Switch observability for a class of inhomogeneous switched DAEs

Ferdinand Küsters*, Deepak Patil†, Stephan Trenn♯

***Abstract*— Necessary and sufficient conditions for switching time and switch observability of a class of inhomogeneous switched differential algebraic equations (DAEs) are obtained. A characterization of initial states and inputs for which switched DAEs are switch unobservable is also provided by using the zeros of an augmented system obtained by combining the output of two modes suitably.**

I. INTRODUCTION

Switch observability in switched system refers to identifying the switching signal from the knowledge of the input and the output. It can be used in fault detection [1], identifying link connections and disconnections in networked systems [2], designing observers for switched systems [3], [4], [5], [6], [7]. In [7] an observer design for switched differential algebraic equations (DAEs) was considered which required the knowledge of which mode the system is currently running in. As a result, detecting the switching signal from the input and output becomes essential for realizing this observer. In this paper, we characterize the initial conditions and inputs for which it is possible to reconstruct the switching signal. This information can then be used for realizing the observer in [7].

We consider two problems, namely, mode detection and identification of the switching instants. Mode detection is feasible, if it is possible to distinguish between outputs of different subsystems, when subjected to the same input. In this paper, we provide a complete characterization of initial conditions and analytic inputs under which it is possible to distinguish between different subsystems. Detecting switching instants require that at the switching instant, the output will behave in a manner so that the switch is apparent. Loss of analyticity of the output at the switching instant for switched systems subject to analytic inputs indicates the change of mode. We also provide a complete characterization of initial conditions and inputs for which switching times can be detected. We will use strong observability notions defined in [8] for switched ordinary differential equations (ODEs) namely, strong σ-, t_S- and σ_1-observability. These notions are strong in the sense that the observability is for all inputs as opposed to standard notion which require observability for “almost all” inputs (generically) [9], [10].

In [8], the switch observability problem was studied for switched homogeneous and inhomogeneous ODEs and then in [11] for switched homogeneous DAEs. Here we consider switch observability for a class of inhomogeneous switched DAEs. Our contribution is twofold, 1) we give necessary and sufficient conditions for switch observability of a class of switched inhomogenous DAEs (with modes that are strictly proper systems) which are generalizations of conditions provided in [11], [8] and 2) we give a characterization of initial conditions and input for which we loose switch observability. Switched DAEs provide richer dynamics compared to ODEs, in terms of appearance of jumps, Dirac impulse and its derivatives in the output. The conditions derived in this paper use this additional information to give necessary and sufficient conditions for mode detection and switching time observability.

This paper is arranged as follows. In section II we formulate the problem by defining observability notions. Further, in section III, we introduce the notation that will be used and assumptions under which we will be operating. We also discuss in brief the distributional solution concept for switched DAEs which will be used throughout the paper. Our main results are presented in Section IV, where we provide necessary and sufficient conditions for strong $\sigma-$, t_s- and σ_1-observability.

II. PROBLEM FORMULATION

We consider switched DAEs of the form

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \\ y &= C_\sigma x, \end{aligned} \tag{1}$$

where switching is governed by a switching function $\sigma : \mathbb{R} \to \mathcal{P} := \{1, \dots, \mathrm{p}\}$, $\mathrm{p} \in \mathbb{N}$, which is piecewise constant and right continuous. On each time interval where σ is constantly p the dynamics are governed by a DAE

$$\Sigma_p : \quad \begin{aligned} E_p \dot{x} &= A_p x + B_p u, \\ y &= C_p x, \end{aligned}$$

with $E_p, A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times n_u}$, $C_p \in \mathbb{R}^{n_y \times n}$. We assume that every matrix pair (E_p, A_p), $p \in \mathcal{P}$, is regular, i.e. $\det(sE_p - A_p)$ is not the zero polynomial. For a given switching signal, the set of switching times is given by

$$T_\sigma := \left\{ \ t \in \mathbb{R} \ \middle| \ \sigma(t^-) \neq \sigma(t^+) \ \right\}.$$

We will assume that all switches occur after the initial time $t = 0$, i.e. $T_\sigma \subseteq \mathbb{R}_{>0}$. Furthermore, we only consider initially consistent solutions, i.e. $x(0) = x_0 \in \mathbb{R}^n$ is contained in the consistency space of mode $\sigma(0)$, see Section III for details on consistency spaces. Under the regularity assumption, existence and uniqueness of *distributional* solutions of the switched DAE (1) is guaranteed [12]; we denote the solution

This work was supported by DFG-grant TR 1223/2-1.

*ferdinand.kuesters@itwm.fraunhofer.de, Fraunhofer ITWM, 67663 Kaiserslautern, Germany

†deepakpatil@ee.iitd.ac.in, Department of Electrical Engineering, Indian Institute of Technology Delhi, India

♯trenn@mathematik.uni-kl.de, FB Mathematik, TU Kaiserslautern, Germany

of (1) with initial condition $x(0) = x_0$ by $x_{(x_0,\sigma,u)}$ and the corresponding output by $y_{(x_0,\sigma,u)}$.

Our goal is to generalize the various observability notions from [11] for switched ODEs to switched DAEs and provide characterizations. For this, we need to briefly discuss the solution formula for DAEs in the framework of piecewise smooth distributions (see [13], [12]) and introduce some notation for our analysis.

III. Preliminaries and Notation

A. DAE preliminaries

For any fixed mode $p \in \mathcal{P}$ let us consider a (nonswitched) DAE Σ_p, where (E_p, A_p) is regular. A useful characterization of regularity which goes back to Weierstrass [14] is the following result:

Theorem 1 (Quasi-Weierstrass form): The matrix pair (E_p, A_p) is regular if, and only if, there exist invertible matrices $S_p \in \mathbb{R}^{n\times n}$ and $T_p \in \mathbb{R}^{n\times n}$ such that

$$(S_pE_pT_p, S_pA_pT_p) = \left(\begin{bmatrix} I_{n-r_p} & 0 \\ 0 & N_p \end{bmatrix}, \begin{bmatrix} J_p & 0 \\ 0 & I_{r_p} \end{bmatrix}\right) \quad (2)$$

where $N_p \in \mathbb{R}^{r_p\times r_p}$, $r_p \in \mathbb{N}$, is a nilpotent matrix, $J_p \in \mathbb{R}^{(n-r_p)\times(n-r_p)}$ and I_{n-r_p}, I_{r_p} denote the identity matrices of corresponding size.

Following [15], we call (2) a *quasi*-Weierstrass form (QWF) because we do not assume that J_p and N_p are in Jordan-canonical form; therein it was shown how to utilize the Wong-sequences [16] to easily obtain the QWF. The nilpotency index of N_p in the QWF of a regular matrix pair (E_p, A_p) is called the *index* of (E_p, A_p).

