Duality of switched ODEs with jumps

Ferdinand Küsters and Stephan Trenn

***Abstract*— Duality between controllability/reachability and determinability/observability of switched systems with jumps is proven. The duality result is based on the recent characterization of controllability for switched differential-algebraic equations (DAEs) which share many properties with switched ordinary differential equations (ODEs) with jumps. Here we view the switching signal as given and fixed, which makes the overall switched system time-varying, in particular controllability and reachability do not coincide anymore.**

I. INTRODUCTION

We study duality of linear *switched ODEs with jumps* of the form

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad && t \notin \{t_1, \ldots, t_m\}, \\ x(t^+) &= G_{\sigma(t^+)}x(t^-), && t \in \{t_1, \ldots, t_m\}, \\ y(t) &= C_{\sigma(t)}x(t) \end{aligned} \tag{1}$$

for a given switching signal $\sigma : \mathbb{R} \to \{0, 1, \ldots, m\}$, $m \in \mathbb{N}$.

Duality is a classical result of system theory. It was first introduced for linear systems by Kalman [1] and later generalized to other system classes, for example switched systems (with switching signal as an input) [9] and non-switched impulsive systems [6]. Duality of hybrid systems (including jumps) was also studied in [7] but the approach was restricted to periodic switching and no distinction between reachability and controllability was made. To our best knowledge, our forthcoming duality result for linear switched systems with jumps and arbitrarily given (but fixed) switching signal is new.

II. SWITCHED ODES WITH JUMPS

A switched ODE with jumps (1) is given by the matrices $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times q}, C_i \in \mathbb{R}^{p\times n}$ and $G_i \in \mathbb{R}^{n\times n}$, $i \in \{0, 1, \ldots, m\}$ for some $n, p, q, m \in \mathbb{N}$ and a switching signal σ as defined below. We call x the *state*, u the *input* and y the *output* of the system. The notion state is also used for $x(t)$, $t \in \mathbb{R}$. Inputs, states and outputs of this system are assumed to be piecewise-smooth functions

$$\mathcal{C}^\infty_{pw} := \left\{ \alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[t_i,t_{i+1})} \;\middle|\; \begin{array}{l} \{t_i | i \in \mathbb{Z}\} \text{ loc. finite} \\ t_i < t_{i+1}, \alpha_i \in \mathcal{C}^\infty, \\ i \in \mathbb{Z} \end{array} \right\}.$$

Another name for this system class is impulsive system (see e.g. [12]). Note that the duality result for impulsive systems given by [6] does not allow for a switching in the system dynamics or different jump maps.

Ferdinand Küsters is with the Fraunhofer Institute for Industrial Mathematics, Kaiserslautern, Germany

Stephan Trenn is with the Technomathematics group, University of Kaiserslautern, Germany, email: trenn@mathematik.uni-kl.de

We will consider switching signals of the following class.

Definition 1 (Switching Signal): The switching signal $\sigma : \mathbb{R} \to \{0, 1, \ldots, m\}$, $m \in \mathbb{N}$ is given by

$$\sigma(t) := \begin{cases} 0, & t < t_1, \\ i, & t \in [t_i, t_{i+1}),\ i = 1, \ldots, m-1, \\ m, & t \geq t_m, \end{cases} \tag{2}$$

where $0 < t_1 < \ldots < t_m$ are the switching times and $\tau_i := t_{i+1} - t_i$ denotes the duration of mode i. Furthermore, to simplify the notation, let $t_0 := 0$ and $t_{m+1} := T$ for some $T > t_m$. Note that t_0 and t_{m+1} are *not* switching times. Finally, the single switch case $m = 1$ will play a crucial role in our analysis and we will denote the single switch switching signal by σ_1.

Remark 1 (Generality of switching signals): Note that assuming that the switching signal has the form (2) is not a practical restriction of generality. First of all, it is reasonable to assume that the switching times do not accumulate towards minus infinity, hence there is a first switching time t_1 and we can chose the time axis such that $t_1 > 0$. Additionally, we assume that there are no finite accumulation points of the switching times. i.e. on every finite interval there are only finitely many switching times and by relabeling the matrices accordingly we arrive at (2). Finally, because of causality the solution behavior of (1) on some finite interval $(0, T)$ is independent of the switching signal on (T, ∞), hence assuming that the switching signal is constant on $[T, \infty)$ is no restriction for our setup where we are only interested in duality with respect to a given finite interval (and not on asymptotic behaviors). As a side effect of the latter assumption, the time reversal of σ (with corresponding relabeling) is a again a switching signal according to our definition.

III. SYSTEM THEORETIC PROPERTIES

The system theoretic properties controllability, reachability, observability and determinability are defined based on the *behavior*

$$\mathcal{B}_\sigma := \left\{ (u, x, y) \in \left(\mathcal{C}^\infty_{pw}\right)^{q\times n\times p} \;\middle|\; (u, x, y) \text{ satisfies (1)} \right\}.$$

Controllability and reachability are analzyed analogous to the work [4]. The results on observability and determinability are based on [8], [10].

