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Duality of Switched DAEs

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Abstract We present and discuss the definition of the adjoint and dual of a switched differential-algebraic equation (DAE). For a proper duality definition it is necessary to extend the class of switch DAE to allow for additional impact terms. For this switched DAE with impacts we derive controllability / reachability / determinability / observability characterizations for a given switching signal. Based on this characterizations, we prove duality between controllability / reachability and determinability / observability for switched DAEs .

Keywords Duality · Switched Systems · Differential-Algebraic Equations

1 Introduction

We study duality of switched differential-algebraic equations (DAEs) of the form

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \\ y &= C_\sigma x, \end{aligned} \tag{1}$$

for a given switching signal $\sigma : \mathbb{R} \to \mathbb{N}$. It will be necessary to generalize this system class later on to allow for dualization.

Duality is a classical research subject in linear system theory and apart from being of theoretical interest it has applications in optimal control. First introduced by Kalman [11], it was later generalized to other system classes, in particular unswitched DAEs [9, 10], linear differential inclusions [2], switched

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linear ODEs (with switching signal as input) [24], linear (continuously) time-varying DAEs [8], non-switched impulsive systems [17] and hybrid systems (including jumps) with periodic switching signal [18]. A concept closely related to duality is adjointness and we will discuss in detail the connection between both in the context of switched DAE. Adjointness for homogeneous (continuously) time-varying DAEs is still an active research field, see the recent article [19] and the references therein. Although switched DAEs with given switching signal are also time-varying DAEs, the discontinuities due to switching pose significant challenges in the theoretical analysis; nevertheless, our approach is inspired by the results on duality / adjointness of linear time-varying DAEs and it may even be possible to unify these results, but this is outside the scope of our paper.

For constant-coefficient DAEs, the recent survey [6] gives duality results for different notions of controllability and observability. Cobb [9] and Frankowska [10] use notions that do not coincide with ours and whose generalization to switched DAEs does not lead to duality. By using more appropriate notations for observability and controllability, our result differs from [9] and [10] even in the unswitched case. In addition to the non-canonical controllability / observability definitions for DAEs, there is also some choice in how to treat the switching signals in the definitions of controllability / observability, see e.g. the survey [21] on different observability concepts for switched systems. Some duality result for switched DAEs is claimed in [20]; however, therein a rigorous solution theory is missing and, furthermore, the observability definition requires to choose the switching signal depending on the initial value.

A priori it is not clear which generalizations of controllability and observability are most natural for switched DAEs; however, with our chosen notions (see Definition 17 and 19) we are able to show the very satisfying duality statement (Theorem 30)

$$\begin{array}{ccc} \text{Observability} & \overset{\text{dual}}{\leftrightsquigarrow} & \text{Reachability} \\ \Downarrow & & \Downarrow \\ \text{Determinability} & \overset{\text{dual}}{\leftrightsquigarrow} & \text{Controllability}. \end{array} \tag{2}$$

There are certain pitfalls towards obtaining this duality result. A first challenge is to find an appropriate definition of the dual system (see Section 4). A "correct" definition of duality should have the following properties:

- D1 The dual of the dual is the original system; in particular, the dual is an element of the same system class (otherwise the original duality definition cannot be applied to the dual system).
- D2 The classical duality between (some form of) controllability and (some form of) observability holds.
- D3 There is some formal justification of duality in terms of the solution trajectories.

The above mentioned works do not elaborate on the derivation of the dual system but merely state the dual system and show that D1 and D2 hold.

Following this approach, a naive definition (motivated by the definition of the dual of a non-switched DAE and indeed proposed in [20]) of the dual system of (1) is

$$\begin{aligned} E_\sigma^\top \dot{z} &= A_\sigma^\top z + C_\sigma^\top u_d, \\ y_d &= B_\sigma^\top z. \end{aligned} \tag{3}$$

Taking into account the time-varying nature of switched DAEs, another possible definition for the dual of (1) would additionally reverse the time, resulting in

$$\begin{aligned} E_{\overline{\sigma}}^\top \dot{z} &= A_{\overline{\sigma}}^\top z + C_{\overline{\sigma}}^\top u_d, \\ y_d &= B_{\overline{\sigma}}^\top z \end{aligned} \tag{4}$$

with $\overline{\sigma}(t) := \sigma(T-t)$ where $[0,T]$, $T > 0$, is the compact time interval of interest. The following example shows that both approaches do not yield a satisfying duality definition as property D2 is not satisfied.

