Modeling water hammers via PDEs and switched DAEs with numerical justification

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Abstract: In water distribution networks instantaneous changes in valve and pump settings may introduces jumps and peaks in the pressure. In particular, a well known phenomenon in response to the sudden closing of a valve is the so called water hammer, which (if not taken into account properly) may destroy parts of the water network. It is classically modeled as a system of hyperbolic partial differential equations (PDEs). After discussing this PDE model we propose a simplified model using switched differential-algebraic equations (DAEs). Switched DAEs are known to be able to produce infinite peaks in response to sudden structural changes. These peaks (in the mathematical form of Dirac impulses) can easily be predicted and may allow for a simpler analysis of complex water networks in the future. As a first step toward that goal, we verify the novel modeling approach by comparing these two modeling techniques numerically for a simple set up consisting of two reservoirs, a pipe and a valve.

Keywords: water distribution networks; water hammer; compressible flow; switched DAEs; Dirac impulses;

1 Introduction

The occurrence of hydraulic transients in the operation of water distribution network is inevitable. Such transients are planned or accidental changes of the network configuration. These sudden structural changes can have dramatic effects in flow regimes, ranging from pump defects to catastrophic pipeline failures. The flow of water in pipes is usually model as system of nonlinear hyperbolic balance laws (i.e. partial differential equations, PDEs), see e.g. Izquierdo et al. (2004), where the sudden structural changes lead to large peaks and fast transients in the solution.

We propose to model such fast transients in the framework of switched differential algebraic equation (switched DAEs). This framework was originally introduced for modeling electrical circuits (Trenn, 2012) and allows a precise mathematical description of peaks and fast transients in the form of Dirac impulses and jumps.

Our focus in this paper is on the so-called water hammer, which results from sudden changes of velocity in pipelines and can cause large pressures magnitudes. It is usually created by rapidly closing valves, shutting off or restarting pumps. Our goal is to show that these pressure peaks occurring in the PDE simulations can be well approximated by a suitable switched DAE model.

The paper is organized as follows. In Section 2 the water network and its components are defined as a graph and the mathematical models of the pipes and other components (like reservoir and valves) are introduced. Section 3 provides a brief introduction to the some mathematical preliminaries for system of conservations laws and switched DAEs. In Section 4 we study in detail a simple water network which exhibits a water hammer; in particular, we derive the corresponding PDE model as well as a switched DAE model. In Section 5 a numerical comparison of the PDE and switched DAE model are presented.

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2 Water network modeling

The structure of a water network can be modeled as a finite undirected graph (V, E) where V is the set of nodes and $E \subseteq V \times V$ is the set of edges. Each edge $e \in E$ corresponds to a pipe of the water network and the nodes $v \in V$ are the connections or endpoints of pipes, including junctions, pumps, valves, or reservoirs.

In the following we will describe the model of some water network element. The two main physical quantities involved in the descriptions are pressure and flow, whose values at the end points of the pipes are related to each other corresponding to the type of node. Furthermore, the modeling of the flow in the pipes also involves density of the water. Usually, water is assumed to be incompressible, i.e. the density is assumed to be constant. However, our focus is on modeling the water hammer effect and for this it is necessary to take into account the (slight) compressibility of water.

2.1 *Models of water flow in pipe*

The water flow in a pipe can be modeled in two different ways depending on whether the compressibility of water is taken into account or not. In order to study transient phenomena like water hammer it is necessary to model compressibility, in particular, density and mass flow become space-dependent quantities. On the other hand, to understand the qualitative behavior, in particular, in large networks, it often suffices to model water as incompressible fluid. We will briefly introduce both models in the following.

2.1.1 Compressible flow in a pipe Following Wylie and Streeter (1978) and Adami et al. (2012) we use the following pressure law for compressible fluids :

$$P(\rho) = P_a + K\frac{\rho - \rho_a}{\rho_a}, \tag{1}$$

where $K > 0$ is the so called *bulk modulus*, $P_a > 0$ is the atmospheric pressure and $\rho_a > 0$ is the density at atmospheric pressure. The bulk modulus is related to the speed of sound $c > 0$ as follows:

$$c^2 = \frac{\partial P}{\partial \rho} = K/\rho_a. \tag{2}$$

Note that $\beta := 1/K$ is the so called compressibility coefficient.

We consider a completely filled pipe of length $L > 0$ with mass density $\rho(x,t) > 0$ and mass flux $q(x,t) \in \mathbb{R}$ both defined on $[0, L] \times \mathbb{R}_+$. The compressible flow of water in the pipe can be modelled by the balance law of the following form (Herty et al., 2010, Sec. 2):

$$\begin{aligned} \partial_t \rho + \partial_x q &= 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + P(\rho) \right) &= -c_f \frac{q\,|q|}{2D\rho}, \end{aligned} \tag{3}$$

with the pressure law $P : \mathbb{R}_+ \to \mathbb{R}_+$ given by (1) and where $c_f > 0$ is the friction against the pipe wall and $D > 0$ is the diameter of the pipe. The initial condition for (3) is:

$$q(x, 0) = q_0(x) \text{ and } P(\rho(x, 0)) = p_0(x) \quad x \in [0, L], \tag{4}$$

for some initial flow $q_0 : [0, L] \to \mathbb{R}$ and some initial pressure $p_0 : [0, L] \to \mathbb{R}_+$. Note that the initial condition is given implicitly in terms of the pressure and not explicitly in terms of the density. The reason is that the pressure is the more relevant physical quantity, in particular, when the pipes are coupled with other water network elements. When the individual pipes are connected with other elements of the overall water distribution network, additional boundary conditions will be imposed.