For a regular matrix pair (E_p, A_p) with QWF (2) we define the following matrices (which are in fact independent of the specific choices of S_p and T_p)

- Consistency projector $\Pi_p := T_p \begin{bmatrix} I_{n-r_p} & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1}$,
- Differential projector $\Pi_p^{\text{diff}} := T_p \begin{bmatrix} I_{n-r_p} & 0 \\ 0 & 0 \end{bmatrix} S_p$,
- Impulse projector $\Pi_p^{\text{imp}} := T_p \begin{bmatrix} 0 & 0 \\ 0 & I_{r_p} \end{bmatrix} S_p$,

Note that only the consistency projector is a projector in the usual sense (i.e. idempotent). Furthermore, let

$$A_p^{\text{diff}} := \Pi_p^{\text{diff}} A_p, \qquad B_p^{\text{diff}} := \Pi_p^{\text{diff}} B_p,$$
$$E_p^{\text{imp}} := \Pi_p^{\text{imp}} E_p, \qquad B_p^{\text{imp}} := \Pi_p^{\text{imp}} B_p.$$

The transfer function of Σ_p is $G_p(s) := C_p(sE_p - A_p)^{-1}B_p$ and it is easily verified that

$$G_p(s) = C_p(sI - A^{\text{diff}})^{-1}B_p^{\text{diff}} + C_p(sE^{\text{imp}} - I)^{-1}B_p^{\text{imp}}.$$

We will call $\mathcal{V}_p^* := \text{im}\ \Pi_p$ the consistency space of Σ_p; it holds that $\dim \mathcal{V}_p^* = n - r_p$. Set $\mathcal{W}_p^* := \ker \Pi_p$.

Furthermore, for $l \in \mathbb{N}$ let

$$\mathcal{O}_p^{[l]} := \mathcal{O}^{[l]}_{(C_p^{\text{diff}}, A_p^{\text{diff}})}, \quad \mathbf{O}_p^{\text{imp},[l]} := \mathcal{O}^{[l]}_{(C_p^{\text{imp}}, E_p^{\text{imp}})} E_p^{\text{imp}},$$

where, for corresponding matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n_y\times n}$,

$$\mathcal{O}^{[l]}_{(C,A)} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{l-1} \end{bmatrix} \in \mathbb{R}^{l\cdot n_y\times n}$$

and

$$\Gamma_p^{[l]} := \Gamma^{[l]}_{(A_p^{\text{diff}}, B_p^{\text{diff}}, C_p^{\text{diff}})} \in \mathbb{R}^{l\cdot n_y\times l\cdot n_u},$$

where, for corresponding matrices A, B, C,

$$\Gamma^{[l]}_{(A,B,C)} = \begin{bmatrix} 0 & & & & & \\ CB & 0 & & & & \\ CAB & CB & \ddots & \ddots & & \\ CA^2B & CAB & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & \\ CA^{l-2}B & CA^{l-3}B & \cdots & \cdots & CB & 0 \end{bmatrix}.$$

B. Explicit DAE solution formula

Consider the switched DAE (1) and assume that on some time interval $[t_p, t_q)$ the switching signal is constantly p and, furthermore, assume that $B_p^{\text{imp}} = 0$. From [12, Thms. 6.4.4&6.5.1] it then follows that the solutions of (1) satisfy, for $t \in [t_p, t_q)$:

$$x(t^+) = e^{A_p^{\text{diff}}(t-t_p)}\Pi_p x(t_p^-) + \int_{t_p}^t e^{A_p^{\text{diff}}(t-\tau)}B_p^{\text{diff}}u(\tau)d\tau, \quad (3a)$$

$$x[t_p] = -\sum_{i=0}^{n-1}(E_p^{\text{imp}})^{i+1}x(t_p^-)\delta_{t_p}^{(i)}, \quad (3b)$$

where $x(t^\pm)$ denotes the left-/right-evaluation of the piecewise-smooth distribution x, $x[t_p]$ denotes the impulsive part of x at t_p and $\delta_{t_p}^{(i)}$ denotes the i-th (distributional) derivative of the Dirac impulse located at t_p. In particular, on the open interval (t_p, t_q), x solves the following ODE:

$$\dot{x} = A_p^{\text{diff}}x + B_p^{\text{diff}}u, \quad x(t_p^+) = \Pi_p x(t_p^-). \quad (4)$$

It is important to note that this is only true, because $B^{\text{imp}} = 0$, otherwise the solution x would not be a solution of a simple ODE, because then x depends also on derivatives of the input u, see [12] for details.

Next we give expressions for the output and its derivatives and the impulse part in the output.

1) Non-impulsive part of output at t_p^+: The output of the switched DAE (1) for $t \in [t_p, t_q)$ is given by

$$y(t^+) = C_p e^{A_p^{\text{diff}}(t-t_p)}\Pi x(t_p^-) + \int_{t_p}^t C_p e^{A_p^{\text{diff}}(t-\tau)}B_p^{\text{diff}}u(\tau)d\tau,$$

in particular, due to (4),

$$y^{[l]}(t_p^+) = \mathcal{O}_p^{[l]}\Pi_p x(t_p^-) + \Gamma_p^{[l]}u^{[l]}(t_p^+) \in \mathbb{R}^{l\cdot n_y}, \quad (5)$$

where $y^{[l]}$ and $u^{[l]}$ denote the vector of y and u together with its $l-1$ consecutive derivatives.

2) Impulse in output at t_p: From (3b) it follows that

$$y[t_p] = -\sum_{i=0}^{n-1} C_p (E_p^{\text{imp}})^{i+1} x(t_p^-)\delta_{t_p}^{(i)} =: -\sum_{i=0}^{n-1} \overline{y}_i \delta_0^{(i)}.$$

Using the notation

$$\overline{y}[t_p] := \begin{bmatrix} \overline{y}_1^\top, & \overline{y}_2^\top, & \ldots, & \overline{y}_n^\top \end{bmatrix}^\top \in \mathbb{R}^{n_y \cdot n},$$

we can also write

$$\overline{y}[t_p] = -\mathbf{O}^{\text{imp},[n]} x(0^-). \tag{6}$$

Note that for the index $\nu_p \leq n$ of (E_p, A_p) we have $0 = (E_p^{\text{imp}})^{\nu_p} = (E_p^{\text{imp}})^{\nu_p+1} = \ldots = (E_p^{\text{imp}})^n$; hence if $\nu_p < n$ (which is usually the case), then the last rows of $\overline{y}[t_p]$ are all zero and do not contain any information. Nevertheless, we use the definition (6) instead of $\overline{y}[t_p] = -\mathbf{O}^{\text{imp},[\nu_p]} x(t_p^-)$ in order to have a consistent mode-independent dimension of $\overline{y}[t_p]$.