Example 1 Consider the switched DAE (1) with switching signal

$$\sigma(t) = \begin{cases} 0, & t \in (-\infty, 1), \\ 1, & t \in [1, 2), \\ 2, & t \in [2, \infty), \end{cases}$$

and modes given by

$$\begin{aligned} (E_0, A_0, B_0, C_0) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right), \\ (E_1, A_1, B_1, C_1) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right), \\ (E_2, A_2, B_2, C_2) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right). \end{aligned}$$

In all three modes it holds $\dot{x}_1 = 0$, hence the solution satisfies $x_1(t) = x_1^0$ for all $t \in \mathbb{R}$ and some $x_1^0 \in \mathbb{R}$. The variable x_2 is zero on $(-\infty, 1)$, on $[1, 2)$ it holds that $x_2 = -x_1$ and afterwards x_2 remains constant because $\dot{x}_2 = 0$. Altogether, we have

$$x(t) = \begin{cases} \begin{pmatrix} x_0^1 \\ 0 \end{pmatrix}, & t \in (-\infty, 1), \\ \begin{pmatrix} x_0^1 \\ -x_0^1 \end{pmatrix}, & t \in [1, \infty). \end{cases}$$

The output is then

$$y(t) = \begin{cases} 0, & t \in (-\infty, 2), \\ -x_0^1, & t \in [2, \infty). \end{cases}$$

Hence, we can uniquely deduce from the output (e.g. observed on the interval $[0,3]$) the value x_0^1 and therefore we can reconstruct the whole state trajectory, i.e. this particular switched DAE is observable.

The naive dual (3) of the switched DAEs satisfies

$$\begin{array}{lll} \text{on } (-\infty,1) & \text{on } [1,2) & \text{on } [2,\infty) \\ \dot z_1 = 0, & \dot z_1 = z_2, & \dot z_1 = 0, \\ 0 = z_2, & 0 = z_2, & \dot z_2 = u_d. \end{array}$$

Clearly, $\dot z_1 = 0$ holds for all times, i.e. $z_1(t) = z_1^0$ for all $t \in \mathbb{R}$ and some $z_0^1 \in \mathbb{R}$. Hence this switched DAE is neither controllable nor reachable. Reversing the switching signal with respect to the interval $[0,3]$ results in

$$\begin{array}{lll} \text{on } (-\infty,1) & \text{on } [1,2) & \text{on } [2,\infty) \\ \dot z_1 = 0, & \dot z_1 = z_2, & \dot z_1 = 0, \\ \dot z_2 = u_d, & 0 = z_2, & 0 = z_2, . \end{array}$$

Again $z_1(t) = z_1^0$ for all $t \in \mathbb{R}$ and some $z_0^1 \in \mathbb{R}$ and also this switched DAE is neither controllable nor reachable. ◁

It will turn out that (in contrast to non-switched system) it is helpful to clearly distinguish between the notion of an *adjoint* and a *dual* system. In view of the desired duality property D3, the definition of the adjoint system is derived based on an adjointness condition in terms of the solution trajectories (Definition 9). This results in the following adjoint system of (1):

$$\tfrac{d}{dt}\left(p^\top E_\sigma\right) = -p^\top A_\sigma + u_a^\top C_\sigma, \quad y_a = p^\top B_\sigma.$$

The resulting system is not causal, i.e. the solution p is not uniquely defined on $[t,\infty)$ by its past $p_{(-\infty,t)}$ and the input u. Reversing the time with respect to an interval $[0,T]$, $T > 0$, we arrive (also paying special attention to the distributional multiplications involved) at a causal system:

$$\tfrac{d}{dt}\left(E_{\overline{\sigma}}^\top z\right) = A_{\overline{\sigma}}^\top z + C_{\overline{\sigma}}^\top u_d, \quad y_d = B_{\overline{\sigma}}^\top z$$

which is then called the T-dual system (Definition 17). This dual system is not a DAE of the form (1), because using the product rule the term $(E_{\overline{\sigma}}^\top)' z$ occurs. Since $E_{\overline{\sigma}}$ has jumps, its derivative contains Dirac impulses. Fortunately, this occurrence of Dirac-impulses in the coefficient matrices is covered by the distributional solution framework in [29] and in view of [31] we call the enlarged system class switched DAEs *with impacts*, given by

$$E_\sigma \dot x = A_\sigma x + B_\sigma u + G[\cdot]x, \quad y = C_\sigma x,$$

where $G[\cdot]$ is a sum of Dirac impulses; details are discussed in Section 3. For this extended system class the above derivation of adjointness and duality have to be repeated (see Section 4) and, furthermore, controllability and observability notions have to be generalized to switched DAEs with impacts (see Section 5). Finally, we are able to precisely state and prove our duality result (2) in Section 6.