2.1.2 Quasi stationary water flow model After some initial transient behaviour, the water flow in the pipe may be assumed to get stationary, i.e. the flow is location independent and we write $Q(t) = \frac{q(x,t)}{A}$ (mass flux is mass flow per unit area), $x \in [0, L]$ and where $A = \pi D^2/4$ is the area of the pipe. Furthermore the density is assumed constant in space and time, i.e. $\rho(x,t) = \rho$ for $(x,t) \in [0, L] \times \mathbb{R}_+$ and the pressure variable $p(x,t)$ is not coupled to the density via (1) anymore (in particular, water is considered incompressible). The remaining dynamical behavior in the variables $Q(t)$, $P_0(t) = p(0,t)$ and $P_L(t) = p(L,t)$ can be described by the following ODE (Jansen and Pade, 2013; Jansen and Tischendorf, 2014; Chaudhry and Mays, 2012):

$$\frac{dQ}{dt} + \frac{A}{L}(P_L - P_0) + \frac{c_f Q\,|Q|}{2DA\rho_a} = 0. \tag{5}$$

2.2 Other network elements

2.2.1 Reservoir A reservoir is a node in the water network graph with arbitrary mass flow but with given pressure. In particular, connecting a pipe with its left side to a reservoir adds a boundary condition for $p(0,t)$ in the PDE model (3) and constraints the variable P_0 in the ODE-model (5).

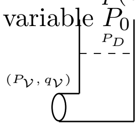

Fig. 1. Valve in combination with reservoir

2.2.2 Valve For simplicity, a valve here is modeled in combination with a reservoir, see Figure 1, which imposes the following algebraic constraints.

$$\text{valve open:} \quad P_\mathcal{V} = P_D, \quad ; \quad \text{valve closed:} \quad Q_\mathcal{V} = 0,$$

where $P_\mathcal{V}, Q_\mathcal{V}$ denotes the pressure and flow through the valve and P_D is the reservoir pressure. In the next section we present some general mathematical preliminaries related to (3) and (5).

3 Preliminaries

3.1 Systems of hyperbolic balance laws

We consider systems of balance laws in one space dimension of the following form

$$\begin{aligned} U_t + (F(U))_x &= S(U), && \text{on } [a, b] \times \mathbb{R}_+, && \text{(6a)} \\ U(x, 0) &= U_0(x), && x \in \Omega, && \text{(6b)} \\ \Psi_1(t, U(t, a)) &= 0, && t > 0, && \text{(6c)} \\ \Psi_2(t, U(t, b)) &= 0, && t > 0,, && \text{(6d)} \end{aligned}$$

where $U : \Omega \times \mathbb{R}_+ \to \mathcal{U} \subseteq \mathbb{R}^n$ with $\Omega := [a, b] \subseteq \mathbb{R}^n$ being the domain and the open connected set $\mathcal{U} \subseteq \mathbb{R}^n$ being the range of the problem, $F : \mathcal{U} \to \mathbb{R}^n$ is the flux function, $S : \mathcal{U} \to \mathbb{R}^n$ is the source term, $U_0 : \Omega \to \mathcal{U}$ is the initial data and $\Psi_i : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}^{b_i}$, $i = 1, 2$, $b_1 + b_2 = n$, are the time-varying, implicit boundary conditions. If the flux F is differentiable, then the PDE (6a) can be written in quasi-linear form by

$$U_t + A(U)U_x = S(U), \tag{7}$$

where $A(u) := \mathbf{D}F(u)$ for $u \in \mathcal{U}$ and $\mathbf{D}F$ denotes the Jacobian of F.

The system (6) does not necessarily admit a classical (i.e. differentiable) solution, even for "well-behaved" initial data U_0. Hence, weak solutions will be considered defined as follows (c.f. Bressan (2013)).

Definition 1. (Weak solution). A function $U : \Omega \times \mathbb{R}_+ \to \mathcal{U}$, is a *weak solution* of (6a) if, and only if, U, $F(U)$ and $S(U)$ are locally integrable and for every $\phi \in \mathcal{C}^1(\mathbb{R}_+ \times \Omega \to \mathbb{R})$ with compact support, one has

$$\int_{\mathbb{R}_+ \times \Omega} (U\phi_t + F(U)\phi_x)dxdt = -\int_{\mathbb{R}_+ \times \Omega} S(U)\phi dxdt. \tag{8}$$

The boundary values are assumed to be satisfied in the Godunov/trace sense, i.e. see [Colombo and Garavello (2008),Colombo and Garavello (2006)] for more details.

When considering weak solutions of (6) uniqueness of solutions cannot be expected in general. Under certain assumptions, uniqueness can be recovered by imposing a so called entropy condition as follows (c.f. Freistuhler (1998)).

Definition 2. (Entropy solution). A differentiable function $\eta \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R})$ is called entropy of the PDE (6a) if there exists an entropy flux $q \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R})$ such that

$$\mathbf{D}\eta(u) \cdot \mathbf{D}F(u) = \mathbf{D}q(u) \quad \forall u \in \mathcal{U}. \tag{9}$$

For a given entropy η with corresponding entropy flux q, a weak solution U of (6a) is called *entropy solution* if it additionally satisfies

$$\eta(U)_t + q(U)_x \leq \mathbf{D}\eta(U) \cdot S(U)$$

in a weak sense.

Remark 3. For any classical (i.e. differentiable) solution U of (6a) it is easily seen that (9) implies

$$\eta(U)_t + q(U)_x = \mathbf{D}\eta(U) \cdot S(U).$$

For existence and uniqueness of weak entropy solutions of the initial/boundary value problem (6) the following well-posedness assumptions are usually imposed (c.f. Borsche et al. (2012)):

(B-I) *Bounded variation*
The initial date U_0 is a function of bounded variation with sufficiently small total variation.