A. Definitions

Definition 2: The switched ODE with jumps (1) with switching signal (2) is called

- *controllable on* $[0,T]$ iff the corresponding behavior $\mathcal{B}_\sigma$ is controllable on $[0,T]$ in the behavioral sense, i.e. $\forall \omega, \hat{\omega} \in \mathcal{B}_\sigma \ \exists \tilde{\omega} \in \mathcal{B}_\sigma$:
$$\omega_{(-\infty,0)} = \tilde{\omega}_{(-\infty,0)},\ \hat{\omega}_{(T,\infty)} = \tilde{\omega}_{(T,\infty)},$$
- *reachable on* $[0,T]$ iff $\forall \omega \in \mathcal{B}_\sigma,\ \hat{\omega} \in \mathcal{B}_{\sigma(T^+)} \ \exists \tilde{\omega} \in \mathcal{B}_\sigma$:
$$\omega_{(-\infty,0)} = \tilde{\omega}_{(-\infty,0)} \text{ and } \hat{\omega}_{(T,\infty)} = \tilde{\omega}_{(T,\infty)},$$
- *observable on* $[0,T]$ iff $\forall (u,x,y), (\hat{u},\hat{x},\hat{y}) \in \mathcal{B}_\sigma$:
$$u = \hat{u} \wedge y_{[0,T]} = \hat{y}_{[0,T]} \ \Rightarrow x = \hat{x},$$
- *determinable on* $[0,T]$ iff $\forall (u,x,y), (\hat{u},\hat{x},\hat{y}) \in \mathcal{B}_\sigma$:
$$u = \hat{u} \wedge y_{[0,T]} = \hat{y}_{[0,T]} \ \Rightarrow x_{(T,\infty)} = \hat{x}_{(T,\infty)}.$$

Because of linearity it is easily seen (c.f. [11, Prop. 7] and [4, Rem. 3.2]) that a switched ODE with jumps is controllable on $[0,T]$ iff for any initial value $x_0 \in \mathbb{R}^n$ there exists a solution $(u,x,y) \in \mathcal{B}_\sigma$ with $x(0^-) = x_0$ and $x(T^+) = 0$. A system is reachable on $[0,T]$ iff for any value $x_T \in \mathbb{R}^n$ there exists a solution $(u,x,y) \in \mathcal{B}_\sigma$ with $x(0^-) = 0$ and $x(T^+) = x_T$. It is observable on $[0,T]$ iff $(0,x,y) \in \mathcal{B}_\sigma$ with $y_{[0,T]} \equiv 0$ implies $x(0^-) = 0$ and determinable on $[0,T]$ iff the same implies $x(T^+) = 0$.

The following spaces do not directly fit these criteria, but turn out to give a more intuitive approach to the duality results.

Definition 3: For $0 \le s < t$ we define

- the *controllable space*
$$\mathcal{C}_\sigma^{(s,t)} := \left\{ x_s \in \mathbb{R}^n \ \middle|\ \begin{array}{l} \exists (u,x,y) \in \mathcal{B}_\sigma : \\ x(s^+) = x_s, x(t^-) = 0 \end{array} \right\},$$
- the *reachable space*
$$\mathcal{R}_\sigma^{(s,t)} := \left\{ x_t \in \mathbb{R}^n \ \middle|\ \begin{array}{l} \exists (u,x,y) \in \mathcal{B}_\sigma : \\ x(s^+) = 0, x(t^-) = x_t \end{array} \right\},$$
- the *unobservable space*
$$\mathcal{UO}_\sigma^{(s,t)} := \left\{ x_s \in \mathbb{R}^n \ \middle|\ \begin{array}{l} \exists (0,x,y) \in \mathcal{B}_\sigma : \\ x(s^+) = x_s, y_{(s,t)} = 0 \end{array} \right\},$$
- the *undeterminable space*
$$\mathcal{UD}_\sigma^{(s,t)} := \left\{ x_t \in \mathbb{R}^n \ \middle|\ \begin{array}{l} \exists (0,x,y) \in \mathcal{B}_\sigma : \\ x(t^+) = x_t, y_{(s,t)} = 0 \end{array} \right\}.$$

In the following we will assume that the interval $[0,T]$ considered for the system theoretic properties corresponds to the switching signal. As all switches lie in $(0,T)$ the spaces defined above fit to the system theoretic properties.

Lemma 1: A switched ODE with jumps and switching signal (2) is

- controllable on $[0,T]$ iff $\mathcal{C}_\sigma^{(0,T)} = \mathbb{R}^n$,
- reachable on $[0,T]$ iff $\mathcal{R}_\sigma^{(0,T)} = \mathbb{R}^n$,
- observable on $[0,T]$ iff $\mathcal{UO}_\sigma^{(0,T)} = \{0\}$,
- determinable on $[0,T]$ iff $\mathcal{UD}_\sigma^{(0,T)} = \{0\}$.

Proof: The system theoretic properties do not change when assuming smooth inputs. As 0 and T are not switching times, u, x and y are smooth at these points. ∎

In the following it will be helpful to consider restrictions of switching signals. For a switching signal σ and $s \ge 0$ we define
$$\sigma_{\ge s}(t) = \begin{cases} \sigma(t), & t \ge s, \\ \sigma(s^+), & t < s. \end{cases}$$
$\sigma_{\ge s}$ is called *restriction to* (s,∞) as it is equal to σ on (s,∞) and has jumps only in this interval.

B. Controllability

This section is based on [4] where the given controllability notion was first discussed for switched DAEs. The ideas for switched ODEs with jumps are quite similar.

Lemma 2 (Single switch, [4, Thm. 3.6]): Consider the switched ODE with jumps (1) with the single switch switching signal σ_1. Then the controllable space is given by
$$\mathcal{C}_{\sigma_1}^{(0,T)} = \mathcal{C}_0 + \mathrm{e}^{-A_0\tau_0} G_1^{-1} \mathcal{C}_1,$$
where $\mathcal{C}_0$ and $\mathcal{C}_1$ are the usual controllability spaces of the unswitched ODEs $\dot{x} = A_0 x + B_0 u$ and $\dot{x} = A_1 x + B_1 u$.

The single switch result is used to derive a formula for the controllable space for general switching signals. Starting from the last switch a recursion for the controllable space is given. To derive this recursion we work with switching signals whose switches are restricted to intervals (t_i, T) as otherwise we would have to care about feasibility. The switching signal $\sigma_{\ge t_i}$ guarantees that any $x_i \in \mathbb{R}^n$ is a feasible state at time t_i^+, i.e. there exists $(u,x,y) \in \mathcal{B}_{\sigma_{\ge t_i}}$ with $x(t_i^+) = x_i$.

