ell=1}^\rho \tag{2}$$

is a polyhedron in $\mathbb{R}^n$. The expression (2) identifies the so-called $\mathcal{V}$-*representation* of the polyhedron. The conical hull $\mathcal{C}_X$ of X is obtained by interpreting the vertices also as rays, i.e.

$$\mathcal{C}_X = \text{cone}\{\{v_\ell\}_{\ell=1}^\lambda, \{r_\ell\}_{\ell=1}^\rho\} \tag{3}$$

and the corresponding ray matrix is

$$R = \begin{pmatrix} v_1 & \cdots & v_\lambda & r_1 & \dots & r_\rho \end{pmatrix}. \tag{4}$$

In the following we assume that in the polyhedron representation (2) all possible redundancies of the set of vertices and rays have been eliminated.

Definition 4: A *partion* of $X \subseteq \mathbb{R}^n$ is a family of full-dimensional sets $\{X_s\}_{s=1}^S$, $S \in \mathbb{N}$, such that $X = \cup_{s=1}^S X_s$ and $\text{int}(X_s) \cap \text{int}(X_m) = \emptyset$ for $s \neq m$, where $\text{int}(X_s)$ denotes the interior of X_s. A *polyhedral partition* is a partition $\{X_s\}_{s=1}^S$ where each X_s is a closed full-dimensional polyhedral and each intersection of two polyhedra is either empty or a common face.

Note that the assumption that for a polyhedral partition the intersection of two adjacent polyhedra is a common face is not restricitve and can always be achieved with suitable subdivision of the regions.

In order to utilize copositiveness and LMIs we will use the homogenization procedure defined below.

Definition 5: Consider a polyhedron $X \subset \mathbb{R}^n$ with the representation (2). For each vertex $v_\ell \in \mathbb{R}^n$, its vertex-homogenization $\widehat{v}_\ell \in \mathbb{R}^{n+1}$ is defined as $\widehat{v}_\ell = \text{col}(v_\ell, 1)$, where $\text{col}(\cdot)$ indicates a vector obtained by stacking in a single column the column vectors in its argument. For each ray $r_\ell \in \mathbb{R}^n$ its direction-homogenization $\widehat{r}_\ell \in \mathbb{R}^{n+1}$ is defined as $\widehat{r}_\ell = \text{col}(r_\ell, 0)$. The resulting conic homogenization of X is then

$$\mathcal{C}_{\widehat{X}} = \text{cone}\{\{\widehat{v}_\ell\}_{\ell=1}^\lambda, \{\widehat{r}_\ell\}_{\ell=1}^\rho\},$$

with corresponding ray matrix

$$\widehat{R} = \begin{pmatrix} v_1 & \cdots & v_\lambda & r_1 & \cdots & r_\rho \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}. \quad (5)$$

We can now present some definitions and results on copositivity and cone-copositivity.

Definition 6: A symmetric matrix $P \in \mathbb{R}^{n \times n}$ is cone-copositive with respect to a cone $\mathcal{C} \subseteq \mathbb{R}^n$ if it is positive semidefinite with respect to that cone, i.e., if $x^\top P x \geq 0$ for any $x \in \mathcal{C}$. A cone-copositive matrix will be denoted by $P \succcurlyeq^\mathcal{C} 0$. If the equality only holds for $x = 0$, then P is strictly cone-copositive and the notation is $P \succ^\mathcal{C} 0$. In the particular case $\mathcal{C} = \mathbb{R}^n_+$, a (strictly) cone-copositive matrix is called (strictly) copositive.

The notation $P \succcurlyeq 0$, i.e., without any superscript on the inequality, indicates that P is positive semidefinite, i.e., $x^\top P x \geq 0$ for any $x \in \mathbb{R}^n$.

The cone-copositivity evaluation of a known symmetric matrix can always be transformed into an equivalent copositive problem and then to an LMI, as stated by the following result.

Lemma 7: Let $P \in \mathbb{R}^{n \times n}$ be a symmetric matrix, $\mathcal{C} \subseteq \mathbb{R}^n$ be a polyhedral cone with ray matrix $R \in \mathbb{R}^{n \times \rho}$ and N be a symmetric (entrywise) positive matrix. Consider the following constrained inequalities

$$P \succ^\mathcal{C} 0, \quad (6a)$$

$$R^\top P R \succ^{\mathbb{R}^\rho_+} 0, \quad (6b)$$

$$R^\top P R - N \succcurlyeq 0. \quad (6c)$$

Then the following conditions hold

i) (6a) $\Longleftrightarrow$ (6b)

ii) (6c) $\Longrightarrow$ (6a).