2 Mathematical preliminaries

2.1 Regular matrix pairs

We first recall properties of the (unswitched) DAE

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned} \tag{5}$$

with matrices $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q}, C \in \mathbb{R}^{r\times n}$ and classical (smooth) solutions (x, u, y). For existence and uniqueness of solutions, the following notion of regularity of the matrix pair (E, A) is crucial:

Definition 2 (Regularity) Let $E, A \in \mathbb{R}^{n\times n}$. The matrix pair (E, A) is called *regular* iff $\det(sE - A) \in \mathbb{R}[s]$ is not the zero polynomial. The DAE (5) is called regular iff the corresponding matrix pair (E, A) is regular. ◁

Lemma 3 (Regularity characterizations, [30, Thm. 6.3.2]) *For a DAE* (5) *the following is equivalent:*

1. *The matrix pair (E, A) is regular.*
2. *There exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ transforming (E, A) into quasi-Weierstrass form (QWF), i.e.*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{6}$$

   with $N \in \mathbb{R}^{n_N\times n_N}$ nilpotent, $J \in \mathbb{R}^{n_J\times n_J}$, $n_N + n_J = n$, and I an identity matrix of appropriate size.
3. *For all smooth $u : \mathbb{R} \to \mathbb{R}^q$ there exists a solution x of* (5) *and x is uniquely determined by $x(t_0)$ for any $t_0 \in \mathbb{R}$.*
4. *The only solution of* (5) *with $u = 0$ and $x(0) = 0$ is $x = 0$.*

In the following, we will assume the DAE (5) to be regular. To obtain the transformation matrices S, T, the *Wong sequences* ([28]) are useful:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), & i &= 0, 1, 2, \ldots, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i), & i &= 0, 1, 2, \ldots. \end{aligned}$$

These sequences converge after finitely many steps. Their limits are denoted by

$$\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i \quad \text{and} \quad \mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}_i.$$

By choosing full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$ we can define $T := [V, W]$, $S = [EV, AW]^{-1}$. These matrices transform (E, A) to

QWF. They can also be used to construct the following "projectors":

$$\text{the } \textit{consistency projector} \qquad \varPi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

$$\text{the } \textit{differential projector} \qquad \varPi^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

$$\text{the } \textit{impulsive projector} \qquad \varPi^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S,$$

where the block structure of all three objects corresponds to the quasi-Weierstrass form. Only the consistency projector is a projector as the other two are not idempotent. The definitions above do not depend on the specific choice of S and T (see [28, Section 4.2.2] for $\varPi$; the proof for $\varPi^{\text{diff}}$ and $\varPi^{\text{imp}}$ is analogous). Using these projectors, the following matrices can be defined:

$$\begin{aligned} E^{\text{diff}} &:= \varPi^{\text{diff}} E, & A^{\text{diff}} &:= \varPi^{\text{diff}} A, & B^{\text{diff}} &:= \varPi^{\text{diff}} B, \\ E^{\text{imp}} &:= \varPi^{\text{imp}} E, & A^{\text{imp}} &:= \varPi^{\text{imp}} A, & B^{\text{imp}} &:= \varPi^{\text{imp}} B, \end{aligned}$$

and

$$C^{\text{diff}} := C\varPi, \qquad C^{\text{imp}} := C(I - \varPi).$$

We call these matrices the *differential* and *impulsive part* of E, A, B, C, respectively.

As *consistency space* we denote the space of all consistent states of the homogeneous system:

$$\{\ x_0 \in \mathbb{R}^n \ \mid \exists \text{ smooth solution } x \text{ of } E\dot{x} = Ax \text{ with } x(0) = x_0\ \}. \tag{7}$$

The consistency space of the inhomogeneous system is called *augmented consistency space*:

$$\{\ x_0 \in \mathbb{R}^n \ \mid \exists \text{ smooth } (x, u) \text{ solving (5) with } x(0) = x_0\ \}. \tag{8}$$

For the DAE (5) the consistency space (7) is given by $\mathcal{V}^* = \operatorname{im} \varPi$ and the augmented consistency space (8) is (see [5, Corollary 4.5])

$$\overline{\mathcal{V}^*} := \mathcal{V}^* \oplus \langle E^{\text{imp}}, B^{\text{imp}} \rangle. \tag{9}$$

All solutions of (5) have the form

$$x(t) = \mathrm{e}^{A^{\text{diff}} t} \varPi c + \int_0^t \mathrm{e}^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s) \mathrm{d}s - \sum_{i=0}^{n-1} \left(E^{\text{imp}}\right)^i B^{\text{imp}} u^{(i)}(t) \tag{10}$$

for some $c \in \mathbb{R}^n$ ([30, Theorem 6.4.4]).