(B-II) *Strict hyperbolicity*
The system of balance laws (6a) in quasi-linear form (7) is *strictly hyperbolic*, i.e., for every $u \in \mathcal{U} \subseteq \mathbb{R}^n$, the Jacobian matrix $A(u)$ of the flux function F

has n real, distinct eigenvalues denoted by $\lambda_i(u)$, $i = 1, \cdots, n$ and are ordered as follows:

$$\lambda_1(u) < \lambda_2(u) < \lambda_3(u) \cdots < \lambda_n(u).$$

(B-III) *Genuine nonlinearity and linear degeneracy*
For a strictly hyperbolic balance law (6a) consider the n eigenvalue/eigenvector pairs $(\lambda_j(u), r_j(u))$, $j = 1, \ldots, n$ with differentiable map $u \mapsto \lambda_j(u)$. Assume that for each j either $(\lambda_j(\cdot), r_j(\cdot))$ is genuinely nonlinear, i.e.

$$\mathbf{D}\lambda_j(u) \cdot r_j(u) \neq 0, \quad \forall u \in \mathbb{R}^n,$$

or $(\lambda_j(\cdot), r_j(\cdot))$ is linearly degenerate, i.e.

$$\mathbf{D}\lambda_j(u) \cdot r_j(u) = 0, \quad \forall u \in \mathbb{R}^n.$$

(B-IV) S is locally Lipschitz.

(B-V) *Feasible boundary conditions*
For a strictly hyperbolic balance law (6a) assume that

$$\lambda_{b_2}(u) < 0 < \lambda_{b_2+1} \quad \forall u \in \mathcal{U}$$

and denote with $r^1_+(u)$, $r^2_+(u)$, $\ldots$, $r^{b_1}_+(u)$ the collection of eigenvectors of $A(u)$ corresponding to the positive eigenvectors of $A(u)$. Assume that for all $u \in \mathcal{U}$ tho following feasibility assumption for the left boundary condition holds (c.f. Colombo and Garavello (2006) for feasibility of general coupling conditions):

$$\forall u \in \mathcal{U}: \quad \det\left[\mathbf{D}_u \Psi_1(t, u) \cdot R^+(u)\right] \neq 0, \tag{10a}$$

where $R^+(u) := [r^1_+(u), \ldots, r^{b_1}_+(u)]$. To formulate a feasibility assumption for the right boundary condition, we can substitute the space variable x by $a + b - x$ in (6) with variable $\overline{U} : \mathbb{R}^+ \times \Omega \to \mathcal{U}$ instead of U. For $\overline{u} \in \mathcal{U}$, let $r^1_-(\overline{u}), \ldots, r^{b_2}_-$ be the eigenvectors corresponding to the positive eigenvalues of $\overline{A}(\overline{u}) = -A(\overline{u})$. With $R^-(\overline{u}) := r^1_-(\overline{u}), \ldots, r^{b_2}_-(\overline{u})]$ we assume

$$\forall \overline{u} \in \mathcal{U}: \quad \det\left[\mathbf{D}_{\overline{u}} \Psi_2(t, \overline{u}) \cdot R^-(\overline{u})\right] \neq 0. \tag{10b}$$

3.2 Switched DAEs

A switched linear differential algebraic equations (DAEs) is of the form

$$E_\sigma \dot{x}(t) = A_\sigma x(t) + f(t) \tag{11}$$

where $(E_\mathbf{p}, A_\mathbf{p}) \in \mathbb{R}^{n \times n}$, $\mathbf{p} \in \{1, \cdots, \mathbf{P}\}$, $\mathbf{P} \in \mathbb{N}$ and $\sigma : \mathbb{R} \to \{1, \cdots, \mathbf{P}\}$ is a piecewise constant switching signal, which is assumed to be right continuous and to have locally finitely many jumps. For existence and uniqueness of solutions of (switched) DAEs, we need the following regularity notion of matrix pairs.

Definition 4. (Regularity of a matrix pair). A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is said to be regular, if the polynomial $\det(sE - A)$ is not the zero polynomial.

Theorem 5. (Trenn (2012, Cor. 6.5.2)). Consider the switched DAE (11) with regular matrix pairs $(E_\mathbf{p}, A_\mathbf{p})$, $\mathbf{p} \in \{1, \cdots \mathbf{P}\}$ and assume $\sigma_{(-\infty,0)}$ is constant. Then for every piecewise-smooth f there exists a globally defined solution x in a distributional sense which is uniquely determined by $x(0^-)$. In particular, jumps and Dirac impulses at the switching times are uniquely determined.

It is not possible to view (11) as an equation in the general space of distributions in the sense of Schwartz (1957, 1959); in fact, it is necessary to restrict the solution space to the so called space of piecewise-smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, see Trenn (2009a,b) for details.

For analyzing switched DAEs and obtaining explicit solution formulas for (11), the following quasi-Weierstrass form (QWF) is helpful see [QWF, c.f. Weierstraß (1868)].

Proposition 6. A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if and only if there exist invertible transformation matrices $S, T \in \mathbb{R}^{n\times n}$ which put (E, A) into QWF [QWF, c.f. Weierstraß (1868)]

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)$$

where $N \in \mathbb{R}^{n_2\times n_2}$, with $0 \leq n_2 \leq n$ is a nilpotent matrix, $J \in \mathbb{R}^{n_1\times n_1}$ with $n_1 := n - n_2$ is some matrix and I is the identity matrix of the appropriate size. Moreover, index of the linear DAE is the minimum value ν for which $N^\nu = 0$. The notion of the index of a DAE can be generlized to nonlinear DAEs in the form of the differential index see Borutzky (2009) for details.