Proof: i) The equivalence is directly obtained by using Def. 1.
ii) From (6c) it is $R^\top P R - N = Q$ with $Q \succcurlyeq 0$ and hence

$$\theta^\top R^\top P R \theta = \theta^\top (Q + N)\theta, \quad (7)$$

since $\theta^\top N \theta$ is strictly positive for $\theta \in \mathbb{R}^\rho_+ - \{0\}$, it follows that also (7) is strictly positive for $\theta \in \mathbb{R}^\rho_+ - \{0\}$. Then (6b) holds and from i) the proof is complete. ∎

Remark 8: By following similar steps it can be easily shown that for any symmetric (entrywise) nonnegative matrix $\overline{N}$, the LMI $R^\top P R - \overline{N} \succcurlyeq 0$ implies $R^\top P R \succcurlyeq^{\mathbb{R}^\rho_+} 0$ which is equivalent by definition to $P \succcurlyeq^\mathcal{C} 0$.

B. Sign of quadratic functions on polyhedra

Let us consider a polyhedron $X \subset \mathbb{R}^n$ represented as (2) and a quadratic function

$$V(x) = x^\top P x + 2\nu^\top x + \omega, \quad x \in X, \quad (8)$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $\nu \in \mathbb{R}^n$ is a vector, ω is a real scalar with $\omega = 0$ if $0 \in X$. Define the symmetric matrix $\widehat{P} \in \mathbb{R}^{(n+1) \times (n+1)}$ as

$$\widehat{P} = \begin{pmatrix} P & \nu \\ \nu^\top & \omega \end{pmatrix}. \quad (9)$$

For the sign analysis of (8) we need the following preliminary result.

Lemma 9: Consider (8), (9) and let $\mathcal{C}_{\widehat{X}} \subset \mathbb{R}^{n+1}$ be the conic homogenization of the polyhedron $X \subset \mathbb{R}^n$. Consider the following constrained inequalities

$$V(x) \geq 0, \quad x \in X, \quad (10a)$$

$$\widehat{P} \succcurlyeq^{\mathcal{C}_{\widehat{X}}} 0, \quad (10b)$$

$$\widehat{R}^\top \widehat{P} \widehat{R} - \overline{N} \succcurlyeq 0, \quad (10c)$$

with $\omega = 0$ if $0 \in X$ and $\overline{N}$ any symmetric (entrywise) nonnegative matrix. Then the following conditions hold

i) (10a) $\Longleftrightarrow$ (10b)

ii) (10c) $\Longrightarrow$ (10a).

Proof: The proof of i) follows as a particular case of Proposition 2 in [14]. The proof of implication ii) is analogously to the proof of Lemma 7 ii). ∎

Remark 10: If $0 \notin X$, the equivalence i) in Lemma 9 is valid also for strict inequalities; the implication ii) in Lemma 9 is valid for strict inequalities if $\overline{N}$ is replaced by a matrix N with (strictly) positive entries, i.e. for the statements

$$V(x) > 0, \quad x \in X, \quad 0 \notin X \quad (11a)$$

$$\widehat{P} \succ^{\mathcal{C}_{\widehat{X}}} 0 \quad (11b)$$

$$\widehat{R}^\top \widehat{P} \widehat{R} - N \succcurlyeq 0, \quad (11c)$$

we have

i) (11a) $\Longleftrightarrow$ (11b)

ii) (11c) $\Longrightarrow$ (11a)

To obtain a strict inequality also for the case that $0 \in X$ an additional condition is required:

Lemma 11: With the notation of Lemma 9 assume $0 \in X$ and $\omega = 0$, i.e. $V(x) = x^\top P x + 2\nu^\top x$. Consider the statements

$$V(x) > 0, \quad x \in X \setminus \{0\}, \quad 0 \in X \quad (12a)$$

$$\left.\begin{array}{l} R^\top P R - N \succcurlyeq 0 \quad \wedge \\ 2\nu^\top R e_i \geq 0, \; i = 1, \dots, \lambda + \rho \end{array}\right\} \quad (12b)$$

where $e_i \in \mathbb{R}^{\lambda+\rho}$ denotes the i-th unit vector and N is a matrix with (strictly) positive entries. Then the following condition holds

$$(12b) \Longrightarrow (12a)$$

Proof: The proof is reported in [3]. ∎

Remark 12: The homogenization procedure adopted for i) in Remark 10, when applied to a polyhedron X which contains the origin, leads to a cone $\mathcal{C}_{\widehat{X}}$ which contains the ray $r_0 = \{\tilde{x} = \mathrm{col}(x, x_{n+1}), x \in X, x_{n+1} \in \mathbb{R}_+ \wedge x = 0\}$. The quadratic form in the augmented space, obtained from the quadratic function in Lemma 11 by using (9) with $\omega = 0$, is identically zero along the ray r_0. As a result, the strict cone-copositive condition has to be restricted to the (not closed) cone $\mathcal{C}_{\widehat{X}} - r_0$, which cannot be treated by our approach and there is no analogue of the equivalence as in Remark 10 i) for polyhedra X with $0 \in X$.

