Controllability characterization of switched DAEs

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We study controllability of switched differential algebraic equations (switched DAEs) with fixed switching signal. Based on a behavioral definition of controllability we are able to establish a controllability characterization that takes into account possible jumps and impulses induced by the switches.

1 Controllability definition

We study *switched DAEs* of the form

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \tag{1}$$

within the space of piecewise smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$, see [1]. The following assumptions are made: 1) the switching signal $\sigma : \mathbb{R} \to \mathcal{P} \subseteq \mathbb{N}$ is piecewise constant without accumulation of jumps and without jumps for $t < 0$; 2) each matrix pair $(E_p, A_p) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, $p \in \mathcal{P}$ is regular, i.e. $\det(sE_p - A_p) \not\equiv 0$. These assumptions guarantee that there exists a solution $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ for any $u \in \mathbb{D}^q_{\mathrm{pw}\mathcal{C}^\infty}$ and it is uniquely defined by $x(0^-)$, see [1]. The behavior of (1), given by

$$\mathcal{B}_\sigma := \left\{ (x,u) \in \mathbb{D}^{n+q}_{\mathrm{pw}\mathcal{C}^\infty} \;\middle|\; E_\sigma \dot{x} = A_\sigma x + B_\sigma u \right\},$$

is a linear subspace of $\mathbb{D}^{n+q}_{\mathrm{pw}\mathcal{C}^\infty}$.

Definition 1.1 A switched DAE (1) is *controllabe*, iff $\mathcal{B}_\sigma$ is controllable in the behavioral sense on some interval $[0,T]$, i.e. iff for all solutions (x_1,u_1) and (x_2,u_2) of (1) there exists a solution (x_{12},u_{12}) such that

$$(x_{12},u_{12})_{(-\infty,0)} = (x_1,u_1)_{(-\infty,0)}\,,$$
$$(x_{12},u_{12})_{(T,\infty)} = (x_2,u_2)_{(T,\infty)}\,.$$

Because of linearity we may assume $(x_2,u_2) = (0,0)$, which motivated the definition of the $[s,t]$-*controllable space*

$$\mathcal{C}^{[s,t]}_\sigma := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \exists (x,u) \in \mathcal{B}_\sigma : \\ x(s^-) = x_0 \wedge x(t^+) = 0 \end{array} \right\}.$$

Clearly, (1) is controllable on $[0,T]$ iff $\mathcal{C}^{[0,T]}_\sigma$ is the set of all feasible states at time $t = 0^-$. $\mathcal{C}^{[0,T]}_\sigma = \mathbb{R}^n$ is not necessary for controllability.

2 Nonswitched DAEs

To characterize controllability for nonswitched (regular) DAEs $E\dot{x} = Ax + Bu$ certain projectors that can be obtained from the *Quasi-Weierstraß-form* (QWF) are helpful. As (E,A) is regular, there exist invertible matrices S,T transforming (E,A) into QWF, i.e. $(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right)$ with N nilpotent [2]. Defining *consistency*, *differential* and *impulsive projector* as $\Pi := T\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]T^{-1}$, $\Pi^{\mathrm{diff}} = T\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]S$, $\Pi^{\mathrm{imp}} = T\left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]S$ and furthermore $A^{\mathrm{diff}} := \Pi^{\mathrm{diff}}A$, $B^{\mathrm{diff}} := \Pi^{\mathrm{diff}}B$, $E^{\mathrm{imp}} := \Pi^{\mathrm{imp}}E$, $B^{\mathrm{imp}} := \Pi^{\mathrm{imp}}B$, the controllable space is given by ([3])

$$\mathcal{C}^{[0,T]} = \langle A^{\mathrm{diff}}, B^{\mathrm{diff}} \rangle \oplus \langle E^{\mathrm{imp}}, B^{\mathrm{imp}} \rangle,$$

where $\langle M,P \rangle := \left[P, MP, M^2 P M^{n-1} P\right]$ for matrices $M \in \mathbb{R}^{n\times n}, P \in \mathbb{R}^{n\times q}$.