4 Analysis of a simple water network

We want to study a simple water network consisting of an upstream reservoirs with given pressure P_U, one pipe of length L and a valve combined with a downstream reservoir with pressure $P_D < P_U$, see Figure 2. This setup is used in Chen et al. (2015), to study optimal boundry condition to mitigate water hammer, and in Szydłowski (2002) for the construction of accurate finite volume scheme. Here, this setup is used to compare the novel modeling approach via switched DAEs with the more classical PDE models with regard to the water hammer effect.

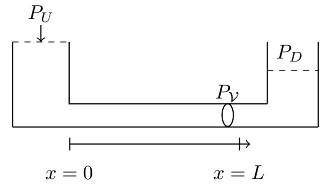

Fig. 2. Simple set up for water hammer

It is well known that such a configuration will result in a water hammer if the valve is instantaneously closed. Our goal is to compare the PDE model and the switched DAE model for this configuration with respect to capturing the water hammer effect. For that we will first discuss the solution concepts for both approaches for the specific example water network.

4.1 PDE solution framework

The balance law in the pipe is given by (3), i.e. in terms of (6a) we have

$$\Omega = [0, L], \quad \mathcal{U} \subseteq \mathbb{R}_+ \times \mathbb{R}, \quad U(x,t) = \begin{pmatrix} \rho(x,t) \\ q(x,t) \end{pmatrix},$$

and, for $u = (\rho, q) \in \mathcal{U}$,

$$F(u) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + P(\rho) \end{pmatrix}, \quad S(u) = \begin{pmatrix} 0 \\ -c_f \frac{q\,|q|}{2D\rho} \end{pmatrix},$$

with pressure law (1). The initial condition is given by (4) where the initial density ρ_0 is induced by the usually considered initial pressure p_0 via the invertible pressure law. The boundary conditions in terms of (6c),(6d) are

$$\Psi_1(t, (\rho, q)) = P(\rho) - P_U, \quad \forall t \in \mathbb{R}_+,$$

and

$$\Psi_2(t, (\rho, q)) = \begin{cases} P(\rho) - P_D, & t \in (0, t_S), \\ q, & t > t_S, \end{cases} \tag{12}$$

Note that the discontinuity induced by the switch only occurs in the boundary condition, therefore wellposedness can be studied for each mode individually as the PDE can be "restarted" at time $t = t_S$ with the initial value given by the final value of the solution on the time interval $[0, t_S]$; in particular, the wellposedness conditions (B-II), (B-III), (B-IV) can be checked independently of the valve's state:

(B-I) For the numerical simulations we will impose constant initial conditions, which has zero total variation.

(B-II) The Jacobian of the flux function in (3) is given by

$$A(\rho, q) = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{\rho^2}+P'(\rho) & 2\frac{q}{\rho} \end{pmatrix}$$

and invoking the pressure law (1) we see that $P'(\rho) = \frac{K}{\rho_a} > 0$ independently of ρ. Hence the eigenvalues of $A(\rho, q)$ are

$$\lambda_{1/2}(\rho, q) = \frac{q}{\rho} \pm \sqrt{\frac{K}{\rho_a}}.$$

Consequently, $\lambda_1(u) > \lambda_2(u)$ for all $u = (\rho, q) \in \mathcal{U}$ and hence (3) with pressure law (1) is strictly hyperbolic.

(B-III) The eigenvectors of $A(u)$ are

$$r_1(u) = \begin{pmatrix} 1 \\ \lambda_1(u) \end{pmatrix}, \quad r_2(u) = \begin{pmatrix} 1 \\ \lambda_2(u) \end{pmatrix}.$$

Hence

$$\begin{aligned}\mathbf{D}\lambda_1(u) \cdot r_1(u) &= \left[-\frac{q}{\rho^2}, \; \frac{1}{\rho}\right] \begin{pmatrix} 1 \\ \frac{q}{\rho}+\sqrt{\frac{K}{\rho_a}} \end{pmatrix} \\ &= \frac{1}{\rho}\sqrt{\frac{K}{\rho_a}} \neq 0 \quad \forall u = (\rho, q) \in \mathcal{U}\end{aligned}$$

and, analogously,

$$\mathbf{D}\lambda_2(u) \cdot r_2(u) = -\frac{1}{\rho}\sqrt{\frac{K}{\rho_a}} \neq 0 \quad \forall u = (\rho, q) \in \mathcal{U}.$$

Consequently, genuine nonlinearity is established.

(B-IV) Clearly S is locally Lipschitz continuous.

(B-V) In order to have sign-definite eigenvalues, i.e.

$$\lambda_1(u) > 0 > \lambda_2(u) \quad \forall u \in \mathcal{U},$$

we have to restrict our attention to the so-called subsonic case, i.e.

$$\mathcal{U} \subseteq \left\{ (\rho, q) \in \mathbb{R}_+ \times \mathbb{R} \;\middle|\; \frac{q}{\rho} < \sqrt{\frac{K}{\rho_a}} \right\}.$$

In that case we have $R^+(u) = [r_1(u)]$ and

$$D_u\Psi_1(t, u) \cdot R^+(u) = \begin{bmatrix} P'(\rho) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1(u) \end{bmatrix} = P'(\rho) \neq 0,$$

i.e. condition (10a) is satisfied for all $u = (\rho, q) \in \mathcal{U}$ and $t > 0$.