III. PWA SYSTEM AND SOLUTION CONCEPT

A. General setup and definitions

Let us consider a polyhedral partition $\{X_s\}_{s=1}^S$ of $\mathbb{R}^n$ and denote by Σ_0 the subset of indices s such that $0 \in X_s$ and Σ_1 its complement, i.e., $\Sigma_0 \cup \Sigma_1 = \{1, \dots, S\} =: \Sigma$. We consider the PWA system

$$\dot{x} = A_s x + b_s, \quad x \in X_s, \quad s = 1, \dots, S \quad (13)$$

where $A_s \in \mathbb{R}^{n \times n}$, $b_s \in \mathbb{R}^n$ with $b_s = 0$ for all $s \in \Sigma_0$. Each X_s is assumed to be a closed set, i.e. neighbouring polyhedra intersect nontrivially and there is some ambiguity in (13) on these intersections. This ambiguity can be resolved by considering solutions in the sense of Caratheodory, i.e. absolutely continuous (in particular, differentiable almost everywhere) functions x which satisfy (13) for *almost all* times. Note that we are not considering sliding modes or Filippov solutions.

Definition 13: A solution x is called *global* iff it is defined on the whole time interval $[0, \infty)$. A solution $x : [0, \omega) \to \mathbb{R}^n$, $\omega \in (0, \infty]$, of (13) is called *maximal* iff either $\omega = \infty$ (i.e. x is global) or there is no solution $\overline{x} : [0, \overline{\omega})$ with $\overline{\omega} > \omega$ which coincided with x on $[0, \omega)$. A non-global solution $x : [0, \omega)$, $\omega \in (0, \infty)$, is called *Zeno-solution* iff it is maximal and the limit of $\dot{x}(t)$ as $t \to \omega$ does not exist.

Note that in general there may be non-global solutions which are not Zeno-solutions.

B. The crossing condition

Recall that we assumed that the intersection

$$X_{ij} := X_i \cap X_j$$

of two polyhedral sets X_i, X_j of the polyhedral partition $\{X_s\}_{s=1}^S$ is either empty or a common face. In particular, any nonempty X_{ij} is again a polyhedron and has a $\mathcal{V}$-representation (2) which simply consists of the common vertices and rays of the $\mathcal{V}$-representations of X_i and X_j (under the nonrestrictive assumption that the rays in the $\mathcal{V}$-representations of X_i and X_j are normalized). Let

$$\Sigma_\cap := \{ (i,j) \mid X_{ij} \neq \emptyset \}$$

In the following we will introduce a crossing condition for which we focus on $n-1$-dimensional intersections, i.e. on the set

$$\Sigma_\cap^{n-1} := \{ (i,j) \in \Sigma_\cap \mid \dim X_{ij} = n - 1 \}.$$

Note that $(i,j) \in \Sigma_\cap$ if, and only if, $(j,i) \in \Sigma_\cap$. Each intersection X_{ij} with $(i,j) \in \Sigma_\cap^{n-1}$ is contained in a unique hyperplane $\mathcal{H}_{ij} = \mathcal{H}_{ji}$ which is given by

$$\mathcal{H}_{ij} = \{ x \in \mathbb{R}^n \mid h_{ij}^\top x + g_{ij} = 0 \}$$

for some normal vector $h_{ij} \in \mathbb{R}^n$ and offset $g_{ij} \in \mathbb{R}$. For any normal vector h_{ij} of $\mathcal{H}_{ij}$ also λh_{ij} for any $\lambda \in \mathbb{R} \setminus \{0\}$ is a normal vector of $\mathcal{H}_{ij}$. Hence it is no restriction of generality to assume that h_{ij} is chosen such that it points from X_i to X_j, i.e. we can assume that

$$h_{ij}^\top x + g_{ij} > 0, \quad x \in X_j - X_{ij}, \quad (14a)$$
$$h_{ij}^\top x + g_{ij} < 0, \quad x \in X_i - X_{ij}. \quad (14b)$$

Note that with this convention the normal vectors h_{ij} and h_{ji} will have opposite directions.

We can now formulate the following *crossing condition* for any $(i,j) \in \Sigma_\cap^{n-1}$ (cf. [2, Sec. 4.3] which presents a slightly weaker assumption):

$$(x^\top A_i^\top + b_i^\top)h_{ij} \cdot h_{ij}^\top (A_j x + b_j) > 0, \quad \forall x \in X_{ij}. \quad (15)$$

Since X_{ij} is connected, (15) implies that each factor $(x^\top A_i^\top + b_i^\top)h_{ij}$ and $h_{ij}^\top (A_j x + b_j)$ in (15) has constant sign. It is easily seen that the crossing condition for $(i,j) \in \Sigma_\cap^{n-1}$ is satisfied if, and only if, the crossing condition for $(j,i) \in \Sigma_\cap^{n-1}$ is satisfied, the only difference is that the positive product in one case results from two positive factors and in the other case from two negative factors. This redundancy can be eliminated by introducing the direction aware index set

$$\Sigma_\cap^{\mathrm{cross}} = \left\{ (i,j) \in \Sigma_\cap^{n-1} \;\middle|\; \begin{array}{l} (15) \text{ holds} \wedge \exists x \in X_{ij} : \\ h_{ij}^\top (A_j x + b_j) > 0 \end{array} \right\}. \quad (16)$$

While the index set $\Sigma_\cap$ is independent of the actual PWA dynamics, the index set $\Sigma_\cap^{\mathrm{cross}}$ depends on the specific system and maybe empty if no intersection X_{ij} satisfies (15). If $(i,j) \in \Sigma_\cap^{\mathrm{cross}}$ than any solution of (13) which crosses X_{ij} does this from X_i to X_j.