IV. Observability Characterizations for Switched DAE

A. Observability Definitions

For $u = 0$, $x_0 = 0$ we have $y_{(x_0,\sigma,u)} = 0$ for any switching signal σ. Hence we cannot expect to determine the switching signal in this case. Similarly, if $u = 0$ and the state jumps to zero at a switch, i.e. $x(t_S^+) = 0$, we cannot expect to determine the switching signal from there onwards. To get a useful notion of switching signal observability, we thus have to exclude such cases. In the recent works [11], [8] different approaches were used: For homogeneous ODEs, it was sufficient to exclude zero initial states. In the inhomogeneous case, we could either make certain assumptions to ensure that the solution stays nonzero or consider equivalence classes of switching signals. For homogeneous switched DAEs, it was also necessary to consider equivalence classes of switching signals.

Here we follow another, more intuitive approach: We define an interval $\mathcal{I}$ on which we want to determine the switching signal. For an initial value x_0, switching signal σ and smooth input u we define the *essential interval* $\mathcal{I}_{(x_0,\sigma,u)}$ as

$$\mathcal{I}_{(x_0,\sigma,u)} = \left\{ t \;\middle|\; \begin{array}{l} \exists i \in \mathbb{N} : u^{(i)}(t) \neq 0 \;\vee \\ x_{(x_0,\sigma,u)}(t^+) \neq 0 \;\vee\; x_{(x_0,\sigma,u)}[t] \neq 0 \end{array} \right\}.$$

In general, $\mathcal{I}_{(x_0,\sigma,u)}$ may be a union of intervals, to avoid this we make the following assumption:

(A1) The input is real analytic.

We then have the following result.

Lemma 2: Consider a regular switched DAE (1) satisfying (A1), then for all (consistent) initial values x_0 and all switching signals we have

$$\begin{aligned} \mathcal{I}_{(x_0,\sigma,u)} &= \mathbb{R}, \quad &\text{for } u \neq 0, \\ \mathcal{I}_{(x_0,\sigma,u)} &= (-\infty, T) \text{ or } (-\infty, T], \quad &\text{for } u = 0, \end{aligned}$$

for some $T \in T_\sigma \cup \{-\infty, \infty\}$.

Proof: It is a well known property of real analytic functions that u is identically zero if there is a single point where all derivatives vanish (this is the key difference from merely smooth functions). Hence, $u \neq 0$ implies that for every t there exists $i \in \mathbb{N}$ with $u^{(i)}(t) \neq 0$, hence $\mathcal{I}_{(x_0,\sigma,u)} = \mathbb{R}$. If u is identically zero, there are three cases possible: the initial value is already zero (which implies the essential interval is empty), the state jumps at some switching time to zero and remains zero afterwards (in which case the essential interval ends at this switching time and depending on whether at this switching time a Dirac impulse occurs or not, the essential interval is open or closed), or the state never jumps to zero (which implies that the essential interval is the whole real axis). ■

Similar as in [11] we make the following assumption on the input matrices in (1):

(A2) For any modes $p, q \in \mathcal{P}$, $p \neq q$, $\ker \begin{bmatrix} B_p^{\text{diff}} \\ B_q^{\text{diff}} \end{bmatrix} = \{0\}$.

This ensures that for a nonzero input u there is at most one mode on which it does not have an effect, which then at least give us the chance to determine the current mode (even when the state has jumped to zero). It is possible to redefine $\mathcal{I}_{(x_0,\sigma,u)}$ in such a way that (A2) is not needed, but this is rather technical.

For $u = 0$, the solution might go to zero and stay zero, making further switches undetectable. In [8] we have seen that this leads to an extra condition (cf. the forthcoming condition (11) in Theorem 5) and an additional term $\mathcal{M}_{i,j,p,q}$ in the characterization of strong σ_1-observability (cf. the forthcoming Theorem 9). For $u \neq 0$, however, such problems do not occur which simplifies the subsequent analysis.

We will now formally define the observability notions studied in this paper:

Definition 3: The switched DAE (1) is called

1) strongly σ-observable if, and only if, for all $(\sigma, \tilde{\sigma})$, all $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\tilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$
$$\sigma \neq \tilde{\sigma} \text{ on } \mathcal{I}_{(x_0,\sigma,u)} \implies y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)} \tag{7}$$
holds for all analytic u, i.e. the switching signal can be determined (on the essential interval) from the knowledge of the output and the input (for all possible inputs and all initial states).
2) strongly (x, σ)-observable if, and only if, for all $(\sigma, \tilde{\sigma})$, all $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\tilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$
$$\begin{pmatrix} \sigma \neq \tilde{\sigma} \text{ on } \mathcal{I}_{(x_0,\sigma,u)} \\ \vee\; x_0 \neq \tilde{x}_0 \end{pmatrix} \implies y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)} \tag{8}$$
holds for all analytic u, i.e. the switching signal and the state can be determined (on the essential interval) from the knowledge of the output and the input (for all possible inputs and initial states).
3) strongly σ_1-observable if and only if for all $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\tilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$, all analytic u and all $(\sigma, \tilde{\sigma})$ with σ having at least one switch in $\mathcal{I}_{(x_0,\sigma,u)}$ the implication (7) holds. This means the switching signal can be determined from the knowledge of the input and the output provided at least one essential switch has occurred.

4) strongly (x, σ_1)-observable if and only if for all $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\tilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$, all analytic u and all $(\sigma, \tilde{\sigma})$ with σ having at least one switch in $\mathcal{I}_{(x_0,\sigma,u)}$ the implication (8) holds. This means the switching signal and the state can be determined from the knowledge of the input and the output provided at least one essential switch has occurred.
5) strongly t_S-observable if and only if for all $(\sigma, \tilde{\sigma})$, all $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\tilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ and all analytic u it holds

$$T_\sigma \neq T_{\tilde{\sigma}} \text{ on } \overline{\mathcal{I}_{(x_0,\sigma,u)}} \implies y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)},$$

i.e. the essential switching times can be determined from the knowledge of the input and the output. We want to detect the switching times in the closure of the essential interval, because at the boundary of the essential interval the non-zero state jumps to zero and this jump should be detectable in the output (independently of the possible presence of Dirac impulses).

Note that we consider *strong* observability notions and not a generic (or weak) notion where observability is only required for almost all (or one) input signal.

With the same arguments as in [11, Lem. 2] the (surprising) equivalence between strong σ- and strong (x, σ)-observability and between strong σ_1- and strong (x, σ_1)-observability can be shown, hence we will focus in the following on σ- and σ_1-observability. Furthermore, it is clear that strong σ-observability is sufficient for strong σ_1-observability which in turn is sufficient for t_S-observability, however the converse is not true in general.