For the right boundary consider $R^-(\overline{u}) = [r_2(\overline{u})]$ and we have

$$\begin{aligned}&D_{\overline{u}}\Psi_2(t, \overline{u}) \cdot R^-(\overline{u}) \\ &= \begin{cases} \begin{bmatrix} P'(\overline{\rho}) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_2(\overline{u}) \end{bmatrix} = P'(\overline{\rho}) \neq 0, & t \in (0, t_S), \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_2(\overline{u}) \end{bmatrix} = \lambda_2(\overline{u}) \neq 0, & t > t_S, \end{cases}\end{aligned}$$

for all $\overline{u} = (\overline{\rho}, \overline{q}) \in \mathcal{U}$; hence, due to the restriction to the subsonic case, condition (10b) is satisfied.

Altogether, we can expect numerical simulations to result in reasonable approximations of solutions of the corresponding initial/boundary value problem (6) as long as the solution remain in the subsonic-case.

4.2 Switched DAE framework

The quasi-stationary model (5) together with the valve-depending boundary conditions (valve open on $[0, t_S)$ and valve closed on $[t_S, \infty)$) for a setup as shown in Figure 2 leads to a switched DAE of the form

$$E_\sigma \dot{x} = A_\sigma x + f + g_\sigma(x), \tag{13}$$

where $x = (Q, P_0, P_L)^\top$, $\quad \sigma(t) = \begin{cases} 1, & t \in [0, t_S), \\ 2, & t \geq t_S, \end{cases}$ and

$$\left.\begin{aligned}&E_1 = E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ &f = \begin{cases} (0, -P_U, -P_D)^\top, & \text{on } [0, t_S), \\ (0, -P_U, 0)^\top, & \text{on } [t_S, \infty), \end{cases} \quad g_1(x) = g_2(x) \\ &= g(x) = \begin{pmatrix} -c_2 Q|Q| \\ 0 \\ 0 \end{pmatrix}, \quad c_1 = \frac{A}{L} > 0, c_2 = \frac{c_f}{2DA\rho_a} > 0\end{aligned}\right\} \tag{14}$$

Note that the switched DAE (13) contains a nonlinear term $g_\sigma(x)$ and therefore the distributional solution framework recalled in Section 3.2 cannot be applied directly. Nonlinear switched DAEs were investigated in Liberzon and Trenn (2012), but this approach excludes Dirac impulses in x by definition, because if a Dirac impulse occurs in the solution x of (13) (which we actually desire to capture the water hammer effect) then it is unclear how $g_\sigma(x)$ has to be evaluated in general (e.g. what is the sine of a Dirac impulse). However, for our specific case we see that the nonlinear term $g(x) = g_1(x) = g_2(x)$ can be written as follows:

$$g(x) = \mathcal{N}\overline{g}(\mathcal{M}x)$$

where

$$\mathcal{M} = [1 \;\, 0 \;\, 0], \qquad \mathcal{N} = \mathcal{M}^\top \qquad \overline{g}(Q) = -c_2 Q|Q|.$$

This special structure of the nonlinearity g allows us to extend the distributional theory from the linear case also to the nonlinear case. In particular, we have a well defined solution concept:

Definition 7. Consider a general nonlinear switched DAE of the form (13) with $E_i, A_i \in \mathbb{R}^{n\times n}$, $i \in \{1, \ldots, \mathbf{P}\}$, and $g_i : \mathbb{R}^n \to \mathbb{R}^n$ and assume that there exists $\mathcal{N} \in \mathbb{R}^{n\times m}$, $\mathcal{M} \in \mathbb{R}^{k\times n}$, $\overline{g}_i : \mathbb{R}^k \to \mathbb{R}^m$ with $m, k \leq n$ such that

$$g_i(\xi) = \mathcal{N}\overline{g}_i(\mathcal{M}\xi) \quad \forall \xi \in \mathbb{R}^n.$$

Then any pair $(x, f)) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \times (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is a solution of (13) if,

1) $\mathcal{M}x$ is impulse-free, i.e. there exists a piecewise-smooth function $\overline{x} : \mathbb{R} \to \mathbb{R}^m$ such that $\mathcal{M}x$ is the distribution induced by the function $\overline{x}$,
2) $\mathcal{N}\overline{g}_\sigma(\overline{x})$ is a piecewise-smooth function, and
3) $E_\sigma\dot{x} = A_\sigma x + f + G_x$ holds as an equality within the space of piecewise-smooth distributions where G_x is the distribution induced by the piecewise-smooth function $\mathcal{N}\overline{g}_\sigma(\overline{x})$.

Characterizing existence and uniqueness of solution in the sense of Definition 7 for general nonlinear switched DAEs (13) is still an open problem and outside the scope of this contribution. However, it is possible to derive an existence and uniqueness result for the specific case considered here:

Lemma 8. Consider the nonlinear initial-trajectory problem (ITP)

$$\begin{aligned}x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (A\dot{x} + f + g(x))_{[0,\infty)}\end{aligned}$$

where either $(E, A) = (E_1, A_1)$ or $(E, A) = (E_2, A_2)$ and $g(x) = g_1(x) = g_2(x)$ as in (14). Then for every initial trajectory $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^3$ and every inhomogeneity f induced by a piecewise-smooth function, there exists a unique solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^3$ of the ITP in the sense of Definition 7.

Proof. Case $(E, A) = (E_1, A_1)$.
The index 1 DAE $E\dot{x} = Ax + f + g(x)$ with $x = (Q, P_0, P_L)^\top$ and $f = (f_1, f_2, f_3)^\top$ reads as

$$\begin{aligned}\dot{Q} &= c_1(P_0 - P_L) + f_1 - c_2 Q|Q|, \\ 0 &= P_0 + f_2, \\ 0 &= P_L + f_3,\end{aligned}$$

which is equivalent to $P_0 = -f_2$, $P_L = -f_3$ and

$$\dot{Q} = c_1(f_3 - f_2) + f_1 - c_2Q|Q|. \tag{15}$$

The latter is an ODE where existence and uniqueness of (local) solutions is guaranteed. Although the right-hand side is not globally Lipschitz-continuous we nevertheless can conclude global existence of solutions, because of the negative sign in the quadratic term. The corresponding ITP inherits the existence and uniqueness result, because inconsistent initial values in P_0 and P_L just lead to simple jumps.