In practise it may be difficult to verify the crossing condition (15) because formally it has to be tested for all $x \in X_{ij}$. However, the following lemma shows that (15) can be verified by finding a solution of an LMI.

Lemma 14: Consider the PWA system (13) and define for every pair $(i,j) \in \Sigma_\cap^{n-1}$

$$Q_{ij} = A_i^\top h_{ij} h_{ij}^\top A_j \tag{17a}$$

$$\mu_{ij} = \frac{1}{2}(A_j^\top h_{ij} h_{ij}^\top b_i + A_i^\top h_{ij} h_{ij}^\top b_j) \tag{17b}$$

$$\zeta_{ij} = b_i^\top h_{ij} h_{ij}^\top b_j, \tag{17c}$$

and

$$\widehat{Q}_{ij} = \begin{pmatrix} Q_{ij} & \mu_{ij} \\ \mu_{ij}^\top & \zeta_{ij} \end{pmatrix}. \tag{18}$$

Furthermore, let $R_{ij} \in \mathbb{R}^{n \times (\lambda_{ij} + \rho_{ij})}$ be the ray matrix of the cone $\mathcal{C}_{X_{ij}}$ and let $\widehat{R}_{ij}$ be the ray matrix of the cone $\mathcal{C}_{\widehat{X}_{ij}}$. Assume that for all $(i,j) \in \Sigma_\cap^{n-1}$ with $0 \in X_{ij}$ the following inequalities hold:

$$2\mu_{ij}^\top R_{ij} e_\ell \geq 0, \quad \ell = 1, \ldots, \lambda_{ij} + \rho_{ij}.$$

Then the crossing condition (15) holds if the following LMIs hold for all $(i,j) \in \Sigma_\cap^{n-1}$

$$R_{ij}^\top Q_{ij} R_{ij} - N_{ij} \succcurlyeq 0, \ 0 \in X_{ij} \tag{19a}$$

$$\widehat{R}_{ij}^\top \widehat{Q}_{ij} \widehat{R}_{ij} - N_{ij} \succcurlyeq 0, \ 0 \notin X_{ij}, \tag{19b}$$

for some symmetric matrix N_{ij} with (strictly) positive entries.

Proof: With simple algebraic manipulations and by using (17), the inequality (15) can be rewritten as

$$x^\top Q_{ij} x + 2\mu_{ij}^\top x + \zeta_{ij} > 0, \quad x \in X_{ij}. \tag{20}$$

Then by applying Lemma 11 [Remark 10], one obtains that (19a) [resp., (19b)] implies (20) and hence (15). ∎

C. Existence and uniqueness of solutions of (13)

As far as we are aware, there is no general characterization of existence and uniqueness of global solutions of the PWA system (13) and in our main stability result we will just assume that existence of global solutions is guaranteed without giving explicit conditions on (13). However, in the following we will show that the above crossing condition ensures that for *almost all* initial values the solutions are unique and any maximal solution is either global or Zeno.

Lemma 15: Consider the PWA system (13) which satisfies the crossing condition (15) for all $(i,j) \in \Sigma_\cap^{n-1}$. Then for almost all initial conditions $x(0) = x_0 \in \mathbb{R}^n$ there exist a unique maximal solution $x : [0, \omega) \to \mathbb{R}^n$ which is either global or Zeno.

Proof: *Step 1:* Existence and uniqueness of local solution for almost all initial values.
If $x_0 \in \text{int}(X_s)$ for some $s \in \Sigma$ then x is a local solution of the PWA system (13) if, and only if, it is a (local) solution of $\dot{x} = A_s x + b_s$, $x(0) = x_0$. Hence existence and uniqueness of solutions is shown in that case. The set $\mathbb{R}^n \setminus \bigcup_{s=1}^S \text{int}(X_s)$ has (Lebesgue) measure zero, hence almost all initial values are covered by the above argument.
Step 2: Finite escape time cannot occur.
Since only finite partitions are considered the nonlinear differential equation is affinely bounded and hence finite