Theorem 1 (General switching signal, cf. [4, Thm. 3.10]): Consider the switched ODE with jumps (1) with switching signal (2). Define
$$\begin{aligned} \mathcal{P}_m^m &:= \mathcal{C}_m, \\ \mathcal{P}_i^m &:= \mathcal{C}_i + \mathrm{e}^{-A_i\tau_i} G_{i+1}^{-1} \mathcal{P}_{i+1}^m \end{aligned} \tag{3}$$
for $i = m-1, \dots, 0$, where $\mathcal{C}_i$ denote the usual controllability space of the unswitched ODE $\dot{x} = A_i x + B_i u$. Then it holds
$$\mathcal{C}_{\sigma_{\ge t_i}}^{(t_i,T)} = \mathcal{P}_i^m \text{ for } i = 0, \dots, m$$
and the system is controllable on $[0,T]$ iff $\mathbb{R}^n = \mathcal{P}_0^m$.

Proof: The first statement says that $\mathcal{P}_i^m$ is the set of all states at time t_i^+ which can be controlled to zero on (t_i, T) when neglecting the switches on $(0, t_i]$, i.e. for a switching signal constant on $(-\infty, t_i]$. The single switch result can be used to obtain
$$\mathcal{C}_{\sigma_{\ge t_{i-1}}}^{(t_{i-1},T)} = \mathcal{C}_{i-1} + \mathrm{e}^{-A_{i-1}\tau_{i-1}} G_i^{-1} \mathcal{C}_{\sigma_{\ge t_i}}^{(t_i,T)}.$$

The statement is then shown by induction. For $i = m$ we have the unswitched case. Using the above equation it holds for the induction step $i \to i-1$:
$$\begin{aligned} \mathcal{C}_{\sigma_{\ge t_{i-1}}}^{(t_{i-1},T)} &= \mathcal{C}_{i-1} + \mathrm{e}^{-A_{i-1}\tau_{i-1}} G_i^{-1} \mathcal{C}_{\sigma_{\ge t_i}}^{(t_i,T)} \\ &\overset{\text{Ind.}}{=} \mathcal{C}_{i-1} + \mathrm{e}^{-A_{i-1}\tau_{i-1}} G_i^{-1} \mathcal{P}_i^m = \mathcal{P}_{i-1}^m. \end{aligned}$$
The last statement follows by Lemma 1. ∎

C. Reachability

Lemma 3 (Single switch): Consider the switched ODE with jumps (1) with switching signal σ_1. The reachable space is given by

$$\mathcal{R}_{\sigma_1}^{(0,T)} = \mathcal{C}_1 + \mathrm{e}^{A_1\tau_1} G_1 \mathcal{C}_0.$$

Proof: "$\subseteq$": Let $x_T \in \mathcal{R}_{\sigma_1}^{(0,T)}$, i.e. there exists $(u,x,y) \in \mathcal{B}_{\sigma_1}$ with $x(0^+) = 0$ and $x(T^-) = x_T$. Define $\bar{u} := u_{(-\infty,t_1)}$, $\hat{u} = u_{[t_1,\infty)}$ and corresponding solutions $\bar{x}, \hat{x}$ with zero initial condition. Clearly, $x = \bar{x} + \hat{x}$. It holds $\bar{x}(t_1^-) \in \mathcal{C}_0$ and hence $\bar{x}(T^-) \in \mathrm{e}^{A_1\tau_1} G_1 \mathcal{C}_0$. For $\hat{x}$ it holds $\hat{x}(t_1^-) = 0$ and hence $\hat{x}(T^-) \in \mathcal{C}_1$. This gives

$$x(T^-) = \bar{x}(T^-) + \hat{x}(T^-) \in \mathrm{e}^{A_1\tau_1} G_1 \mathcal{C}_0 + \mathcal{C}_1.$$

"$\supseteq$": Let $x_T \in \mathcal{C}_1 + \mathrm{e}^{A_1\tau_1} G_1 \mathcal{C}_0$. Hence there exists $x_1 \in \mathcal{C}_0$ such that $x_T - \mathrm{e}^{A_1\tau_1} G_1 x_1 \in \mathcal{C}_1$. Define $\bar{u}$ on $[0,t_1)$ such that $(\bar{u},\bar{x},\bar{y}) \in \mathcal{B}_{\sigma_1}$ with zero initial condition and $\bar{x}(t_1^-) = x_1$ and define $\hat{u}$ on $[t_1,T)$ such that $(\hat{u},\hat{x},\hat{y}) \in \mathcal{B}_{\sigma_1}$ with zero initial condition and $\hat{x}(T^-) = x_T - \mathrm{e}^{A_1\tau_1} G_1 x_1$. Note that $\bar{u}$ is zero outside $(0,t_1)$ and $\hat{u}$ is zero outside (t_1,T). It holds for $(u,x,y) := (\bar{u}+\hat{u}, \bar{x}+\hat{x}, \bar{y}+\hat{y}) \in \mathcal{B}_{\sigma_1}$: $x(0^+) = \bar{x}(0^+) + \hat{x}(0^+) = 0$ and

$$\begin{aligned} x(T^-) &= \bar{x}(T^-) + \hat{x}(T^-) \\ &= \mathrm{e}^{A_1\tau_1} G_1 x_1 + x_T - \mathrm{e}^{A_1\tau_1} G_1 x_1 = x_T. \end{aligned}$$

Hence $\mathcal{C}_1 + \mathrm{e}^{A_1\tau_1} G_1 \mathcal{C}_0 \subseteq \mathcal{R}_{\sigma_1}^{(0,T)}$. ∎

The single switch result is now used to derive a recursion giving the reachable space on $(0,t_{k+1})$ based on the reachable space on $(0,t_k)$. As we are going forward in time, feasibility is not an issue and thus a restriction of the switching signal - as for controllability - is not necessary.