2.2 Distributional solutions

A switched DAE (1) usually does not have a classical solution as the consistency spaces of different modes do not need to coincide. Thus a switch might result in an inconsistent initial condition, which - even for homogeneous systems - may produce jumps or even Dirac impulses in the state variables [30]. A distributional solution framework is therefore necessary. Recall that the space of distributions (or generalized functions) is defined as (following Schwartz [23]):

$$\mathbb{D} := \{ \ D : \mathcal{C}_0^\infty \to \mathbb{R} \ \mid D \text{ is linear and continuous } \},$$

where $\mathcal{C}_0^\infty$ is the space of smooth functions with compact support (so called test functions) and is equipped with a suitable locally convex topology. Every distribution $D \in \mathbb{D}$ has a derivative in $\mathbb{D}$ given by $D'(\varphi) := -D(\varphi')$, $\varphi \in \mathcal{C}_0^\infty$. However, it turns out that the whole space of distributions is not an appropriate solution space for (1) because it is "too large" [28]. To overcome this problem we follow [28] and introduce the space of piecewise-smooth distributions. The latter can be seen as the "differential closure" of the space of *piecewise-smooth functions* defined as follows:

$$\mathcal{C}_{\text{pw}}^\infty := \left\{ \ \alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[t_i,t_{i+1})} \ \middle| \ \begin{array}{l} \{ \ t_i \ \mid i \in \mathbb{Z} \ \} \text{ locally finite with} \\ t_i < t_{i+1}, \ \alpha_i \in \mathcal{C}^\infty, \ i \in \mathbb{Z} \end{array} \right\},$$

where $\mathcal{C}^\infty$ denotes the space of smooth functions $\alpha : \mathbb{R} \to \mathbb{R}$ and α_I denotes the restriction of the function α to the interval $I \subseteq \mathbb{R}$ given by $\alpha_I(t) = \alpha(t)$ for $t \in I$ and $\alpha_I(t) = 0$ otherwise. The precise definition of piecewise-smooth distributions is then:

Definition 4 A distribution $a \in \mathbb{D}$ is called *piecewise-smooth* iff

$$a = \alpha_{\mathbb{D}} + a[\cdot] := \alpha_{\mathbb{D}} + \sum_{t\in\Gamma} a_t,$$

where

- $\alpha_{\mathbb{D}}$ is the regular distribution induced by a piecewise-smooth function $\alpha \in \mathcal{C}_{\text{pw}}^\infty$, i.e. $\alpha_{\mathbb{D}} : \mathcal{C}_0^\infty \ni \varphi \mapsto \int_{\mathbb{R}} \alpha(t)\varphi(t)\mathrm{d}t$,
- $a_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \dots\}$ where $\delta_t : \mathcal{C}_0^\infty \ni \varphi \mapsto \varphi(t)$ is the Dirac impulse with support $\{t\}$,
- $\Gamma \subseteq \mathbb{R}$ is locally finite.

We denote $a(t^-) = \lim_{s\nearrow t} \alpha(s)$, $a(t^+) = \lim_{s\searrow t} \alpha(s)$ and $a[t] = a_t$ if $t \in \Gamma$ and $a[t] = 0$ otherwise. These "evaluations" of a are well-defined [28]. The space of piecewise-smooth distributions is denoted by $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$. ◁

As mentioned above $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ is a subspace of $\mathbb{D}$ which is closed under differentiation and, additionally, for which restrictions to intervals are well defined [28]. Furthermore, there exist exactly two (noncommutative) multiplications $*$ on $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ that satisfy

1. $\forall \alpha, \beta \in \mathcal{C}^\infty_{\text{pw}} : \alpha_\mathbb{D} * \beta_\mathbb{D} = (\alpha\beta)_\mathbb{D}$ (generalization of multiplication on $\mathcal{C}^\infty_{\text{pw}}$),
2. $\forall a, b, c \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} : (a * b) * c = a * (b * c)$ (associativity),
3. $\forall a, b \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} : (a * b)' = a' * b + a * b'$ (differentiation rule of multiplication),
4. $\forall t \in \mathbb{R}\ \forall \varphi \in \mathcal{C}^\infty_0 : \left(\mathbb{1}_{[0,\infty)_\mathbb{D}} * \delta_0\right)(\varphi) = \left(\mathbb{1}_{[t,\infty)_\mathbb{D}} * \delta_t\right)\left(\varphi(\cdot - t)\right)$ (condition for shift-invariance),

see [28, Section 2.4.1]. Here, $\mathbb{1}_{[t,\infty)}$ denotes the characteristic function of the interval $[t, \infty)$, i.e. $\mathbb{1}_{[t,\infty)}(\tau) = 0$ if $\tau < t$ and $\mathbb{1}_{[t,\infty)}(\tau) = 1$ otherwise. We denote these two multiplications by $*_c, *_{ac}$. They are uniquely characterized by

$$\begin{aligned} \mathbb{1}_{[0,\infty)_\mathbb{D}} *_c \delta_0 &= \delta_0, \\ \mathbb{1}_{[0,\infty)_\mathbb{D}} *_{ac} \delta_0 &= 0, \end{aligned}$$

and are called *causal and anticausal Fuchssteiner multiplication*, respectively. If not stated otherwise, the causal Fuchssteiner multiplication will be used in the following. We will shortly write ab instead of $a *_c b$ and αa instead of $\alpha_\mathbb{D} *_c a$.