escape is not possible.
Step 3: Almost all non-global solutions are Zeno-solutions. Consider a non-global maximal solution $x : [0, \omega) \to \mathbb{R}^n$. By Step 2 we know that x remains bounded, hence from (13) and finiteness of the partition it also follows that $\dot{x}$ remains bounded on $[0, \omega)$. Hence $x_\omega := \lim_{t \to \omega} x(t)$ is well defined. By Step 1, the limit x_ω must be on the boundary of some X_s. If x_ω is contained in the intersection of exactly two regions X_i and X_j with $(i,j) \in \Sigma_\cap^{n-1}$, then x can be extended by a local solution of $\dot{x} = A_j x + b_j$ because the crossing condition ensures that $A_j x_\omega + b_j$ points into X_j and x_ω has a positive distance from any other boundary of X_j (otherwise x_ω would be in the intersection of at least three regions). Hence the solution x can be continued further, contradicting maximality of x. As a consequence, if x is maximal but not global, then x_ω must be contained in the intersection of at least three regions. These intersections have dimension of at most $n - 2$. Assume now x is not a Zeno-solution, which implies that $\dot{x}_\omega := \lim_{t \to \omega} \dot{x}(t)$ is well defined. In particular, there is at least one index $s \in \Sigma$ for which $\dot{x}_\omega = A_s x_\omega + b_s$ and $-(A_s x_\omega + b_s)$ points into X_s. Consequently, for each region X_s there is at most one trajectory ending in x_ω and the set of initial values which end in x_ω is therefore one-dimensional. Altogether the union of all initial values ending up in the at most $(n - 2)$-dimensional intersections of at least three polyhedra is at most $(n-1)$-dimensional and has therefore measure zero. ∎

Remark 16: Unfortunately, the crossing condition alone does not exclude Zeno-solutions for a large set of initial values. As an example consider the following PWA system on $\mathbb{R}^2$ (see also Figure 1):

$$\dot{x} = \begin{cases} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, & x_1 \geq 0, \ x_2 \geq 0, \\ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, & x_1 \leq 0, x_2 \geq 0, \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, & x_1 \leq 0, \ x_2 \leq 0, \\ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, & x_1 \leq 0, x_2 \leq 0. \end{cases}$$
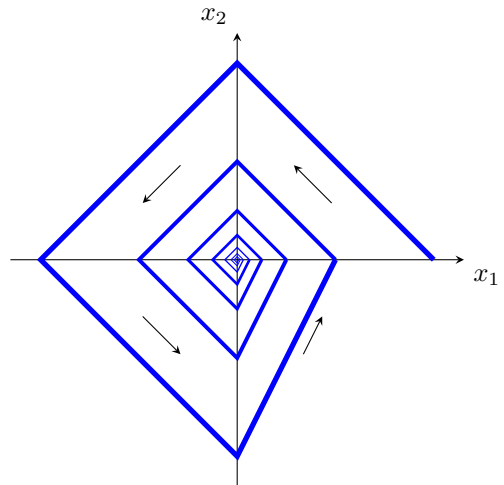

Fig. 1: Example which satisfies the crossing condition, but still exhibits Zeno-behavior.

The trajectories of this example move around the origin with constant speed and since the length halves after each round, the origin is reached in finite time. This specific example is already excluded by the assumption that $b_s = 0$ for all $s \in \Sigma_0$, however the same Zeno behavior occurs when the example is shifted away from the origin.

IV. ASYMPTOTIC STABILITY OF PWA SYSTEMS

Let us consider the quadratic functions

$$V_s(x) = x^\top P_s x + \nu_s^\top x + \omega_s, \quad x \in X_s \qquad (21)$$

with $s = 1, \dots, S$, $P_s \in \mathbb{R}^{n \times n}$ symmetric matrix, $\nu_s \in \mathbb{R}^n$, $\omega_s \in \mathbb{R}$ with $\omega_s = 0$ if $s \in \Sigma_0$. Define the (possibly discontinuous) candidate Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x) = V_s(x), \quad x \in \text{int}(X_s) \qquad (22)$$

with $s = 1, \dots, S$ and arbitrary elsewhere.

Lemma 11 can be applied to the different polyhedra of the polyhedral partition of $\mathbb{R}^n$ in order to get a set of LMIs which provides a sufficient condition for the positive sign of each (21) and negative sign of each corresponding time derivative along the system solution. Then we exploit Lemma 9 for providing a sufficient condition on the negative sign of the possible jumps of the candidate Lyapunov function when the solution crosses the boundaries shared by different polyhedra.