B. σ-observability

For the switched DAE (1) and two modes $i, j \in \mathcal{P}$ consider first the following augmented system

$$\Sigma_{(i,j)}: \begin{bmatrix} E_i & 0 \\ 0 & E_j \end{bmatrix} \dot{\xi} = \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \xi + \begin{bmatrix} B_i \\ B_j \end{bmatrix} u, \qquad \chi = \begin{bmatrix} C_i & -C_j \end{bmatrix} \xi, \tag{9}$$

where

$$\xi = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}.$$

The role of the augmented systems for σ-observability is expressed by the following Lemma.

Lemma 4: For the switched system (1) the two constant switching signals $\sigma \equiv i$ and $\tilde{\sigma} \equiv j$ can be distinguished for any initial values and common input on the essential interval if, and only if, the augmented system $(\Sigma_{(i,j)})$ is unknown-input-(ui-)observable (see Appendix).

Proof: The two constant switching signals cannot be distinguished if, and only if, there are initial vales $x_0, \tilde{x}_0$ and an input such that $y_{(x_0,i,u)} = y_{(\tilde{x}_0,j,u)}$. We can assume $x_0 \neq 0$ (otherwise the essential interval $\mathcal{I}_{(x_0,i,u)} = \emptyset$), hence the non-zero initial value $\xi_0 = \left(\begin{smallmatrix} x_0 \\ \tilde{x}_0 \end{smallmatrix}\right)$ for $(\Sigma_{(i,j)})$ results in a vanishing output, i.e. the augmented system is not ui-observable. ■

In order to characterize ui-observability of the augmented system $(\Sigma_{(i,j)})$ we make the following assumption:

(A3) The transfer matrix from u to x for each DAE (Σ_p) given by $(sE_p - A_p)^{-1} B_p$ is strictly proper, or, equivalently, $B^{\text{imp}}_p = 0$ for all $p \in \mathcal{P}$.

Assumption (A3) means that in the QWF (2) the input is not effecting the nilpotent part. In other words there is no feedthrough from the input (or its derivatives) to the state.

Lemma 11 in the Appendix together with (A3) now shows that

$$\operatorname{rk} \begin{bmatrix} \mathcal{O}^{[2n]}_i & \mathcal{O}^{[2n]}_j & \Gamma^{[2n]}_i - \Gamma^{[2n]}_j \end{bmatrix} = \dim \mathcal{V}^*_i + \dim \mathcal{V}^*_j + \operatorname{rk} \left(\Gamma^{[2n]}_i - \Gamma^{[2n]}_j \right) \tag{10}$$

is characterizing ui-observability. Hence we already have derived a *necessary* condition for σ-observability. In contrast to switched ODEs condition (10) is however *not sufficient*, this can already be seen in the homogeneous case, see e.g. [8, Ex. 7]. In fact, only for $u = 0$ the condition (10) is not sufficient for σ-observability, because $u \neq 0$ already implies that the essential interval is the whole real axis and ui-observability of the augmented systems $(\Sigma_{(i,j)})$ for all $i, j \in \mathcal{P}$ with $i \neq j$ implies the ability to determine the active mode on all open intervals between the switching times, which implies that σ can be determined. Hence by recalling the condition

$$\mathcal{V}^*_i \cap \mathcal{W}^*_j \cap \mathcal{W}^*_k \cap \ker \left(\mathcal{O}^{\text{imp}}_i - \mathcal{O}^{\text{imp}}_j \right) \subseteq \ker \begin{bmatrix} E^{\text{imp}}_j \\ E^{\text{imp}}_k \end{bmatrix} \tag{11}$$

derived in [8] for homogeneous switched DAEs we have our first main result:

Theorem 5: The switched DAE (1) satisfying (A1), (A2) and (A3) is strongly σ-observable, or equivalently, strongly (x, σ)-observable if, and only if, (10) and (11) hold for all pairwise different $i, j, k \in \mathcal{P}$.

In case the switched DAE is not strongly σ-observable, we now want to investigate the (nontrivial) inputs and initial conditions which lead to σ-unobservability.

Lemma 6: Consider $\lambda \in \mathbb{C}$ for which there exists $\widehat{x}_0 \in \mathbb{C}^n, \widehat{\tilde{x}}_0 \in \mathbb{C}^n, u_0 \in \mathbb{C}^m$ such that

$$\begin{bmatrix} \lambda E_i - A_i & 0 & -B_i \\ 0 & \lambda E_i - A_j & -B_j \\ C_i & -C_j & 0 \end{bmatrix} \begin{bmatrix} \widehat{x}_0 \\ \widehat{\tilde{x}}_0 \\ u_0 \end{bmatrix} = 0. \tag{12}$$

Let $u(t) = e^{\operatorname{Re} \lambda t} (\cos(\operatorname{Im} \lambda t) \operatorname{Re} u_0 - \sin(\operatorname{Im} \lambda t) \operatorname{Im} u_0)$. Then $y_{(x_0,p,u)}(t) = y_{(\tilde{x}_0,q,u)}(t)$ for all $t \geq 0$ for initial conditions $x_0 = \operatorname{Re} \widehat{x}_0$ and $\tilde{x}_0 = \operatorname{Re} \widehat{\tilde{x}}_0$.

Proof: By applying Theorem 13 to the augmented system $\Sigma_{(i,j)}$ the proof follows easily. ■

Lemma 6 gives us a way to characterize the initial conditions and inputs which lead to σ-unobservability. Consider a matrix pencil

$$P_{(i,j)}(\lambda) = \begin{bmatrix} \lambda E_i - A_i & 0 & -B_i \\ 0 & \lambda E_j - A_j & -B_j \\ C_i & -C_j & 0 \end{bmatrix}.$$

Then its generalized eigenstructure gives us inputs and state-trajectory for which we loose σ-observability for $\sigma = i$ and

$\tilde{\sigma} = j$. The generalized eigenstructure of $P_{(i,j)}(\lambda)$ can be computed by using quasi-Kronecker form (see [13] and [12]).

Example 1: Consider the switched DAE (1) with $\mathcal{P} = \{1, 2, 3\}$ and

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$
$$E_3 = E_2, \qquad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_3 = B_2, \quad C_3 = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}$$

This system is not σ-observable because (10) is not satisfied for mode pairs $(1, 3)$ and $(2, 3)$. We characterize the initial conditions and inputs for which it is not σ-observable from the following matrix pencils created for pairs $(1, 2)$, $(2, 3)$ and $(1, 3)$ respectively.

$$P_{1,2}(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}$$

$$P_{2,3}(\lambda) = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -0.5 & -0.5 & 0 & -1 & 2 & 0 & 0 \end{bmatrix}$$

$$P_{1,3}(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 & 0 \end{bmatrix}$$

Among these three matrix pencils $P_{1,2}(\lambda)$ remains full rank for any $\lambda \in \mathbb{C}$. Hence, modes 1 and 2 are distinguishable for all initial conditions and analytic inputs.