The solution formula (10) for DAEs still holds true when allowing $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$-solutions if one introduces the notion of antiderivative for piecewise-smooth distributions [30, Remark 6.4.5 (3)].

3 Switched DAEs with impacts

Switched DAEs are now considered within the space of piecewise-smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$. This makes it necessary to slightly restrict the set of switching signals:

Definition 5 (Switching Signal) $\sigma : \mathbb{R} \to \mathbb{N}$, $t \mapsto \sigma(t)$ is called a *suitable switching signal* iff it is right-continuous, piecewise constant with locally only finitely many discontinuities (*jumps*), and constant on $(-\infty, 0)$. Without restriction (e.g. by appropriate relabeling of the matrices) we can assume that

$$\begin{aligned} \sigma_{(-\infty,t_1)} &= 0, \\ \sigma_{[t_i,t_{i+1})} &= i \quad \text{for } i = 1, 2, \ldots, \end{aligned} \tag{11}$$

where $0 \leq t_1 < t_2 < \ldots$ are the *switching times* of σ with *dwell times* $\tau_i := t_{i+1} - t_i$ (with the convention $t_0 := 0$ and $t_{m+1} := T$ when the discontinuities are only allowed in the open interval $(0, T)$ for some $T > 0$). Note that this notation does not exclude an artificial introduction of switching times because $(E_i, A_i, B_i, C_i) = (E_j, A_j, B_j, C_j)$ for $i \neq j$ is allowed. Finally, for some $r \geq 0$, we define the restriction $\sigma_{>r}$ of a switching signal σ by

$$\sigma_{>r}(t) = \begin{cases} \sigma(t), & t > r, \\ \sigma(r^+), & t \leq r. \end{cases} \tag{12}$$

In particular, the restriction $\sigma_{>r}$ does not have a jump at r. ◁

For a switched DAE (1) the corresponding solution *behavior* is given by

$$\mathcal{B}_\sigma = \left\{ (u,x,y) \;\middle|\; (u,x,y) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{q+n+r} \text{ solves (1)} \right\}.$$

Theorem 6 ([30, Theorem 6.5.1 and Corollary 6.5.2]) *The switched DAE (1) with suitable switching signal σ has a solution x for any input $u \in \mathbb{D}^q_{\mathrm{pw}\mathcal{C}^\infty}$, which is uniquely determined by $x(0^-) \in \overline{\mathcal{V}^*_{\sigma(0^-)}}$. For any consistent initial state $x_0 \in \overline{\mathcal{V}^*_{\sigma(0^-)}}$ there exists $(u,x,y) \in \mathcal{B}_\sigma$ with $x(0^-) = x_0$. If $u_{[t_i,t_i+\varepsilon)} = 0$ for some $\varepsilon > 0$, it holds*

$$\begin{aligned} x(t_i^+) &= \Pi_i x(t_i^-), \\ x[t_i] &= -\sum_{j=0}^{n-1} \left(E_i^{\mathrm{imp}}\right)^{j+1} (I - \Pi_i)\, x(t_i^-) \delta_{t_i}^{(j)}. \end{aligned}$$

As motivated in the introduction, switched DAEs of the form (1) are not general enough to define a dual within the same system class. Therefore we introduce the following larger system class:

Definition 7 A *switched DAE with impacts* is a system of the form

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u + \sum_{i\geq 1} G_{t_i} \delta_{t_i} x, \\ y &= C_\sigma x \end{aligned} \tag{13}$$

where $E_i, A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times q}, C_i \in \mathbb{R}^{r\times n}$ for all $i \in \mathbb{N}$ and some $n,q,r \in \mathbb{N}$; $\sigma : \mathbb{R} \to \mathbb{N}$ is a suitable switching signal according to Definition 5 with switching times t_i, $i \in \mathbb{N}$ and $G_{t_i} \in \mathbb{R}^{n\times n}$. Furthermore, we assume that (E_i, A_i) is regular for each $i \in \mathbb{N}$ and that x,u,y are vectors of piecewise-smooth distributions, i.e. $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^q_{\mathrm{pw}\mathcal{C}^\infty}$ and $y \in \mathbb{D}^r_{\mathrm{pw}\mathcal{C}^\infty}$. $\triangleleft$

The behavior for a switched DAE with impact is defined in the same way as for switched DAEs. Note that for a restricted switching signal $\sigma_{>r}$ also the impacts G_{t_i} are restricted to the interval (r,∞).