Case $(E, A) = (E_2, A_2)$.
The index 2 DAE $E\dot{x} = Ax + f + g(x)$ with $x = (Q, P_0, P_L)^\top$ and $f = (f_1, f_2, f_3)^\top$ reads as

$$\begin{aligned}\dot{Q} &= c_1(P_0 - P_L) + f_1 - c_2Q|Q|\\ 0 &= P_0 + f_2\\ 0 &= Q + f_3\end{aligned}$$

For the corresponding ITP with initial trajectory $x^0 = (Q^0, P_0^0, P_L^0)$ we can directly derive the following unique solution:

$$\begin{aligned}Q &= Q^0_{(-\infty,0)} - f_{3[0,\infty)}\\ P_0 &= P^0_{0\,(-\infty,0)} - (f_2)_{[0,\infty)}\\ P_L &= P^0_{L(-\infty,0)} + \frac{1}{c_1}(-\dot{Q} + c_1P_0 + f_1 - c_2Q|Q|)_{[0,\infty)}\\ &= P^0_{L(-\infty,0)} + \tfrac{1}{c_1}Q^0(0^-)\delta_0\\ &\quad + \tfrac{1}{c_1}\tfrac{\mathrm{d}}{\mathrm{d}t}(f_{3[0,\infty)}) + (-f_2 + \tfrac{1}{c_1}f_1 + \tfrac{c_2}{c_1}f_3|f_3|)_{[0,\infty)},\end{aligned}$$

where δ_0 denotes the Dirac impulse located at $t = 0$. □

Corollary 9. The switched nonlinear DAE (13), (14) has for every initial condition $Q(0) = Q^0 \in \mathbb{R}$ a unique solution in the sense of Definition 7. In particular, the jump and the Dirac impulse in P_L at t_S are given by:

$$P_L(t_S^+) = P_U, \qquad P_L[t_S] = \frac{1}{c_1}Q(t_S^-)\delta_{t_s}.$$

5 Comparison of both modeling approaches

Our longterm goal is to rigorously prove that the solution $(q, P(\rho))$ of the balance law (6) given by (3), (1), (4), (12) converges in an appropriate sense to the solutions (Q, P_0, P_L) of the switched DAE (13),(14) as the compressibility coefficient β approaches zero. Here we will focus on the jump and Dirac impulse in the pressure due to closing the valve. In particular, we assume that the initial condition (4) is such that the PDE solution on $[0, t_S)$ is stationary, i.e. $q(t, x)$ is constant in time and space (or in other words, when the valve is closed the dynamics in the pipe have settled down). For the numerical simulations we use a HEOC type ADER scheme of order 3 (Borsche and Kall, 2014). For the detailed characteristics analysis see [chapter 5, Kall (2016)]. The upper part of Figure 3 shows the results for the pressure value at the valve over the time interval $[0.4s, 4s]$ with initial values

$$q_0(x) \equiv 0, \quad \rho_0(x) \equiv 1.4115 \times 10^3$$

and pipe parameters

$$\begin{aligned}P_a &= 1.01 \times 10^6, &\beta &= \frac{1}{K} = 4 \times 10^{-9}, &\rho_a &= 1000,\\ L &= 5, &D &= 0.5, &c_f &= 0.02.\end{aligned}$$

Clearly, there is a strong pressure spike just after the switching time $t_S = 0.5s$ and then the pressure periodically settles to a new pressure value. The frequency of this periodic behavior is determined by the pipe length L (the larger L the lower the frequency) and the speed of sound (higher for smaller compressibility coefficients β).

In order to compare both results, we first have to obtain an approximation of the pressure value $p(t, L)$ as t tends to infinity. Instead of running the simulation for a very long time, we just chose a settling time $\varepsilon > 0$ and take the average of $p(t, L)$ on the interval $(t_S+\varepsilon, T]$ where $T > t_S+\varepsilon$ is our overall simulation time, i.e.

$$\overline{P}_L := \frac{1}{T - (t_S + \varepsilon)} \int_{t_S+\varepsilon}^{T} p(t, x)\mathrm{d}t.$$

with

$$\varepsilon = 1.5, \qquad T = 4.$$

we obtain

$$\overline{P}_L \approx 8.23 \times 10^8$$

The value predicted by the switched DAE is

$$P_L(t_S^+) = P_U \approx 8.23 \times 10^8$$

In Table 1 the relative error between $\overline{P}_L$ and $P_L(t_S^+)$ is presented for decreasing compressibility coefficients β.

β	$\overline{P}_L$	$\frac{\lvert\overline{P}_L - P_L(t_S^+)\rvert}{P_L(t_S^+)}$
$4.0 \cdot 10^{-9}$	$8.2336 \cdot 10^8$	$5.4678 \cdot 10^{-04}$
$2.0 \cdot 10^{-9}$	$8.2329 \cdot 10^8$	$4.4046 \cdot 10^{-04}$
$5.0 \cdot 10^{-10}$	$8.2305 \cdot 10^8$	$7.5942 \cdot 10^{-05}$
$2.5 \cdot 10^{-10}$	$8.2303 \cdot 10^8$	$4.5565 \cdot 10^{-05}$
$1.25 \cdot 10^{-10}$	$8.2299 \cdot 10^8$	$1.5188 \cdot 10^{-05}$

Table 1. Pressure at valve comparison for long after switching.