However for $P_{2,3}(\lambda)$, we note from its quasi-Kronecker form that it is not full rank for $\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then the generalized eigenvectors of $P_{2,3}(\lambda)$ corresponding to the complex eigenvalues give us initial conditions $x_0 = (-u_0, 0, 0)$ and $\tilde{x}_0 = (-\frac{1}{2}u_0, -\frac{1}{2}u_0, 0)$ for modes 2 and 3 respectively. For these initial conditions, we get identical outputs if an input $u(t) = e^{\frac{t}{2}} \left(-\frac{1}{2} \cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{\sqrt{3}}{6} \sin\left(\frac{\sqrt{3}t}{2}\right) \right) u_0$ is applied to modes 2 and 3. Thus, the calculated $(x_0, \tilde{x}_0, u(t))$ makes the mode changes between 2 and 3 unobservable.

Similarly $P_{1,3}(\lambda)$ looses its rank for $\lambda = 0$. The kernel of $P_{1,3}(0)$ gives us $x_0 = (c, 0, 0)$ and $\tilde{x}_0 = (-c, 0, 0)$ and input $u(t) \equiv 0$.

C. t_S-observability

Note that, by definition, t_S-observability is equivalent to the ability to determine switching time t_S from the output. Assume that at time t_S, the mode changes from i to j for some $i, j \in \mathcal{P}$ i.e.,

$$\sigma = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S. \end{cases}$$

Then, the switching time t_S can be detected from the output in two ways namely, 1) detecting an impulse in output at t_S i.e., $y[t_S] \neq 0$ or 2) the output is not smooth at t_S i.e., there exists $l \in \mathbb{N}$ such that $y^{[l]}(t_S^-) \neq y^{[l]}(t_S^+)$.

For characterizing strong t_S-observability, we need a notion of the set of controllable weakly unobservable states of system Σ (see (17)) denoted as $\mathcal{R}(\Sigma)$ (see [17]).

$$\mathcal{R}(\Sigma) := \left\{ x_0 \in \mathcal{V}^* \;\middle|\; \begin{array}{c} \exists u(\cdot), T > 0 : y_{(x_0,u)} \equiv 0 \\ \text{and } x_{(x_0,u)}(T) = 0 \end{array} \right\}.$$

This notion enables us to characterize strong t_S-observability as follows:

Lemma 7: A switched DAE (1) satisfying A1, A2 and A3 is strongly t_S-observable if and only if it holds

$$\mathcal{R}\left(\Sigma_{i,j}\right) = \{0\} \tag{13}$$

and

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} \mathcal{O}_i^{[2n]} - \mathcal{O}_j^{[2n]} \Pi_i & \Gamma_i^{[2n]} - \Gamma_j^{[2n]} \\ \mathbf{O}_j^{\text{imp}} \Pi_i & 0 \end{bmatrix} \\ = \dim \mathcal{V}_i^* + \operatorname{rank}\left(\Gamma_i^{[2n]} - \Gamma_j^{[2n]}\right), \end{aligned} \tag{14}$$

hold for all $i \neq j$.

Before proving this lemma, we relate the conditions to those obtained in [11] and [8]. (13) generalizes a corresponding condition for inhomogeneous switched ODEs. As apparent from the definition of $\mathcal{R}(\Sigma)$, it is not required in the homogeneous case. (14) generalizes both the conditions for inhomogeneous switched ODEs and homogeneous switched DAEs. The terms $\mathbf{O}_j^{\text{imp}}$ and Π_i in (14) correspond to an impulse and state jump, respectively. Hence they do not appear for ODEs. In the homogeneous DAE case, the second column block (as well as the term $\operatorname{rank}\left(\Gamma_i^{[2n]} - \Gamma_j^{[2n]}\right)$) will not appear.

Proof: Necessity of (13) and (14) (without $\Gamma_i - \Gamma_j$) have been shown in [8]. Now let us assume that (14) is not satisfied with $\operatorname{rank} \begin{bmatrix} \mathcal{O}_i^{[2n]} - \mathcal{O}_j^{[2n]} \Pi_i \\ \mathbf{O}_j^{\text{imp}} \Pi_i \end{bmatrix} = \dim \mathcal{V}_i^*$. This means there exist x_1 and $U \neq 0$ with

$$\begin{bmatrix} \mathcal{O}_i^{[2n]} - \mathcal{O}_j^{[2n]} \Pi_i & \Gamma_i^{[2n]} - \Gamma_j^{[2n]} \end{bmatrix} \begin{pmatrix} x_1 \\ U \end{pmatrix} = 0$$

and $\mathbf{O}_j^{\text{imp}} \Pi_i x_1 = 0$. Then,

$$\begin{bmatrix} \mathcal{O}_i^{[2n]} & -\mathcal{O}_j^{[2n]} & \Gamma_i^{[2n]} - \Gamma_j^{[2n]} \end{bmatrix} \begin{pmatrix} x_1 \\ \Pi_j x_1 \\ U \end{pmatrix} = 0.$$

This implies that the augmented system $\Sigma_{i,j}$ is not ui-observable and that there exists an analytic function $\hat{u}$ with $\hat{u}^{[n]}(0) = U$ and initial condition $\xi_0 = \begin{bmatrix} x_1 \\ \Pi_j x_1 \end{bmatrix}$ with $\chi_{(\xi_0, \hat{u})} = 0$. Now let $t_S > 0$, $u(\cdot) := \hat{u}(\cdot - t_S)$ and x_0 consistent such that $x_{(x_0,i,u)}(t_S^-) = x_1$. Then

$$\sigma(t) = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S, \end{cases}$$

gives $y_{(x_0,\sigma,u)}^{[\infty]}(t_S^-) = y_{(x_0,\sigma,u)}^{[\infty]}(t_S^+)$. As $\mathbf{O}_j^{\text{imp}} \Pi_i x_1 = 0$ we have $y_{(x_0,\sigma,u)}[t_S] = 0$. Hence $y_{(x_0,\sigma,u)}$ equals $y_{(x_0,\tilde{\sigma},u)}$ for

$\tilde{\sigma} = i$. As $u \neq 0$ analytic we have $\mathcal{I}_{(x_0,\sigma,u)} = \mathbb{R}$. Hence the system cannot be strongly t_S-observable.

Now, we show sufficiency of (13) and (14): For $u = 0$ it has been shown in [8]. Let both assumptions be satisfied and σ, $\tilde{\sigma}$, $u \neq 0$, $x_0 \in \mathcal{V}^*_{\sigma(0)}$ and $\tilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ be given with $T_\sigma \neq T_{\tilde{\sigma}}$. W.l.o.g. assume $t_S \in T_\sigma \setminus T_{\tilde{\sigma}}$. Then there exists an open interval $\mathcal{I}$ in which σ undergoes exactly one switch at time instant $t_S \in \mathcal{I}$ and $\tilde{\sigma}$ is constant throughout.