With

$$G = G[\cdot] := \sum_{i\geq 1} G_{t_i} \delta_{t_i} \in \mathbb{D}^{n\times n}_{\mathrm{pw}\mathcal{C}^\infty},$$

we can rewrite (13) as a distributional DAE [29]

$$E_\sigma \dot{x} = \mathcal{A} x + B_\sigma u,$$

where $\mathcal{A} := A_{\sigma\mathbb{D}} + G[\cdot] \in \mathbb{D}^{n\times n}_{\mathrm{pw}\mathcal{C}^\infty}$. Hence by [29, Thm. 21] existence and uniqueness of solutions, as stated in Theorem 6, follows for switched DAEs with impacts, independently of the choice of G. For (13) a solution formula similar to the one given in Theorem 6 holds:

$$x(t_i^+) = \Pi_i^{\text{diff}} (E_i + G_{t_i}) \, x(t_i^-), \tag{14a}$$

$$x[t_i] = -\sum_{j=0}^{n-1} \left(E_i^{\text{imp}}\right)^j \Pi_i^{\text{imp}} (E_i + G_{t_i}) \, x(t_i^-) \delta_{t_i}^{(j)}. \tag{14b}$$

This can be seen as follows: We can restrict our attention to the initial trajectory problem with $u = 0$ and $G[\cdot] = G_0\delta_0$

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)}, \\ (E\dot{x})_{[0,\infty)} &= (Ax + G_0\delta_0 x)_{[0,\infty)} \, . \end{aligned} \tag{15}$$

Note that it holds $G_0\delta_0 x = G_0\delta_0 x^0(0^-)$. By linearity one can write $x = \tilde{x} + \hat{x}$, where $\tilde{x}$ solves (15) without the term $G_0\delta_0 x$ and $\hat{x}$ solves (15) with $\hat{x}_{(-\infty,0)} = 0$. The solution formula for $\hat{x}$ follows by [30, Theorem 6.4.4 and Remark 6.4.5 (3)]:

$$\hat{x}(0^+) = \Pi^{\text{diff}} G_0 x^0(0^-), \qquad \hat{x}[0] = -\sum_{i=0}^{n-1} \left(E^{\text{imp}}\right)^i \Pi^{\text{imp}} G_0 x^0(0^-) \delta_0^{(i)}.$$

Rewriting the solution of $\tilde{x}$ as given in Theorem 6 gives

$$\tilde{x}(0^+) = \Pi^{\text{diff}} E x^0(0^-), \qquad \tilde{x}[0] = -\sum_{i=0}^{n-1} \left(E^{\text{imp}}\right)^i \Pi^{\text{imp}} E x^0(0^-) \delta_0^{(i)}.$$

Remark 8 The expression *impact* is related to the corresponding switched ODEs. In [31] it was shown that the ODE with jumps

$$\dot{x} = Ax + Bu, \; x(t_i^+) = J_i x(t_i^-) \; \text{ for } i \in \mathbb{Z} \tag{16}$$

and the distributional ODE

$$\dot{x} = Ax + Bu + \left(\sum_{i \in \mathbb{Z}} (J_i - I) \, \delta_{t_i} \right) x \tag{17}$$

have the same solutions. Solutions of (16) are assumed to be piecewise-smooth functions. [31] showed that (17) has the same solutions (in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$) as (16) if we assume the input to be piecewise smooth. Duality for switched ODEs with jumps has been dealt with in [14]. ◁

The *feasible space* at time $t^\pm$ is the set of all values the system can obtain at time $t^\pm$ [15, Remark 2.10], i.e.

$$\left\{ \; x_t \in \mathbb{R}^n \; \middle| \; \exists (u,x,y) \in \mathcal{B}_\sigma \text{ with } x(t^\pm) = x_t \; \right\}.$$

A consequence of Theorem 6 and the subsequent considerations is that for $t = 0^-$ the feasible space of (1) and (13) is given by $\overline{\mathcal{V}^*_{\sigma(0^-)}}$. This does not hold true in general for $t > 0$ for either of the systems.

4 Adjoint and dual system

The dual of a non-switched DAE is given in [9, 10] as

$$E^\top \dot{x} = -A^\top x + C^\top u, \quad y = B^\top x$$

or, in the (continuously) time-varying case [8, 19], as

$$\tfrac{d}{dt}\left(E^\top x\right) = -A^\top x + C^\top u, \quad y = B^\top x.$$

The references lack a motivation that would suffice to generalize the dual to switched systems. As we have seen in the introduction, a naive dualization does not work. In contrast to the unswitched case, time-inversion is crucial. To point this out, we distinct between the adjoint and the dual system in the subsequent derivation.