Theorem 2 (General switching signal): Consider the switched ODE with jumps (1) with switching signal (2). Define

$$\begin{aligned} \mathcal{Q}_0^0 &:= \mathcal{C}_0, \\ \mathcal{Q}_0^i &:= \mathcal{C}_i + \mathrm{e}^{A_i\tau_i} G_i \mathcal{Q}_0^{i-1} \end{aligned} \tag{4}$$

for $i = 1,\dots,m$. Then it holds

$$\mathcal{R}_\sigma^{(0,t_{i+1})} = \mathcal{Q}_0^i \quad \text{for } i = 0,1,\dots,m$$

and the system is reachable on $[0,T]$ iff $\mathbb{R}^n = \mathcal{Q}_0^m$.

Proof: $i = 0$ describes an unswitched system with mode 0, whose reachable space is given by $\mathcal{C}_0 = \mathcal{Q}_0^0$. The case $i = 1$ is the single switch case shown in Lemma 3.

Analogously to Lemma 3 it holds for $i \geq 1$

$$\mathcal{R}_\sigma^{(0,t_{i+1})} = \mathcal{C}_i + \mathrm{e}^{A_i\tau_i} G_i \mathcal{R}_\sigma^{(0,t_i)}.$$

Hence it holds by induction

$$\mathcal{R}_\sigma^{(0,t_{i+1})} = \mathcal{Q}_0^i \quad \text{for } i = 0,1,\dots,m$$

and the reachability criterion follows by Lemma 1. ∎

D. Observability

Lemma 4 (Single switch, [8, Lem. 8.9]): Consider a switched ODE with jumps (1) with single switch signal σ_1. The unobservable space is then given by

$$\mathcal{UO}_{\sigma_1}^{(0,T)} = \mathcal{U}_0 \cap \mathrm{e}^{-A_0\tau_0} G_1^{-1} \mathcal{U}_1,$$

where $\mathcal{U}_0$ and $\mathcal{U}_1$ are the usual unobservable spaces of the unswitched DAE $\dot{x} = A_0 x$, $y = C_0 x$ and $\dot{x} = A_1 x$, $y = C_1 x$.

Theorem 3 (General switching signal, [10, Thm. 1]): Consider a switched ODE with jumps (1) with switching signal (2) and define

$$\begin{aligned} \mathcal{M}_m^m &:= \mathcal{U}_m, \\ \mathcal{M}_i^m &:= \mathcal{U}_i \cap \mathrm{e}^{-A_i\tau_i} G_{i+1}^{-1} \mathcal{M}_{i+1}^m \end{aligned} \tag{5}$$

for $i = m-1,\dots,0$, where $\mathcal{U}_i$ denotes the usual unobservable spaces of the unswitched ODEs $\dot{x} = A_i x$, $y = C_i x$. Then it holds

$$\mathcal{UO}_{\sigma_{\geq t_i}}^{(t_i,T)} = \mathcal{M}_i^m \text{ for } i = 0,\dots,m$$

and the system is observable on $[0,T]$ iff $\mathcal{M}_0^m = \{0\}$.

E. Determinability

Lemma 5 (Single switch, [8, Lemma 8.11]): Consider a switched ODE with jumps (1) with single switch signal σ_1. The undeterminable space is then given by

$$\mathcal{UD}_{\sigma_1}^{(0,T)} = \mathrm{e}^{A_1\tau_1} G_1 \mathcal{U}_0 \cap \mathcal{U}_1.$$

Theorem 4 (General switching signal, cf. [10, Thm. 2]): For a switched ODE with jumps (1) with switching signal (2) define

$$\begin{aligned} \mathcal{N}_0^0 &:= \mathcal{U}_0, \\ \mathcal{N}_0^i &:= \mathcal{U}_i \cap \mathrm{e}^{A_i\tau_i} G_i \mathcal{N}_0^{i-1} \end{aligned} \tag{6}$$

for $i = 1,\dots,m$. It holds

$$\mathcal{UO}_\sigma^{(0,t_i)} = \mathcal{N}_0^i \text{ for } i = 0,\dots,m$$

and the system is determinable on $[0,T]$ iff $\{0\} = \mathcal{N}_0^m$.

Proof: The statement is shown by induction. $i = 0$ is the unswitched case. The induction step is analogous to the single switch result:

For the induction step $i \to i+1$ let

$$x_{i+1} \in \mathcal{UD}_\sigma^{(0,t_{i+1})}.$$

Hence there exists $(0,x,y) \in \mathcal{B}_\sigma$ with $x(t_{i+1}^-) = x_{i+1}$ and $y_{(0,t_{i+1})} \equiv 0$. $y_{(0,t_i)} \equiv 0$ gives $x(t_i^-) \in \mathcal{UD}_\sigma^{(0,t_i)}$, which is $\mathcal{N}_0^i$ by induction. Hence $x(t_{i+1}^-) \in \mathrm{e}^{A_i\tau_i} G_i \mathcal{N}_0^i$. $y_{(t_i,t_{i+1})} \equiv 0$ yields $x(t_{i+1}^-) \in \mathcal{U}_i$. All in all we obtain

$$x_{i+1} = x(t_{i+1}^-) = \mathcal{U}_i \cap \mathrm{e}^{A_i\tau_i} G_i \mathcal{N}_0^{i-1} = \mathcal{N}_0^i.$$