Clearly, $y_{(\tilde{x}_0,\tilde{\sigma},u)}$ is analytic on $\mathcal{I}$. We show that $y_{(x_0,\sigma,u)}$ is not analytic on $\mathcal{I}$, i.e. that $y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)}$. If we have $x_{(x_0,\sigma,u)}(t_S^-) \neq 0$, (14) implies $y_{(x_0,\sigma,u)}[t_S] \neq 0$ or $y^{[l]}_{(x_0,\sigma,u)}(t_S^-) \neq y^{[l]}_{(x_0,\sigma,u)}(t_S^+)$ for some $l \in \mathbb{N}$. Now assume that $x_{(x_0,\sigma,u)}(t_S^-) = 0$. Define $\bar{\sigma}$ by

$$\bar{\sigma}(t) = \begin{cases} \sigma(t), & t < t_S, \\ \sigma(t_S^-), & t \geq t_S. \end{cases}$$

As $\bar{\sigma}$ is constant on $\mathcal{I}$, we have analyticity of $y_{(x_0,\bar{\sigma},u)}$ on $\mathcal{I}$. By A1, A2 and $u \neq 0$, at least one of the solutions $x_{(x_0,\sigma,u)}$, $x_{(x_0,\bar{\sigma},u)}$ becomes nonzero on $\mathcal{I} \cap [t_S, \infty)$. Then (13) implies $y_{(x_0,\sigma,u)} \neq y_{(x_0,\bar{\sigma},u)}$ on $\mathcal{I}$. In particular, $y_{(x_0,\sigma,u)}$ is not analytic, i.e. $y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)}$.

■

Remark 8: By defining the set of weakly unobservable states as

$$\mathcal{W}(\Sigma) := \left\{ \, x_0 \in \mathcal{V}^* \;\middle|\; \exists u(\cdot) : y_{(x_0,u)} \equiv 0 \, \right\}.$$

we see that condition (14) is in fact equivalent to

$$\mathcal{W}(\Sigma_{(i,j)}) \cap \begin{bmatrix} I \\ \Pi_i \end{bmatrix} \ker \mathbf{O}_j^{\text{imp}} \Pi_i = \{0\}.$$

In particular, the initial value pair $\begin{pmatrix} x_0 \\ \tilde{x}_0 \end{pmatrix}$ which leads to t_s-unobservability are exactly those which are in the set $\left(\mathcal{W}(\Sigma_{(i,j)}) \cap \text{image } \begin{bmatrix} I \\ \Pi_i \end{bmatrix} \ker \mathbf{O}_j^{\text{imp}} \Pi_i \right) \cup \mathcal{R}(\Sigma_{i,j})$.

D. σ_1-*observability*

The notion of σ_1-observability is a weaker version of σ-observability. In σ-observability, the information obtained at the switching instants is not utilized, since we are comparing output of the system for two distinct but constant switching signals. Instead of comparing two constant switching signals, in σ_1-observability, two distinct switching signals with one of them having at least one switch are compared. Note that from by definition strong t_S-observability is necessary for strong σ_1-observability.

To obtain a characterization for σ_1-observabiliy, we compare two switching signals

$$\sigma(t) = \begin{cases} i, & t < T_{ij}, \\ j, & t \geq T_{ij}. \end{cases}$$

and

$$\tilde{\sigma}(t) = \begin{cases} p, & t < T_{pq}, \\ q, & t \geq T_{pq}. \end{cases}$$

and give following necessary and sufficient condition which must be followed by all $i, j, p, q \in \mathcal{P}$. Due to strong t_S-observability, it turns out later in the proof of Theorem 9, that we only need to consider the case $T_{ij} = T_{pq} = t_S$. As a result, we end up comparing augmented systems $\Sigma_{(i,p)}$ and $\Sigma_{(j,q)}$ for strong σ-observability. Thus, to examine the σ_1-observability, we consider ui-observability of the bigger augmented system $\Sigma_{i,j,p,q}$ formed by combining $\Sigma_{(i,p)}$ and $\Sigma_{(j,q)}$ which is defined as follows

$$\begin{bmatrix} E_i & 0 & 0 & 0 \\ 0 & E_j & 0 & 0 \\ 0 & 0 & E_p & 0 \\ 0 & 0 & 0 & E_q \end{bmatrix} \dot{\xi} = \begin{bmatrix} A_i & 0 & 0 & 0 \\ 0 & A_j & 0 & 0 \\ 0 & 0 & A_p & 0 \\ 0 & 0 & 0 & A_q \end{bmatrix} \xi + \begin{bmatrix} B_i \\ B_j \\ B_p \\ B_q \end{bmatrix} u,$$

$$\chi = \begin{bmatrix} C_i & 0 & -C_p & 0 \\ 0 & C_j & 0 & -C_q \end{bmatrix} \xi + \begin{bmatrix} D_i - D_p \\ D_j - D_q \end{bmatrix} u.$$

For ui-observability of $\Sigma_{i,j,p,q}$ it suffices to consider the matrices $\mathcal{O}^{[4n]}_{i,j,p,q}$ and $\Gamma^{[4n]}_{i,j,p,q}$. Note that these matrices are given by

$$\mathcal{O}^{[\nu]}_{i,j,p,q} = T \begin{bmatrix} \mathcal{O}^{[\nu]}_i & -\mathcal{O}^{[\nu]}_p & 0 & 0 \\ 0 & 0 & \mathcal{O}^{[\nu]}_j & -\mathcal{O}^{[\nu]}_q \end{bmatrix},$$

$$\Gamma^{[\nu]}_{i,j,p,q} = T \begin{bmatrix} \Gamma^{[\nu]}_i - \Gamma^{[\nu]}_j \\ \Gamma^{[\nu]}_p - \Gamma^{[\nu]}_q \end{bmatrix}$$

for some permutation matrix T.