4.1 Adjointness

To derive an adjointness condition we will first recall adjointntess for linear systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned} \tag{18}$$

In [22] the following adjointness condition for (18) was derived:

$$\tfrac{d}{dt}\left(p^\top x\right) - y_a^\top u + u_a^\top y = 0 \;\; \forall\, (u, x, y) \in \mathcal{B}. \tag{19}$$

The set of all (u_a, p, y_a) solving (19) is precisely the behavior of the system

$$\begin{aligned} \dot{p} &= -A^\top p - C^\top u_a, \\ y_a &= B^\top p. \end{aligned} \tag{20}$$

Thus we call (20) the *adjoint system* of (18).

Adjointness for homogeneous time-varying DAEs (with continuous coefficients) was considered in [1, 16]. The condition given there was (adapted to our notion)

$$p^\top E_\sigma x = const.$$

Together with (19), this leads us to the following adjointness condition for switched DAEs with impacts (13):

$$\boxed{\tfrac{d}{dt}\left(p^\top E_\sigma x\right) - y_a^\top u + u_a^\top y = 0 \;\; \forall\, (u, x, y) \in \mathcal{B}_\sigma.} \tag{21}$$

We call any linear subspace $\mathcal{B} \subseteq \mathbb{D}^{r+n+q}$ a *behavioral adjoint* of (13) if for any $(u_a, p, y_a) \in \mathcal{B}$ the adjointness condition (21) holds. Invoking the differentiation rule of the Fuchssteiner multiplication and inserting (13) we obtain the equivalent adjointness condition

$$\begin{aligned} \left(\tfrac{d}{dt}\left(p^\top E_\sigma\right) + p^\top A_\sigma + u_a^\top C_\sigma + p^\top G[\cdot]\right) x + \left(p^\top B_\sigma - y_a^\top\right) u = 0 \\ \forall\, (u, x, y) \in \mathcal{B}_\sigma. \end{aligned} \tag{22}$$

This motivated the following definition of *the* adjoint system of (13):

Definition 9 For the switched DAE with impacts (13) and suitable switching signal σ the *adjoint system* is

$$\boxed{\begin{aligned} \tfrac{d}{dt}\left(p^\top E_\sigma\right) &= -p^\top A_\sigma - u_a^\top C_\sigma - p^\top G[\cdot],\\ y_a^\top &= p^\top B_\sigma. \end{aligned}} \tag{23}$$

The corresponding behavior is denoted by $\mathcal{B}_\sigma^{\text{adj}}$. ◁

Obviously, $\mathcal{B}_\sigma^{\text{adj}}$ is a behavioral adjoint of (13). But in contrast to ODEs the condition (22) does not uniquely yield the adjoint system (23) and this is already a problem in the unswitched case:

Example 10 Consider the DAE $0 = x + 0\,u$, $y = 0$ which has only the zero solution, i.e. $\mathcal{B}_\sigma = \left\{ (u,0,0) \;\middle|\; u \in \mathbb{D}^q_{\text{pw}\mathcal{C}^\infty} \right\}$. Hence condition (21) reduces to $y_a = 0$ and there are no constraints on u_a and p, i.e. the largest behavior $\mathcal{B}$ which is a behavioral adjoint of the above DAE is given by

$$\left\{ (u_a,p,0) \;\middle|\; u_a \in \mathbb{D}^r_{\text{pw}\mathcal{C}^\infty}, p \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty} \right\}.$$

This is not the solution behavior of any regular DAE. ◁

Another problem of the adjoint system (23) is that it is not in the form of (13) for two reasons:

1) The coefficient matrix E_σ is inside of the derivative operator.
2) The matrix-vector product is reversed in order.

The first problem can be resolved easily as $E_\sigma^\top$ is piecewise constant and $p^\top \frac{d}{dt} E_\sigma^\top$ fits to the impact term in (23) (which has been introduced precisely for this reason). The second problem is more severe because for the Fuchssteiner multiplication it is not true in general that $(AB)^\top = B^\top A^\top$ for A, B matrices over $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$. In fact, the reversed order leads to an acausal behavior as the following two examples illustrate.

Example 11 Consider

$$\tfrac{d}{dt}\left(p\mathbb{1}_{(-\infty,0)}\right) = p\mathbb{1}_{[0,\infty)}.$$

or, equivalently, $\dot p\mathbb{1}_{(-\infty,0)} = p\mathbb{1}_{[0,\infty)} + p\delta_0$. Invoking the calculus of piecewise-smooth distribution we can conclude

$$\begin{aligned} \dot p &= 0 && \text{on } (-\infty,0),\\ 0 &= p && \text{on } (0,\infty),\\ \dot p[0] &= p\delta_0. \end{aligned}$$

Since $p\delta_0 = p(0^+)\delta_0 = 0$ it follows that $\dot p[0] = 0$, which implies that p cannot have a jump at $t = 0$ and $p = 0$ is the only solution. Hence the past $(p_{(-\infty,0)})$ is restricted by the future $(p_{[0,\infty)} = 0)$, i.e. the system is not causal. ◁