For the other inclusion let $x_{i+1} \in \mathcal{N}_0^i = \mathcal{U}_i \cap \mathrm{e}^{A_i\tau_i} G_i \mathcal{N}_0^{i-1}$. Thus there exists $x_i \in \mathcal{N}_0^{i-1}$ with $x_{i+1} = \mathrm{e}^{A_i\tau_i} G_i x_i$. By the induction assumption it holds $\mathcal{N}_0^i = \mathcal{UD}_\sigma^{(0,t_i)}$, hence there exists $(0,x,y) \in \mathcal{B}_\sigma$ with $x(t_i^-) = x_i$ and $y_{(0,t_i)} \equiv 0$. $x(t_{i+1}^-) \in \mathcal{U}_i$ gives $y_{(t_i,t_{i+1})} \equiv 0$. All in all, we have $y_{(0,t_{i+1})} \equiv 0$ and thus $x_{i+1} = x(t_{i+1}^-) \in \mathcal{UD}_\sigma^{(0,t_{i+1})}$.

The determinability criterion follows by Lemma 1. ∎

Remark 2: It follows that the system theoretic properties do not depend on $\tau_m > 0$. Hence a system is controllable/reachable/observable/determinable on $[0,T]$ iff it has the property on $[0,T']$ for $T,T' > t_m$. In particular, the system theoretic properties are the same for all T corresponding to the switching signal σ. The corresponding spaces might, however, depend on the precise time-interval.

IV. Dual system

To derive an adjoint system for switched ODEs with jumps we recapitulate the derivation of the adjoint system for ODEs given in [2], [13] and [14]. The distinction between adjoint and dual system is not necessary for ODEs but turns out to be helpful for switched ODEs with jumps.

Although we defined switched ODEs for $\mathcal{C}^\infty_{pw}$-solutions, the following operators will be defined on $\mathcal{L}^2$ as we want to use their $\mathcal{L}^2$-adjoints. However, for the interpretation of these adjoint operators we restrict ourselves to $\mathcal{C}^\infty_{pw}([0,T],\mathbb{R}^n) \subset \mathcal{L}^2([0,T],\mathbb{R}^n) =: X^n$.

A. Ajoint of ODEs

Consider first a linear ODE

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{7}$$

The following three mappings characterize the system (7) uniquely.

1) input-state-map
$$\eta_T : X^q \to X^n,\ u(\cdot) \mapsto Bu(\cdot),$$
mapping an input to the corresponding system's inhomogenity;
2) state-state-map
$$\omega_T : \mathbb{R}^n \times X^n \to \mathbb{R}^n \times X^n,\ (x_0, g(\cdot)) \mapsto (x(T), x(\cdot)),$$
where $x(t)$ is given by $\mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-t')}g(t')\mathrm{d}t'$. ω_T maps an initial state and an inhomogenity to a final state and the trajectory;
3) state-output-map
$$\tau_T : X^n \to X^p,\ x(\cdot) \mapsto Cx(\cdot) = y(\cdot),$$
mapping the state trajectory to the output trajectory.

Lemma 6 ([13, Section 2.2]): The adjoint operators of η_T, ω_T, τ_T are given by:

$$\begin{aligned} \eta_T^* :&\quad X^n \to X^q, && p(\cdot) \mapsto B^\top p(\cdot), \\ \omega_T^* :&\quad \mathbb{R}^n \times X^n \to \mathbb{R}^n \times X^n, && (p_T, h(\cdot)) \mapsto (p(0), p(\cdot)), \\ \tau_T^* :&\quad X^p \to X^n, && u_a(\cdot) \mapsto C^\top u(\cdot), \end{aligned}$$

where $p(t)$ is given by $\mathrm{e}^{-A^\top(t-T)}p_T - \int_T^t \mathrm{e}^{-A^\top(t-t')}h(t')\mathrm{d}t'$ η_T^*, τ_T^* and ω_T^* characterize the system

$$\begin{aligned} \dot{p} &= -A^\top p - C^\top u_a, \\ y_a &= \quad B^\top p. \end{aligned} \tag{8}$$

on $[0,T]$ going backwards in time ([2, II, Kapitel 11.3]). We call this system *adjoint system*.

B. Adjoint of switched ODEs with jumps

Generalizing η_T, ω_T and τ_T to switched ODEs with jumps gives

$$\begin{aligned} \eta_T :&\quad X^q \to X^n, && u(\cdot) \mapsto B_\sigma u(\cdot), \\ \omega_T :&\quad \mathbb{R}^n \times X^n \to \mathbb{R}^n \times X^n, && (x_0, g(\cdot)) \mapsto (x(T), x(\cdot)), \\ \tau_T :&\quad X^n \to X^p, && x(\cdot) \mapsto C_\sigma x(\cdot) = y(\cdot), \end{aligned}$$

where $x(t)$, $t \in [t_i, t_{i+1})$, is defined by

$$\begin{aligned} x(t) = \mathrm{e}^{A_i(t-t_i)}G_i\Big[\mathrm{e}^{A_{i-1}\tau_{i-1}}G_{i-1}\Big(\ldots \\ \Big(\mathrm{e}^{A_0(t_1-0)}x_0 + \int_0^{t_1}\mathrm{e}^{A_0(t_1-t')}g(t')\mathrm{d}t'\Big)\cdots\Big) \\ + \int_{t_{i-1}}^{t_i}\mathrm{e}^{A_{i-1}(t_i-t')}g(t')\mathrm{d}t'\Big] + \int_{t_i}^{t}\mathrm{e}^{A_i(t-t')}g(t')\mathrm{d}t'. \end{aligned}$$