Theorem 9: A switched DAE (1) satisfying A1, A2 and A3 is strongly σ_1-observable if and only if it is strongly t_S-observable and satisfies

$$\begin{aligned} &\operatorname{rank} \begin{bmatrix} \mathcal{O}^{[4n]}_i & \mathcal{O}^{[4n]}_p & \Gamma^{[4n]}_i - \Gamma^{[4n]}_p \\ \mathcal{O}^{[4n]}_j \Pi_i & \mathcal{O}^{[4n]}_q \Pi_p & \Gamma^{[4n]}_j - \Gamma^{[4n]}_q \\ \mathbf{O}^{\text{imp}}_j \Pi_i & \mathbf{O}^{\text{imp}}_q \Pi_p & 0 \end{bmatrix} \\ &= \dim \mathcal{V}^*_i + \dim \mathcal{V}^*_p + \operatorname{rank} \begin{bmatrix} \Gamma^{[4n]}_i - \Gamma^{[4n]}_p \\ \Gamma^{[4n]}_j - \Gamma^{[4n]}_q \end{bmatrix} \\ &\qquad - \dim \mathcal{M}_{i,j,p,q} \end{aligned} \tag{15}$$

for all i, j, p, $q \in \mathcal{P}$ with $i \neq j$, $p \neq q$ and $(i,j) \neq (p,q)$. Here $\mathcal{M}_{i,j,p,q}$ is given by

$$\mathcal{M}_{i,j,p,q} = \begin{cases} \mathcal{V}^*_i \cap \ker E_j \cap \ker E_q, & \text{for } i = p, \\ \{0\}, & \text{else.} \end{cases}$$

Proof: By definition, strong t_S-observability is a necessary condition for strong σ_1-observability. Assume that (15) is not satisfied. Then, there exist some i, j, p, q, some $(x_1, \tilde{x}_1) \notin \begin{bmatrix} I \\ I \end{bmatrix} \mathcal{M}_{i,j,p,q}$ and some U with

$$\begin{bmatrix} \mathcal{O}^{[4n]}_i & \mathcal{O}^{[4n]}_p & \Gamma^{[4n]}_i - \Gamma^{[4n]}_p \\ \mathcal{O}^{[4n]}_j \Pi_i & \mathcal{O}^{[4n]}_q \Pi_p & \Gamma^{[4n]}_j - \Gamma^{[4n]}_q \\ \mathbf{O}^{\text{imp}}_j \Pi_i & \mathbf{O}^{\text{imp}}_q \Pi_p & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \tilde{x}_1 \\ U \end{pmatrix} = 0. \tag{16}$$

For $U = 0$ we get a counterexample for strong σ_1-observability as in [8]. Thus we consider only $U \neq 0$. Let $t_S > 0$,

$$\sigma(t) = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S, \end{cases} \quad \text{and} \quad \tilde{\sigma}(t) = \begin{cases} p, & t < t_S, \\ q, & t \geq t_S. \end{cases}$$

As $U \neq 0$ will give $u \neq 0$, we get $\mathcal{I}_{(x_0,\sigma,u)} = \mathbb{R}$ and σ and $\tilde{\sigma}$ differ on this interval. The last rowblock of (16) implies that solutions with $x_{(x_0,\sigma,u)}(t_S^-) = x_1$ and $x_{(\tilde{x}_0,\tilde{\sigma},u)}(t_S^-) = \tilde{x}_1$ satisfy $y_{(x_0,\sigma,u)}[t_S] = y_{(x_0,\tilde{\sigma},u)}[t_S]$. The first two rowblocks of (16) imply that the augmented system $\Sigma_{i,j,p,q}$ is not ui-observable. For $\xi_1 = \begin{pmatrix} x_1^\top & \tilde{x}_1^\top & x_1^\top & \tilde{x}_1^\top \end{pmatrix}^\top$ there exits an analytic input u with $\chi_{(\xi_1,u)} = 0$. From this we can conclude that the non-impulsive parts of $y_{(x_0,\sigma,u)}$ and $y_{(\tilde{x}_0,\tilde{\sigma},u)}$ coincide. Hence the system cannot be strongly σ_1-observable.

Sufficiency of strong t_S-observability and (15): Let the system be strongly t_S-observable and (15) hold true. The case $u = 0$ is covered by [8]. Let σ, $\tilde{\sigma}$, $u \neq 0$, $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\tilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ be given with either σ or $\tilde{\sigma}$ having at least one switch. We can assume $T_{\sigma_{(x_0,u)}} = T_{\tilde{\sigma}_{(\tilde{x}_0,u)}}$ as otherwise strong t_S-observability would give $y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)}$. Since either of the switching signals must have at least one switch, there exist $t_S \in T_{\sigma_{(x_0,u)}} \neq \emptyset$.

Let t_S be a common switching time with $\sigma(t_S^-) \neq \tilde{\sigma}(t_S^-)$ or $\sigma(t_S^+) \neq \tilde{\sigma}(t_S^+)$. If $x_{(x_0,\sigma,u)}(t_S^-)$ or $x_{(\tilde{x}_0,\tilde{\sigma},u)}(t_S^-)$ is nonzero, (15) gives

$$\begin{bmatrix} y^{[4n]}_{(x_0,\sigma,u)}(t_S^-) \\ y^{[4n]}_{(x_0,\sigma,u)}(t_S^+) \\ y_{(x_0,\sigma,u)}[t_S] \end{bmatrix} \neq \begin{bmatrix} y^{[4n]}_{(\tilde{x}_0,\tilde{\sigma},u)}(t_S^-) \\ y^{[4n]}_{(\tilde{x}_0,\tilde{\sigma},u)}(t_S^+) \\ y_{(\tilde{x}_0,\tilde{\sigma},u)}[t_S] \end{bmatrix}.$$

In particular, it implies $y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)}$.

Now assume that at such a switching time t_S we had $x_{(x_0,\sigma,u)}(t_S^-) = x_{(\tilde{x}_0,\tilde{\sigma},u)}(t_S^-) = 0$. Assume that $\sigma(t_S^+) \neq \tilde{\sigma}(t_S^+)$. From the assumptions A1, A2 and $u \neq 0$ we can conclude that at least one of the solutions becomes nonzero. Hence (13) implies $y_{(x_0,\sigma,u)} \neq y_{(\tilde{x}_0,\tilde{\sigma},u)}$. The same argument holds for $\sigma(t_S^-) \neq \tilde{\sigma}(t_S^-)$ backwards in time. ∎

V. Conclusion

We have shown that ui-observability of an augmented DAE system obtained from combining any two modes $i, j \in \mathcal{P}$ is necessary and sufficient for strong σ observability and a rank condition in terms of observability matrices was obtained. A relation between weakly unobservable states and weak σ-unobservability is brought out. A rank condition which determines if it is possible to detect switching time is provided. Finally a weak notion of σ_1-observability which does not require individual modes to be observable is considered and a necessary and sufficient condition in terms of observability matrices is provided.

Appendix

Unknown-input-observability, weakly unobservable states, controlled weakly unobservable states for DAEs

In this section, we introduce and characterize the notions unknown-input-observability, weakly unobservable and controlled weakly unobservable states for unswitched DAEs; these are straightforward generalizations from the linear ODE-case [17], [18].

Definition 10: A linear DAE

$$\Sigma: \quad \begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{17}$$

is called *unknown-input-observable (ui-observable)* if and only if $y \equiv 0$ implies $x \equiv 0$ (independently of the input u).