Already for the largest value of β, the value $P_L(t_S^+)$ is a very good approximation of $\overline{P}_L$ and the approximation gets better for decreasing β.

In order to compare the peak in $p(t, L)$ just after the valve is closed with the Dirac impulse $P_L[t_S]$ in response to the switching time, we recall that a Dirac impulse δ_{t_s} at $t_s > 0$ can be approximated by a sequence of functions $t \mapsto \delta^\varepsilon_{t_s}(t)$ such that $\delta^\varepsilon(t) = 0$ for $t \neq [t_s, t_s + \varepsilon]$ and $\int_{t_s}^{t_s+\varepsilon} \delta^\varepsilon_{t_s}(t)\mathrm{d}t = 1$. We therefore make the Ansatz

$$p(t, L) \approx \overline{P}^{\mathrm{imp}}_{t_S}\delta^\varepsilon(t) + \overline{P}_L, \quad t \in (t_S, T],$$

hence we can approximate the magnitude of the "smoothed-out" Dirac impulse occurring in the PDE model as follows:

$$\overline{P}^{\mathrm{imp}}_{t_S} := \int_{t_S}^{t_S+\varepsilon} p(t, L) - \overline{P}_L\mathrm{d}t.$$

The Dirac impulse induced by the switched DAE is (see Corollary 9):

$$P_L[t_S] = \frac{1}{c_1}Q(t_S^-)\delta_{t_s} =: P^{\mathrm{imp}}_{t_S}\delta_{t_s}.$$

Similar as for the PDE simulations we assume that the DAE is stationary before we switch, i.e. $Q(t_S^-)$ is obtained by solving

$$0 = \dot{Q} = c_1(P_U - P_D) - c_2Q|Q|$$

With the above chosen parameters we get:

$$Q(t_s^-) = 1.2830 \cdot 10^6, \quad P^{\mathrm{imp}}_{t_S} = \frac{1}{c_1}Q(t_s^-) = 3.2670 \cdot 10^7.$$

A comparison between $\overline{P}^{\mathrm{imp}}_{t_S}$ and $P^{\mathrm{imp}}_{t_S}$ for different values of the compressibility coefficient β is presented in Table 2. For large β the approximation is not very accurate, however, for decreasing compressibility the accuracy of the approximation drastically improves. Finally, we want to stress that the choice of ε influences the approximation accuracy, see Table 3.
However, the qualitative behavior of a decreasing error for decreasing compressibility coefficient remains valid.

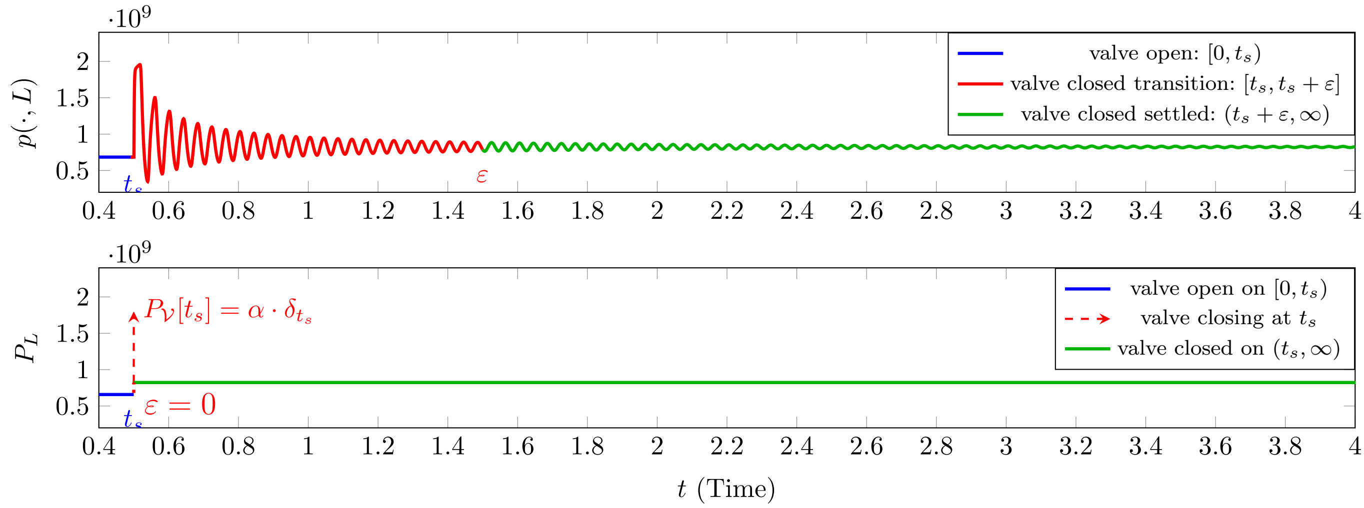

Fig. 3. Comparison of pressure profile at valve (P_L) with PDE models (above) and switched DAE model (below)

β	$\overline{P}^{\text{imp}}_{t_S}$	$P^{\text{imp}}_{t_S}$	$RE := \frac{\lvert \overline{P}^{\text{imp}}_{t_S} - P^{\text{imp}}_{t_S} \rvert}{P^{\text{imp}}_{t_S}}$
$4.0 \cdot 10^{-9}$	$4.1251 \cdot 10^{7}$	$3.2670 \cdot 10^{7}$	0.2626
$2.0 \cdot 10^{-9}$	$3.4758 \cdot 10^{7}$	$3.2670 \cdot 10^{7}$	0.0639
$5.0 \cdot 10^{-10}$	$3.1817 \cdot 10^{7}$	$3.2670 \cdot 10^{7}$	0.0261
$2.5 \cdot 10^{-10}$	$3.2069 \cdot 10^{7}$	$3.2670 \cdot 10^{7}$	0.0184
$1.25 \cdot 10^{-10}$	$3.2398 \cdot 10^{7}$	$3.2670 \cdot 10^{7}$	0.0083