Theorem 17: Consider the system (13) with the polyhedral partition $\{X_s\}_{s=1}^S$ where each X_s is expressed according to (2) with corresponding matrices $\{R_s\}_{s \in \Sigma_0}$ and $\{\widehat{R}_s\}_{s \in \Sigma_1}$ as in (4) and (5), respectively. Assume that all maximal solutions are global, furthermore let $\Sigma_\cap^{\text{cross}} \subseteq \Sigma_\cap$ be given by (16). For $s \in \Sigma_0$ consider the set of LMIs

$$R_s^\top P_s R_s - N_s \succcurlyeq 0, \qquad (23a)$$
$$-R_s^\top (A_s^\top P_s + P_s A_s) R_s - M_s \succcurlyeq 0, \qquad (23b)$$

and, for $s \in \Sigma_1$,

$$\widehat{R}_s^\top \widehat{P}_s \widehat{R}_s - N_s \succcurlyeq 0, \qquad (24a)$$
$$-\widehat{R}_s^\top (\widehat{A}_s^\top \widehat{P}_s + \widehat{P}_s \widehat{A}_s) \widehat{R}_s - M_s \succcurlyeq 0, \qquad (24b)$$

where

$$\widehat{A}_s = \begin{pmatrix} A_s & b_s \\ 0 & 0 \end{pmatrix} \qquad (25)$$

and $\widehat{P}_s \in \mathbb{R}^{(n+1) \times (n+1)}$ are symmetric matrices in the form (9); N_s, M_s are symmetric (entrywise) positive matrices of appropriate dimensions. Furthermore, for $(i,j) \in \Sigma_\cap^{\text{cross}}$ consider the decreasing-jump-LMI

$$-\widehat{R}_{ij}^\top \widehat{P}_{ij} \widehat{R}_{ij} - \overline{N}_{ij} \succcurlyeq 0, \qquad (26a)$$

and, for $(i,j) \in \Sigma_\cap \setminus (\Sigma_\cap^{\text{cross}} \cup \{ (i,j) \mid (j,i) \in \Sigma_\cap^{\text{cross}} \})$, the continuity equality constraint

$$\widehat{R}_{ij}^\top \widehat{P}_{ij} \widehat{R}_{ij} = 0, \qquad (26b)$$

where $\widehat{R}_{ij}$ is the ray matrix corresponding to the cone $\mathcal{C}_{\widehat{X}_{ij}}$, $\overline{N}_{ij}$ is a symmetric matrix with nonnegative entries and

$$\widehat{P}_{ij} := \begin{pmatrix} P_j - P_i & \nu_j - \nu_i \\ (\nu_j - \nu_i)^\top & \omega_j - \omega_i \end{pmatrix}.$$

Assume that the LMIs (23), (24), (26), and, for $s \in \Sigma_0$, the inequalities

$$\left. \begin{array}{r} 2\nu_s^\top R_s e_i \geq 0, \\ -2\nu_s^\top A_s R_s e_i \geq 0, \end{array} \right\} \quad i = 1, \dots, \lambda_s + \rho_s, \qquad (27)$$

have a solution $\{P_s, \nu_s, \omega_s, N_s, M_s\}_{s=1}^S$ with $\omega_s = 0$ for $s \in \Sigma_0$. Then all solutions of the PWA system (13) converge asymptotically to zero.

The proof of Theorem 17 is based on the following Lemma highlighting specific properties of the functions V_s.

Lemma 18: Let V_s be given by (21), let

$$\dot{V}_s(x) := x^\top (A_s^\top P_s + P_s A_s) x + (2b_s^\top P_s + \nu_s^\top A_s) x + \nu_s^\top b_s$$

and assume that (23), (24), (26) and (27) hold. Then V_s is positive and $\dot{V}_s$ is negative definite in the following sense:

$$V_s(x) > 0 \quad \forall x \in X_s \setminus \{0\}, \qquad (28a)$$
$$\dot{V}_s(x) < 0 \quad \forall x \in X_s \setminus \{0\}. \qquad (28b)$$

Furthermore, each V_s is radially unbounded in the following sense: for all $\overline{v} \in V_s(X_s) \subseteq \mathbb{R}_+$ the preimage

$$V_s^{-1}([0, \overline{v}]) := \{ x \in X_s \mid V_s(x) \leq \overline{v} \} \text{ is compact} \qquad (29)$$

and for all $\varepsilon > 0$ exists $\overline{v} > 0$ such that

$$V_s^{-1}([0, \overline{v}]) \subseteq \{ x \in X_s \mid \|x\| \leq \varepsilon \}. \qquad (30)$$

Note that for $s \in \Sigma_1$ and for sufficiently small $v > 0$ the set $V_s^{-1}([0, \overline{v}])$ will be empty.

Proof: From Remark 10 and Lemma 11 together with (24a), (23a) and (27) we conclude that V_s is positive definite and the same Remark and Lemma together with (24a), (23a) and (27) ensure that $\dot{V}_s(x)$ is negative definite. Radial unboundedness (29) follows from the quadratic nature of V_s and (30) is a consequence of continuity and positive definiteness of V_s. ∎