Note that in [17], [18] ui-observability is actually called strong observability; however, in the context of DAEs the notion of strong observability is used for a different concept, see e.g. the survey [19]. Therefore we use the less ambiguous notion ui-observability as, see e.g. [20].

Lemma 11: The regular system (17) with $B^{\text{imp}} = 0$ is ui-observable if, and only if,

$$\operatorname{rk}\begin{bmatrix} \mathcal{O}^{[n]}_{(A^{\text{diff}},C^{\text{diff}})} & \Gamma^{[n]}_{(A^{\text{diff}},B^{\text{diff}},C^{\text{diff}})} \end{bmatrix} = \dim \mathcal{V}^* + \operatorname{rank}(\Gamma^{[n]}).$$

Proof: It suffices to observe that all solutions of (17) evolve within the consistency space $\mathcal{V}^*$ (because B^{imp} is assumed to be zero), then the proof of [18, Prop. 2] can be easily adapted to obtain the claimed characterization. ∎

Another characterization of ui-observability is given by the absence of zeros of (17), where $\lambda \in \mathbb{C}$ is called a zero of (17) if, and only if,

$$\operatorname{rk}\begin{bmatrix} \lambda E - A & -B \\ C & 0 \end{bmatrix} < n + \operatorname{rk}\begin{bmatrix} -B \\ 0 \end{bmatrix}. \tag{18}$$

Lemma 12 (cf. [21] for ODE case): Consider the regular DAE (17) with $B^{\text{imp}} = 0$ and let $\mathcal{Z}(\Sigma)$ be the set of its zeros. Then (17) is ui-observable if, and only if, $\mathcal{Z}(\Sigma) = \emptyset$.

Proof: The proof is similar to the one in [21] and therefore omitted. ∎

For a DAE which is not ui-observable, the input and initial conditions that lead to outputs being zero are characterized by the following result.

Lemma 13 (cf. [17] for ODE case): Consider the regular DAE (17) with $B^{\text{imp}} = 0$ and for $\lambda \in \mathbb{C}$ let $\widehat{x}_0 \in \mathbb{C}^n$ and $u_0 \in \mathbb{C}^m$ be such that

$$\begin{bmatrix} \lambda E - A & -B \\ C & 0 \end{bmatrix} \begin{pmatrix} \widehat{x}_0 \\ u_0 \end{pmatrix} = 0.$$

Then, for initial condition $x(0) = \operatorname{Re} \widehat{x}_0$, the output of (17) is identically zero for input

$$u(t) = e^{\operatorname{Re}\lambda t}(\cos(\operatorname{Im}\lambda t)\operatorname{Re} u_0 - \sin(\operatorname{Im}\lambda t)\operatorname{Im} u_0).$$

Proof: The proof is similar to the one in [17] and therefore omitted. ∎

References

[1] A. Tanwani, A. D. Domínguez-García, and D. Liberzon, “An inversion-based approach to fault detection and isolation in switching electrical networks,” *IEEE Trans. Control Systems Technology*, vol. 19, pp. 1059 –1074, sept. 2011.

[2] G. Battistelli and P. Tesi, “Detecting topology variations in dynamical networks,” in *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 3349–3354, IEEE, 2015.

[3] A. Tanwani and S. Trenn, “On observability of switched differential-algebraic equations,” in *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA*, pp. 5656–5661, 2010.

[4] A. Tanwani and S. Trenn, “Observability of switched differential-algebraic equations for general switching signals,” in *Proc. 51st IEEE Conf. Decis. Control, Maui, USA*, pp. 2648–2653, 2012.

[5] A. Tanwani and S. Trenn, “An observer for switched differential-algebraic equations based on geometric characterization of observability,” in *Proc. 52nd IEEE Conf. Decis. Control, Florence, Italy*, pp. 5981–5986, 2013.

[6] M. Petreczky, A. Tanwani, and S. Trenn, "Observability of switched linear systems," in *Hybrid Dynamical Systems* (M. Djemai and M. Defoort, eds.), vol. 457 of *Lecture Notes in Control and Information Sciences*, pp. 205–240, Springer-Verlag, 2015.

[7] A. Tanwani and S. Trenn, "Determinability and state estimation for switched differentialalgebraic equations," *Automatica*, vol. 76, pp. 17 – 31, 2017.

[8] F. Küsters, S. Trenn, and A. Wirsen, "Switch observability for homogeneous switched daes," accepted for publication.

[9] M. Babaali and M. Egerstedt, "Observability of switched linear systems," in *Hybrid Systems: Computation and Control*, vol. 2993 of *Lecture Notes in Computer Science*, pp. 48–63, Springer, 2004.

[10] M. Babaali and G. J. Pappas, "Observability of switched linear systems in continuous time," in *Hybrid Systems: Computation and Control*, vol. 3414 of *LNCS*, pp. 103–117, Springer, Berlin, 2005.

[11] F. Küsters and S. Trenn, "Switch observability for switched linear systems," submitted.

[12] S. Trenn, "Switched differential algebraic equations," in *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters* (F. Vasca and L. Iannelli, eds.), ch. 6, pp. 189–216, London: Springer-Verlag, 2012.

[13] S. Trenn, *Distributional differential algebraic equations*. PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany, 2009.

[14] K. Weierstraß, "Zur Theorie der bilinearen und quadratischen Formen," *Berl. Monatsb.*, pp. 310–338, 1868.

[15] T. Berger, A. Ilchmann, and S. Trenn, "The quasi-Weierstraß form for regular matrix pencils," *Linear Algebra Appl.*, vol. 436, no. 10, pp. 4052–4069, 2012.

[16] K.-T. Wong, "The eigenvalue problem $\lambda Tx + Sx$," *J. Diff. Eqns.*, vol. 16, pp. 270–280, 1974.

[17] H. L. Trentelman, A. A. Stoorvogel, and M. L. J. Hautus, *Control Theory for Linear Systems*. Communications and Control Engineering, London: Springer-Verlag, 2001.

[18] W. Kratz, "Characterization of strong observability and construction of an observer," *Linear algebra and its applications*, vol. 221, pp. 31–40, 1995.

[19] T. Berger, T. Reis, and S. Trenn, "Observability of linear differential-algebraic systems: A survey," in *Surveys in Differential-Algebraic Equations IV* (A. Ilchmann and T. Reis, eds.), Differential-Algebraic Equations Forum, pp. 161–219, Berlin-Heidelberg: Springer-Verlag, 2017.

[20] G. Basile and G. Marrq, "A new characterization of some structural properties of linear systems: unknown-input observability, invertibility and functional controllability," *Int. J. Control*, vol. 17, no. 5, pp. 931–943, 1973.

[21] M. L. J. Hautus, "Strong detectability and strong observers," *Linear Algebra Appl.*, vol. 50, pp. 353–368, 1983.