Lemma 7 (Adjoint mappings for switched ODEs with jumps): The adjoint mappings to η_T, ω_T, τ_T are given by

$$\begin{aligned} \eta_T^* :&\quad X^n \to X^q, && p(\cdot) \mapsto B_\sigma^\top p(\cdot), \\ \omega_T^* :&\quad \mathbb{R}^n \times X^n \to \mathbb{R}^n \times X^n, && (p_T, h(\cdot)) \mapsto (p(0), p(\cdot)), \\ \tau_T^* :&\quad X^p \to X^n, && u_a(\cdot) \mapsto C_\sigma^\top u_a(\cdot), \end{aligned}$$

where $p(t)$, $t \in [t_i, t_{i+1})$, is given by

$$\begin{aligned} p(t) = \mathrm{e}^{A_i^\top(t_{i+1}-t)}G_{i+1}^\top\Big[\ldots\Big(\mathrm{e}^{A_m^\top\tau_m}p_T \\ + \int_{t_m}^{T}\mathrm{e}^{A_m^\top(t'-t_m)}h(t')\mathrm{d}t'\Big)\cdots\Big] + \int_t^{t_{i+1}}\mathrm{e}^{A_i^\top(t'-t)}h(t')\mathrm{d}t'. \end{aligned}$$

Proof: For the proof we have to show that

$$\begin{aligned} \langle \eta_T(u(\cdot)), p(\cdot)\rangle_{\mathcal{L}_2} &= \langle u(\cdot), \eta_T^*(p(\cdot))\rangle_{\mathcal{L}_2}, \\ \langle \tau_T(x(\cdot)), u_a(\cdot))\rangle_{\mathcal{L}_2} &= \langle x(\cdot), \tau_T^*(u_a(\cdot))\rangle_{\mathcal{L}_2}, \end{aligned}$$

which directly follows from the definition, and

$$\begin{aligned} &\langle \omega_T(x_0, g(\cdot)), (p_T, h(\cdot))\rangle_{\mathbb{R}^n \times \mathcal{L}_2} \\ &\qquad = \langle (x_0, g(\cdot)), \omega_T^*(p_T, h(\cdot))\rangle_{\mathbb{R}^n \times \mathcal{L}_2}, \end{aligned}$$

which can be shown with straightforward (but lengthly) calculations, for details see [3, Lem. 5.3.1]. ■

These mappings describe the system

$$\begin{aligned} \frac{d}{dt}p(t) &= -A_\sigma^\top p(t) - C_\sigma^\top u_a(t), && t \notin \{t_1, \ldots, t_m\}, \\ p(t^-) &= G_{\sigma(t^+)}^\top p(t^+), && t \in \{t_1, \ldots, t_m\}, \\ y_a(t) &= B_\sigma^\top p(t) \end{aligned} \tag{9}$$

going backward in time. This system is called *adjoint system*. While a switched ODE can be inverted in time, this does not hold true for switched ODEs with jumps as the jump mappings might be singular. The jumps $p(t_i^-) = G_{t_i}^\top p(t_i^+)$ lead to the condition $p(t_i^-) \in \operatorname{im} G_{t_i}^\top$ for all switching times t_i. Hence (9) does not have a solution for every initial condition $p(0) = p_0$. Furthermore, the solution of the adjoint system for an initial value problem does not have to be unique.

C. Alternative derivation

Another approach to deriving the adjoint system is by considering the equation

$$\frac{d}{dt}\left(p^\top x\right) = 0$$

giving adjointness for homogeneous ODEs ([2]) and generalize it to systems with inputs and outputs.

This approach was taken by [14]. It leads to the condition

$$\frac{d}{dt}\left(p^\top x\right) - y_a^\top u + u_a^\top y = 0. \tag{10}$$

and the corresponding adjoint behavior

$$\mathcal{B}_\sigma^{\text{ad}} := \left\{\ (u_a, p, y_a)\ \mid \text{(10) holds for all } (u,x,y) \in \mathcal{B}_\sigma\ \right\}.$$

To interpret (10) for switched ODEs with jumps one has to reformulate the switched ODE with jumps in the space of piecewise smooth distributions, see [12]. However, the adjoint systems equations are not uniquely defined by the adjoint behavior (see [3]). Nevertheless, this approach is of some interest because it can be used to study duality for switched DAEs (see [5]), where the approach based on adjoint mappings is not applicable because the underlying distributional solution space cannot be embedded into $\mathcal{L}^2$.

D. Dual system of switched ODEs with jumps

The adjoint of a switched ODE with jumps is not causal, i.e. it does not have unique solutions for initial value problems. It is however anticausal, i.e. causal backwards in time. Thus we consider the time inverted version of (9), which we denote as *dual system*:

$$\begin{aligned} \frac{d}{ds} z(s) &= A_{\bar\sigma}^\top z(s) + C_{\bar\sigma}^\top u_d(s), && s \notin \{s_1, \dots, s_m\}, \\ z(s^+) &= G_{\bar\sigma(s^-)}^\top z(s^-), && s \in \{s_1, \dots, s_m\}, \\ y_d(s) &= B_{\bar\sigma}^\top z(s) \end{aligned} \tag{11}$$

with inverted time $s := T - t$, $s_i := T - t_i$ for $i = 1, \dots, m$ and inverted switching signal $\bar\sigma(s) = \sigma(t)$.

The dual of a switched ODE with jumps is again a switched ODE with jumps (after relabeling the impact matrices of the dual as $\widehat{G}_{\sigma(s^+)} := G_{\sigma(t^+)}^\top$), the dual of a switched ODE is again a switched ODE. For σ_m this relabeling gives $\widehat{G}_i = G_{i+1}^\top$. Furthermore, the dual of a dual system is the original system.

It turns out that it would be preferable to label the jump matrices G according to the switching time and not to the mode that is switched to. We stick to the notation of [8] as this makes it in fact more intuitive to apply the results on the system theoretic properties on the dual system.