Proof of Theorem 17. Choose the PWQ function V as in (8) as a candidate Lyapunov function; in fact, V can be defined also on $X \setminus \bigcup_s \text{int} X_s = \bigcup_{(i,j) \in \Sigma_\cap} X_{ij}$ as follows:

$$V(x) = \begin{cases} V_j(x), & x \in X_{ij} \text{ with } (i,j) \in \Sigma_\cap^{\text{cross}}, \\ V_i(x) = V_j(x), & x \in X_{ij} \text{ with } (i,j), (j,i) \notin \Sigma_\cap^{\text{cross}}; \end{cases}$$

where the equality $V_i(x) = V_j(x)$ in the second case is a consequence from condition (26b). Note furthermore, that due to condition (26a) we have for all $x \in X_{ij}$ with $(i,j) \in \Sigma_\cap^{\text{cross}}$:

$$V(x) = V_j(x) \leq V_i(x). \qquad (31)$$

Step 1: We show V is decreasing along solutions.
Let $x : [0, \infty) \to \mathbb{R}^n$ be any (global) solution of the PWA system (13) and let $v(t) := V(x(t))$. By definition x is

differentiable almost everywhere, in particular for almost all $t \in [0, \infty)$ there exists $s \in \Sigma$, such that

$$\dot{x}(t) = A_s x(t) + b_s \quad \text{and} \quad x(t) \in X_s$$

and invoking (28b) we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t} v(t) = \tfrac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \dot{V}_s(x(t)) < 0. \tag{32}$$

It remains to show that v does not jump upwards at those time points where v is not continuous. Therefore let $t_* \in [0, \infty)$ be some point where v is discontinuous. Since x is continuous it follows that $x(t_*) \in X_{ij}$ for some $(i, j) \in \Sigma_\cap$. If $(i, j) \in \Sigma_\cap^{\text{cross}}$ [or $(j, i) \in \Sigma_\cap^{\text{cross}}$] then the solution crosses X_{ij} from region X_i towards region X_j [or vice versa] and (31) ensures that $v(t_*^-) = V_i(x(t^*)) \geq V_j(x(t^*) = v(t_*^+)$ [or $v(t_*^-) = V_j(x(t^*)) \geq V_i(x(t^*) = v(t_*^+)$]. For all other cases, V is continuous on X_{ij} by (26b) which contradicts our assumption that v is discontinuous at t_*.

Step 2: We show that all solutions converge to zero. Let $\underline{v} := \lim_{t \to \infty} v(t)$, which is well-defined because v is monotonically decreasing. Seeking a contradiction assume $\underline{v} > 0$ then x evolves within the set

$$K := \bigcup_{s=1}^{S} V_s^{-1}([\underline{v}, v(0)])$$

which, due to positive definiteness of all V_s does not contain the origin. Furthermore, K is compact because of (29) and continuity of V_s. Hence, for almost all $t \geq 0$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} v(t) \leq \min_{s \in \Sigma} \min_{x \in K \cap X_s} \dot{V}_s(x) =: \delta < 0$$

which implies that

$$v(t) \leq v(0) - \delta t,$$

contradicting $v(t) \geq \underline{v}$ for all $t \geq 0$ and we have shown that $V(x(t)) \to 0$ as $t \to \infty$.

Step 3: $x(t)$ converges to zero. Seeking again a contradiction assume $x(t) \not\to 0$. Then there exists $\varepsilon > 0$ and an increasing unbounded sequence $\{t_k\}_{k \in \mathbb{N}}$ with $\|x(t_k)\| \geq \varepsilon$. Since $V(x(t_k)) \to 0$ as $k \to \infty$, this contradicts (30). ∎

Remark 19: The current formulation of Theorem 17 requires continuity of the PWQ-LF on all intersections X_{ij} where the crossing condition (17) is not satisfied. This assumption can significantly be relaxed, for example intersections X_{ij} where solutions can never pass through (for example when all vector fields point away from X_{ij}) are irrelevant and the corresponding constraint (26b) can be omitted.

Remark 20: Local stability in a bounded (polyhedral) region which is an invariant set containing the origin, can be proved with straightforward reformulations of Theorem 17.

V. SIMULATION RESULTS

Consider the switched second order system $\dot{x} = A(x)x$ with $A(x) = [1, -5; 0.2, 1]$ for x being in the first and third quadrants of the state space, and $A(x) = [1, -0.2; 5, 1]$ when x is in the second and forth quadrants. The set of LMIs (23) with the decreasing jump LMIs (26) was solved

by considering a uniform partition into 108 cones. It easy to verify that the crossing conditions (15) are satisfied for all the common boundaries.
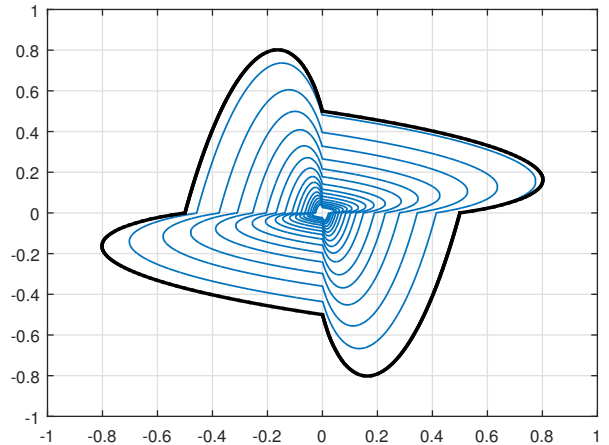

Fig. 2: State space trajectory (blue line) for the switched second order system and a Lyapunov function level curve (black line).