The *augmented consistency space*, i.e. the set of all consistent initial values for $E\dot{x} = Ax + Bu$, is given by ([3])

$$\overline{\mathcal{V}^*} = \mathcal{V}^* \oplus \operatorname{im}\langle E^{\mathrm{imp}}, B^{\mathrm{imp}} \rangle,$$

where it holds for the *consistency space* $\mathcal{V}^* = \operatorname{im}\Pi$.

Thus the nonswitched DAE is controllable iff $\operatorname{im}\Pi = \operatorname{im}\langle A^{\mathrm{diff}}, B^{\mathrm{diff}} \rangle$. This condition depends neither on $T > 0$ nor on the impulsive part of $E\dot{x} = Ax + Bu$. We will see that these simplifications do not hold true for switched DAEs.

3 Switched DAEs

Denote by

$$\mathcal{C}_i := \langle A_i^{\mathrm{diff}}, B_i^{\mathrm{diff}} \rangle \oplus \langle E_i^{\mathrm{imp}}, B_i^{\mathrm{imp}} \rangle \quad \text{for } i \in \mathcal{P}$$

the *local controllable space*.

Lemma 3.1 ([4, Thm. 3.6]) *The controllable space for a switched DAE with single switch signal* $\sigma_1 = \mathbb{1}_{[t_s,\infty)}$ *and* $T > t_s$ *is given by*

$$\mathcal{C}^{[0,T]}_{\sigma_1} = \Pi_1^{-1}\mathcal{C}_1 \cap \overline{\mathcal{V}_0^*} \qquad \textit{for } t_s = 0,$$
$$\mathcal{C}^{[0,T]}_{\sigma_1} = \left(\mathcal{C}_0 \cap \mathrm{e}^{-A_0^{\mathrm{diff}} t_s}\Pi_1^{-1}\mathcal{C}_1\right) \cap \overline{\mathcal{V}_0^*} \qquad \textit{for } t_s > 0.$$

Hence the system is controllable iff $\Pi_1^{-1}\mathcal{C}_1 \supseteq \overline{\mathcal{V}_0^*}$ *for* $t_s = 0$ *and* $\mathcal{C}_0 + \Pi_1^{-1}\mathcal{C}_1 \supseteq \overline{\mathcal{V}_0^*}$ *for* $t_s > 0$*, respectively.*

Note that the precise switching time $t_s > 0$ does not have any influence on controllability. This does not hold true for general switching signals [4, Ex. 3.11].

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Remark 3.2 In [5, Prop. 3.1] a sufficient condition for controllabilty of the single switch case (with $t_s > 0$) was given, namely

$$\operatorname{im}\langle A_0^{\text{diff}}, B_0^{\text{diff}}\rangle + \Pi_1^{-1}\operatorname{im}\langle A_1^{\text{diff}}, B_1^{\text{diff}}\rangle \supseteq \mathcal{V}_0^*. \quad (2)$$

The condition itself is correct as $\mathcal{C}_0 + \Pi_1^{-1}\mathcal{C}_1 \supseteq \overline{\mathcal{V}_0^*}$ can be concluded from (2) by adding $\operatorname{im}\langle E_0^{\text{imp}}, B_0^{\text{imp}}\rangle$ on both sides. However, the proof of [5, Prop. 3.1] is incorrect. The statement can either be seen as a corollary of Lemma 3.1 or proven with basically the same lines as the proof of this lemma.

To show that (2) is not a necessary condition, the following example can be employed:

$$E_0 = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right],\ A_0 = \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right],\ B_0 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],$$
$$E_1 = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],\ A_1 = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right],\ B_1 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right].$$

It holds

$$\Pi_0 = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right],\ \operatorname{im}\langle A_0^{\text{diff}}, B_0^{\text{diff}}\rangle = \operatorname{im}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],$$
$$\Pi_1 = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],\ \operatorname{im}\langle A_1^{\text{diff}}, B_1^{\text{diff}}\rangle = \operatorname{im}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right].$$

Hence condition (2) is not fulfilled, but it holds $\mathcal{C}_0 + \Pi_1^{-1}\mathcal{C}_1 \supseteq \overline{\mathcal{V}_0^*}$ as $\operatorname{im}\langle E_0^{\text{imp}}, B_0^{\text{imp}}\rangle = \operatorname{im}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. To steer $x_0 = \left[\begin{smallmatrix}x_{01}\\0\end{smallmatrix}\right] \in \mathcal{V}_0^*$ to zero it is necessary to have $x(t_s^+) \in \mathcal{C}_1$, i.e. $x(t_s^-) = \left[\begin{smallmatrix}x_{01}\\x_{01}\end{smallmatrix}\right]$. This can only be acchieved by controlling the impulsive part of the first mode. In contrast to this, condition (2) means that a system can be controlled without using this impulsive part.