Table 2. Impulse length comparison

β	RE $\varepsilon = 1$	RE $\varepsilon = 1.5$	RE $\varepsilon = 2$	RE $\varepsilon = 3$
$4.0 \cdot 10^{-9}$	0.1812	0.2626	0.2616	0.2697
$2.0 \cdot 10^{-9}$	0.0508	0.0639	0.0809	0.0808
$5.0 \cdot 10^{-10}$	0.0438	0.0261	0.0253	0.0233
$2.5 \cdot 10^{-10}$	0.0228	0.0184	0.0016	0.0160
$1.25 \cdot 10^{-10}$	0.0084	0.0083	0.0078	0.0053

Table 3. Error comparison with different ε

6 Conclusion

We have presented a switched DAEs model for water hammer on a simple setup, which we compared with a compressible nonlinear system of balance laws. With the support of numerical simulations of the PDE model we justified our conjecture that a switched DAE model is a good approximation for the PDE model with small compressibility coefficient. Future work will focus on a formal proof of convergence as well as the treatment of larger networks.

References

Adami, S., Hu, X., and Adams, N. (2012). Simulating three-dimensional turbulence with sph. In *Proceedings of the Summer Program*, 177. Citeseer.

Borsche, R., Colombo, R.M., and Garavello, M. (2012). Mixed systems: Odes–balance laws. *Journal of Differential equations*, 252(3), 2311–2338.

Borsche, R. and Kall, J. (2014). Ader schemes and high order coupling on networks of hyperbolic conservation laws. *Journal of Computational Physics*, 273(0), 658 – 670. doi:10.1016/j.jcp.2014.05.042.

Borutzky, W. (2009). *Bond graph methodology: development and analysis of multidisciplinary dynamic system models*. Springer Science & Business Media.

Bressan, A. (2013). Hyperbolic conservation laws: an illustrated tutorial. In B. Piccoli and M. Rascle (eds.), *Modelling and Optimisation of Flows on Networks*, volume 2062 of *Lecture Notes in Mathematics*, 157–245. Springer, Berlin, Heidelberg. doi:10.1007/978-3-642-32160-3_2.

Chaudhry, M.H. and Mays, L. (2012). *Computer modeling of free-surface and pressurized flows*. Springer Science & Business Media.

Chen, T., Xu, C., Lin, Q., Loxton, R., and Teo, K.L. (2015). Water hammer mitigation via pde-constrained optimization. *Control Engineering Practice*, 45, 54–63.

Colombo, R.M. and Garavello, M. (2006). A well posed riemann problem for the p-system at a junction. *Networks and heterogeneous media*, 1(3), 495–511.

Colombo, R.M. and Garavello, M. (2008). On the cauchy problem for the p-system at a junction. *SIAM Journal on Mathematical Analysis*, 39(5), 1456–1471.

Freistuhler, H. (1998). *Analysis of systems of conservation laws*, volume 99. CRC Press.

Herty, M., Mohring, J., and Sachers, V. (2010). A new model for gas flow in pipe networks. *Mathematical Methods in the Applied Sciences*, 33(7), 845–855.

Izquierdo, J., Pérez, R., and Iglesias, P.L. (2004). Mathematical models and methods in the water industry. *Mathematical and Computer Modelling*, 39(11), 1353–1374.

Jansen, L. and Pade, J. (2013). Global unique solvability for a quasi-stationary water network model. Preprint series: Institut für Mathematik, Humboldt-Universität zu Berlin (ISSN 0863-0976), 2013-11.

Jansen, L. and Tischendorf, C. (2014). A unified (P)DAE modeling approach for flow networks. In S. Schöps, A. Bartel, M. Günther, W.E.J. ter Maten, and C.P. Müller (eds.), *Progress in Differential-Algebraic Equations: Deskriptor 2013*, 127–151. Springer-Verlag, Berlin, Heidelberg. doi:10.1007/978-3-662-44926-4_7.

Kall, J. (2016). *ADER schemes for system of conservations laws on networks*. Ph.D. thesis, University of Kaiserslautern.

Liberzon, D. and Trenn, S. (2012). Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability. *Automatica*, 48(5), 954–963. doi:10.1016/j.automatica.2012.02.041.

Schwartz, L. (1957, 1959). *Théorie des Distributions*. Hermann, Paris.

Szydłowski, M. (2002). Finite volume method for water hammer simulation. *TIASWiK, Poland*.

Trenn, S. (2009a). *Distributional differential algebraic equations*. Ph.D. thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany. URL `http://www.db-thueringen.de/servlets/DocumentServlet?id=13581`.

Trenn, S. (2009b). Regularity of distributional differential algebraic equations. *Math. Control Signals Syst.*, 21(3), 229–264. doi:10.1007/s00498-009-0045-4.

Trenn, S. (2012). Switched differential algebraic equations. In F. Vasca and L. Iannelli (eds.), *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, chapter 6, 189–216. Springer-Verlag, London. doi:10.1007/978-1-4471-2885-4_6.

Weierstraß, K. (1868). Zur Theorie der bilinearen und quadratischen Formen. *Berl. Monatsb.*, 310–338.

Wylie, E.B. and Streeter, V.L. (1978). *Fluid transients*. McGraw-Hill International Book Co., New York.