Figure 2 shows the state space trajectory and a level curve of the Lyapunov function. In Fig. 3 it is reported the discontinuous (in some time instants) Lyapunov function computed along the trajectory together with the time evolutions of the state variables.

As a second example let us consider the opinion dynamics model in [15] which can be represented in the form (13) with the partition reported in Fig. 4.
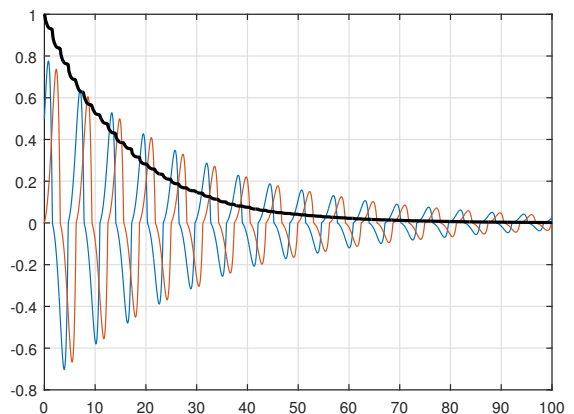

Fig. 3: Time evolutions of the state variables (blue and red lines) and Lyapunov function (black line) which present discontinuities at some time instants.
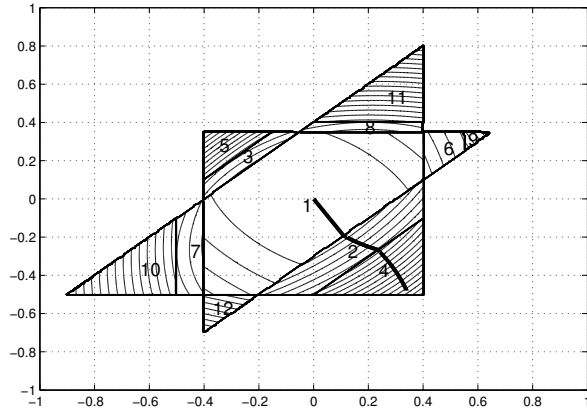
Fig. 4: State space for the opinion dynamics model: a state trajectory (black line) and some level curves of the PWQ-LF.
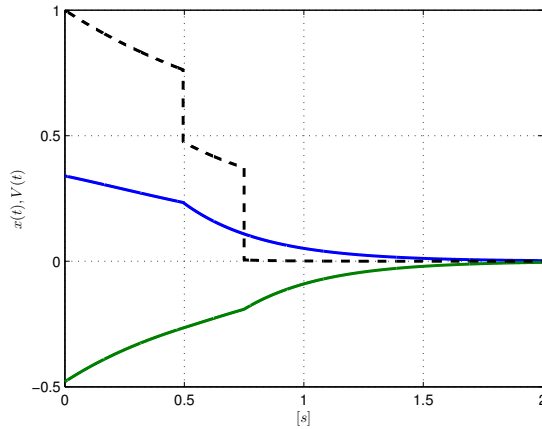

Fig. 5: Time evolutions of the state variables and discontinuous PWQ-LF (dashed line).

The system matrices are

$$A_1 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}, \ A_2 = A_3 = \begin{pmatrix} -3 & 0 \\ -1 & -2 \end{pmatrix},$$

$$A_4 = A_5 = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}, \ A_6 = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix}, \ A_8 = \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix},$$

$$A_9 = A_{10} = \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix}, \ A_{11} = A_{12} = \begin{pmatrix} -3 & 1 \\ 0 & -1 \end{pmatrix},$$

and $b_s = 0$ for all $s \in \{1, \ldots, 12\}$. Depending on the initial conditions, the convergence to an equilibrium or a clustering with different steady state values can occur [9].

The local asymptotic stability of the origin of (13) can be analyzed by using the proposed PWQ-LF approach. By applying Remark 20, we found a PWQ-LF for the PWA dynamics in the polyhedral region shown in Fig. 4 which is contained in the feasibility domain. In the same region, which is an invariant set, the crossing conditions are satisfied

and hence the origin is asymptotically stable for any initial condition belonging to that region. Fig. 5 shows the time evolutions of state variables and PWQ-LF. It is evident the discontinuity of the PWQ-LF when the trajectory crosses the polyhedra boundaries.

VI. CONCLUSIONS

Piecewise quadratic Lyapunov functions (PWQ-LFs) have been used in the literature for the analysis of the asymptotic stability for piecewise affine (PWA) systems. In this paper, by exploiting the cone-copositivity approach, the problem of finding a PWQ-LF has been formulated in terms of a set of LMIs. The resulting PWQ-LF is not required to be continuous and non-increasing conditions at the polyhedra boundaries are included in the problem in terms of further LMIs. Simulation results have shown the effectiveness of the proposed approach.

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