Note that it is wrongly claimed in [5, Prop. 3.1] that

$$\ker \Pi_0 + \operatorname{im}\langle A_0^{\text{diff}}, B_0^{\text{diff}}\rangle + \Pi_1^{-1}\operatorname{im}\langle A_1^{\text{diff}}, B_1^{\text{diff}}\rangle = \mathbb{R}^n$$

is equivalent to (2). A counter example is the above example with $B_1 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

In order to extend the result from Lemma 3.1 to general switching signals, we use the following relabeling

$$\sigma(t) = \begin{cases} -1, & t < t_0, \\ k, & t \in [t_k, t_{k+1}), \end{cases} \quad (3)$$

and the restriction of a switching signal

$$\sigma_{>s}(t) = \begin{cases} \sigma(s^+), & t \le s, \\ \sigma(t), & t > s. \end{cases}$$

One can conclude from the single switch result

$$\mathcal{C}_{\sigma>t_{k-1}}^{[t_{k-1},t_\ell]} = \left(\mathcal{C}_{k-1} + \mathrm{e}^{-A_{k-1}^{\text{diff}}(t_k - t_{k-1})}\Pi_k^{-1}\mathcal{C}_{\sigma>t_k}^{[t_k,t_\ell]}\right) \cap \overline{\mathcal{V}_{k-1}^*}$$

for $k \le \ell$. This gives rise to the following recursion

$$\mathcal{C}_\ell^\ell := \mathcal{C}_\ell,$$
$$\mathcal{C}_{k-1}^\ell := \mathcal{C}_{k-1} + \mathrm{e}^{-A_{k-1}^{\text{diff}}(t_k - t_{k-1})}\Pi_k^{-1}\mathcal{C}_k^\ell,$$

for $k = \ell, \ldots, 2, 1$.

Theorem 3.3 ([4, Them. 3.6]) *For a switched DAE* (1) *with switching signal* (3) *it holds*

$$\mathcal{C}_\sigma^{[0,t_\ell]} = \Pi_0^{-1}\mathcal{C}_0^\ell \cap \overline{\mathcal{V}_{-1}^*}$$

and the system is controllable iff there exists $\ell \in \mathbb{N}$ such that

$$\Pi_0^{-1}\mathcal{C}_0^\ell \supseteq \overline{\mathcal{V}_{-1}^*}.$$

4 A remark on duality

With the given definition of controllability (on $[0,T]$) it is possible to show a duality result, see [6]. It turns out that the dual is not a switched DAE anymore and that time-inversion has to be applied to get a causal system. Thus, the dual property to controllability is not observability but determinability (see e.g. [7] for a definition). The property dual to observability is reachability.

Definition 4.1 A switched DAE (1) is *reachable* on $[0,T]$, iff for any solutions (x_1, u_1) of (1) and (x_2, u_2) of (1) with $\widetilde{\sigma} = \sigma(T^+)$ there exists a solution $(x_{12}, u_{12}) \in \mathcal{B}_\sigma$ such that

$$(x_{12}, u_{12})_{(-\infty,0)} = (x_1, u_1)_{(-\infty,0)},$$
$$(x_{12}, u_{12})_{(T,\infty)} = (x_2, u_2)_{(T,\infty)}.$$

A system is reachable iff any trajectory can be connected to a trajectory of the unswitched system of the last mode. Thus, not only those states $x(T^+)$ that are feasible for the switched system have to be considered (as for controllability), but all consistent values $x_T \in \overline{\mathcal{V}_{\sigma(T^+)}^*}$. An equivalent condition to reachability is that any state $x_T \in \overline{\mathcal{V}_{\sigma(T^+)}^*}$ can be reached form zero. Clearly, reachability implies controllability.

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