V. Duality of switched ODEs with jumps

In order to derive a duality statement for switched ODEs with jumps the recursions for the system theoretic properties are revised for the dual system (11). By " ˆ " we denote matrices and subspaces referring to the dual system. Note that the switching signal for the dual system is inverted, i.e. it goes from mode m to mode 0.

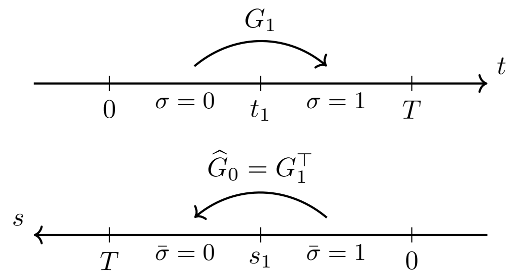

Fig. 1. Jump matrices G_1, $\widehat{G}_0 = G_1^\top$ for a switched ODE with jumps and its dual.

Recall that for each mode $i \in \{0, \dots, m\}$ we have

$$\widehat{\mathcal{C}}_i = \mathcal{U}_i^\perp \ \text{ and } \ \widehat{\mathcal{U}}_i = \mathcal{C}_i^\perp$$

A recursive equation for the controllability of the dual system is given by

$$\begin{aligned} \widehat{\mathcal{P}}_0^0 &= \widehat{\mathcal{C}}_0 = (\mathcal{U}_0)^\perp, \\ \widehat{\mathcal{P}}_i^0 &= \widehat{\mathcal{C}}_i + \mathrm{e}^{-\widehat{A}_i \tau_i} \widehat{G}_{i-1}^{-1} \widehat{\mathcal{P}}_{i-1}^0 \\ &= (\mathcal{U}_i)^\perp + \mathrm{e}^{-A_i^\top \tau_i} G_i^{-\top} \widehat{\mathcal{P}}_{i-1}^0 \end{aligned} \tag{12}$$

for $i = 1, \dots, m$. Analogously, for the reachability of the dual system we have

$$\begin{aligned} \widehat{\mathcal{Q}}_m^m &= \widehat{\mathcal{C}}_m = (\mathcal{U}_m)^\perp, \\ \widehat{\mathcal{Q}}_m^i &= \widehat{\mathcal{C}}_i + \mathrm{e}^{\widehat{A}_i \tau_i} \widehat{G}_i \widehat{\mathcal{Q}}_m^{i+1} \\ &= (\mathcal{U}_i)^\perp + \mathrm{e}^{A_i^\top \tau_i} G_{i+1}^\top \widehat{\mathcal{Q}}_m^{i+1} \end{aligned} \tag{13}$$

for $i = m-1, \dots, 0$. Observability of the dual can be described by

$$\begin{aligned} \widehat{\mathcal{M}}_0^0 &= \widehat{\mathcal{U}}_0 = (\mathcal{C}_0)^\perp, \\ \widehat{\mathcal{M}}_i^0 &= \widehat{\mathcal{U}}_i \cap \mathrm{e}^{-\widehat{A}_i \tau_i} \widehat{G}_{i-1}^{-1} \widehat{\mathcal{M}}_{i-1}^0 \\ &= (\mathcal{C}_i)^\perp \cap \mathrm{e}^{-A_i^\top \tau_i} G_i^{-\top} \widehat{\mathcal{M}}_{i-1}^0 \end{aligned} \tag{14}$$

for $i = 1, \dots, m$. Finally, for the determinability we get

$$\begin{aligned} \widehat{\mathcal{N}}_m^m &= \widehat{\mathcal{U}}_m = (\mathcal{C}_m)^\perp, \\ \widehat{\mathcal{N}}_m^i &= \widehat{\mathcal{U}}_i \cap \mathrm{e}^{\widehat{A}_i \tau_i} \widehat{G}_i^{-1} \widehat{\mathcal{N}}_m^{i+1} \\ &= (\mathcal{C}_i)^\perp \cap \mathrm{e}^{A_i^\top \tau_i} G_{i+1}^{-\top} \widehat{\mathcal{N}}_m^{i+1} \end{aligned} \tag{15}$$

for $i = m-1, \dots, 0$.

These recursions are illustrated in Figure 2.

Theorem 5 (Duality of switched ODEs with jumps): For a switched ODE with jumps (1) with switching signal (2) and dual system (11) it holds:

$$\begin{array}{ccc} \textit{Observability} & \overset{dual}{\leftrightsquigarrow} & \textit{Reachability} \\ \Downarrow & & \Downarrow \\ \textit{Determinability} & \overset{dual}{\leftrightsquigarrow} & \textit{Controllability.} \end{array}$$

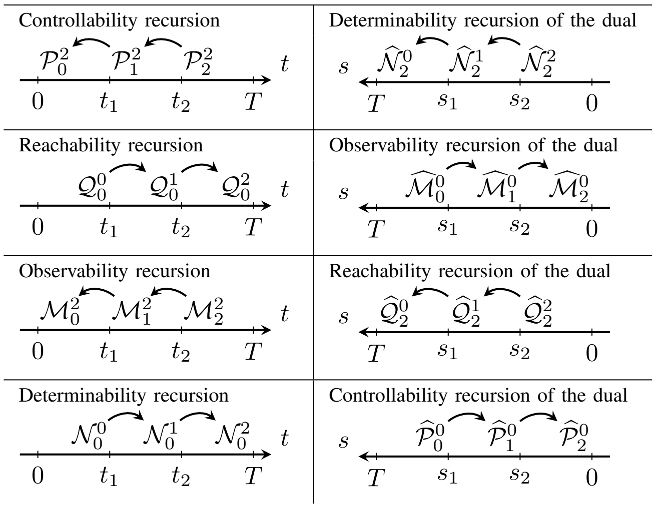

Fig. 2. Recursions for switched ODE with jumps and its dual for σ_2.

Proof: It holds for the recursions of the original switched ODE with jumps (3)-(6) and the recursions for its dual (12)-(15):

$$(\mathcal{P}_i^m)^\perp = \widehat{\mathcal{N}}_m^i, \quad \left(\mathcal{N}_0^i\right)^\perp = \widehat{\mathcal{P}}_i^0,$$
$$(\mathcal{M}_i^m)^\perp = \widehat{\mathcal{Q}}_m^i, \quad \left(\mathcal{Q}_0^i\right)^\perp = \widehat{\mathcal{M}}_i^0. \qquad \blacksquare$$

Remark 3: In Remark 1 we restricted our attention to switches on the interval $[0, T]$. Considering also switches $t_i > T$ destroys the duality. First of all, the dual system does not have a switching signal corresponding to Definition 1. Furthermore, the switch $t_i > T$ does not influence the system theoretic properties of the original system on $[0, T]$, but it will reduce the feasibility set at time $s = 0^-$ for the dual system: Consider the switched ODE with jumps with $(B_0, C_0) = (0, 0)$ and $G_1 = 0$ with switching time $t_1 > T$ (system matrices that are not relevant were not specified). The system is neither controllable, reachable, observable nor determinable on $[0, T]$. The dual system has a switch at $s_1 = T - t_1 < 0$ with jump matrix $G_1^\top = 0$. As $C_0 = 0$ it holds $z(s) = 0$ for $s \geq s_1$. Hence $z(0^-) = 0$ for every solution and the dual is controllable, observable and determinable on $[0, T]$. However, it is not reachable on $[0, T]$.

A final example might illustrate the importance of time-inversion:

Example 1: Consider the single switch problem with

$$A_{\sigma_1} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$C_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The system is controllable and determinable, but neither observable nor reachable. Hence its dual is by Theorem 5 determinable and controllable, but not reachable or observable.

A naive computation of the dual omitting the time-inversion would lead to a system that is neither controllable, reachable, observable nor determinable. Hence the time-inversion is in fact crucial for the duality result.

VI. Conclusions

We have introduced the dual of a switched ODE with jumps via a time-inversion of the adjoint systems. The latter was defined formally via adjointness of the input-state-, state-state- and state-output-maps. The time-inversion is necessary to obtain a causal system again, i.e. in contrast to switched ODEs (without jumps) the dual and adjoint are different in nature. Based on recent controllability and observability characterization it is then possible to proof a duality result for switched ODEs with jumps. It should be noted that here controllability and reachability as well as observability and determinability are not equivalent and the duality pairs are controllability$\leftrightsquigarrow$determinability and reachability$\leftrightsquigarrow$observability (and not as usually controllability$\leftrightsquigarrow$observability). Since the solution behavior of switched ODEs with jumps are very similar to the solution behavior of switched DAEs the ideas presented here open the door to also establish a duality result for switched DAEs [5].

References

[1] Rudolf E. Kalman. On the general theory of control systems. In *Proceedings of the First International Congress on Automatic Control, Moscow 1960*, pages 481–493, London, 1961. Butterworth's.

[2] H. W. Knobloch and F. Kappel. *Gewöhnliche Differentialgleichungen*. Teubner, Stuttgart, 1974.

[3] Ferdinand Küsters. On duality of switched DAEs. Master's thesis, TU Kaiserslautern, 2015.

[4] Ferdinand Küsters, Markus G.-M. Ruppert, and Stephan Trenn. Controllability of switched differential-algebraic equations. *Syst. Control Lett.*, 78(0):32 – 39, 2015.

[5] Ferdinand Küsters and Stephan Trenn. Duality of switched DAEs. submitted for publication, preprint available from the website of the authors, 2015.

[6] Douglas A. Lawrence. Duality properties of linear impulsive systems. In *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA*, pages 6028–6033, Dec 2010.

[7] Zhengguo Li, Cheong Boon Soh, and Xinhe Xu. Controllability and observability of impulsive hybrid dynamic systems. *IMA J. Math. Control & Information*, 16(4):315–334, 1999.

[8] Mihály Petreczky, Aneel Tanwani, and Stephan Trenn. Observability of switched linear systems. In Mohamed Djemai and Michael Defoort, editors, *Hybrid Dynamical Systems*, volume 457 of *Lecture Notes in Control and Information Sciences*, pages 205–240. Springer-Verlag, 2015.

[9] Zhendong Sun and Shuzhi Sam Ge. *Switched linear systems*. Communications and Control Engineering. Springer-Verlag, London, 2005.

[10] Aneel Tanwani, Hyungbo Shim, and Daniel Liberzon. Observability implies observer design for switched linear systems. In *Proc. ACM Conf. Hybrid Systems: Computation and Control*, pages 3 – 12, 2011.

[11] Aneel Tanwani and Stephan Trenn. On observability of switched differential-algebraic equations. In *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA*, pages 5656–5661, 2010.

[12] Stephan Trenn and Jan C. Willems. Switched behaviors with impulses - a unifying framework. In *Proc. 51st IEEE Conf. Decis. Control, Maui, USA*, pages 3203–3208, December 2012.

[13] Jochen Trumpf. *On the geometry and parametrization of almost invariant subspaces and observer theory*. PhD thesis, Universität Würzburg, Fakultät für Mathematik und Informatik, 2003.

[14] Arjan van der Schaft. Duality for linear systems: External and state space characterization of the adjoint system. In Bernard Bonnard, Bernard Bride, Jean-Paul Gauthier, and Ivan Kupka, editors, *Analysis of Controlled Dynamical Systems*, volume 8 of *Progress in Systems and Control Theory*, pages 393–403. Birkhäuser Boston, 1991.