Example 12 Consider now

$$\tfrac{d}{dt}\left(p\mathbb{1}_{[0,\infty)}\right) = p\mathbb{1}_{(-\infty,0)},$$

or, equivalently, $\dot{p}\mathbb{1}_{[0,\infty)} = p\mathbb{1}_{(-\infty,0)} - p\delta_0$. This gives

$$\begin{aligned} 0 &= p && \text{on } (-\infty,0),\\ \dot{p} &= 0 && \text{on } (0,\infty),\\ p[0] &= p\delta_0. \end{aligned}$$

Hence $p[0] = p(0^+)\delta_0$ and there is no constraint on the jump at $t = 0$, i.e. from a unique past $p_{(-\infty,0)}$ the future $p_{[0,\infty)}$ is not determined uniquely. ◁

The system in Example 11 does not have a solution for every initial trajectory $p_{(-\infty,0)}$, while the system in Example 12 has multiple solutions for the same initial trajectory $p_{(-\infty,0)}$. Therefore, we have to add a third problem concerning the adjoint system (23) to our list:

3) In general the adjoint system is not causal, i.e. a solution p is not uniquely defined on $[t,\infty)$ by its past $p_{(-\infty,t)}$ and the input u.

We will resolve the third problem by reverting time in the next section. As a by-product this will also resolve problem 2) mentioned above.

Remark 13 Another approach to derive the adjoint of the ODE (18) was given in [12, 32]. There the system is identified with three mappings (input-to-state, initial-state-to-final-state and state-to-output) and the adjoint is then defined by the adjoint operators of these mappings. While the approach works well for switched ODEs with jumps [14], it seems to fail for the more general class of switched DAEs [13]. ◁

The dual system given in [6, 8, 9, 10] fits to our notion of adjoint system (23) - despite the order of multiplication and possibly some signs, on which the references also do not agree. A main difference to these references is that the adjoint system which we have derived so far is not causal. This is a consequence of the considered solution space. For the noncausal system (23) it does not make sense to consider system properties such as controllability and observability. Therefore, we introduce a time-inversion of the adjoint system to arrive at a causal system of the form (13), which we then call the dual system.

4.2 Time-inversion

For linear systems (18) the adjoint can be considered as a system going backwards in time [12]. We therefore define a time-inversion for distributions:

Definition 14 Let $D \in \mathbb{D}$ a distribution and $T \in \mathbb{R}$. The *time-inversion of D at time T* is defined by

$$\mathscr{T}_T\{D\}(\varphi) := D(\varphi(T-\cdot)) \text{ for all } \varphi \in \mathcal{C}_0^\infty.$$

$\mathscr{T}_T$ is a linear operation on the space of distributions. For the differentiation it holds

$$\mathscr{T}_T\{D'\} = -\left(\mathscr{T}_T\{D\}\right)' \text{ for } D \in \mathbb{D}. \tag{24}$$

The space of piecewise-smooth distributions is closed under time-inversion as it holds $\mathscr{T}_T\{\alpha_{\mathbb{D}}\} = \alpha(T-\cdot)_{\mathbb{D}}$ for a regular distribution $\alpha_{\mathbb{D}}$ and $\mathscr{T}_T\left\{\delta_t^{(k)}\right\} = (-1)^k \delta_{T-t}^{(k)}$ for the Dirac impulse and its derivatives. In particular, it holds

$$\mathscr{T}_T\{a\}\left((T-t)^{\pm}\right) = a(t^{\mp}) \text{ for } t \in \mathbb{R}. \tag{25}$$

and $\mathscr{T}_T\{a\}[T-t] = a[t]$ if a does not contain derivatives of Dirac impulses. Applying the time-inversion to a product of piecewise-smooth distributions yields an anticausal multiplication:

Lemma 15 *Let $a, b \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ and $T \in \mathbb{R}$. Then it holds*

$$\mathscr{T}_T\{a *_c b\} = \mathscr{T}_T\{a\} *_{ac} \mathscr{T}_T\{b\} \text{ and } \mathscr{T}_T\{a *_{ac} b\} = \mathscr{T}_T\{a\} *_c \mathscr{T}_T\{b\}.$$

Proof See Appendix B. □

The following lemma will be helpful for rewriting the time-inversion of the adjoint system.

Lemma 16 *Let $A \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n_1 \times n_2}, B \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n_2 \times n_3}$ matrices over $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ for some $n_1, n_2, n_3 \in \mathbb{N}$, then it holds*

$$(A *_c B)^\top = B^\top *_{ac} A^\top.$$

Proof See Appendix B. □

4.3 Definition of the dual system

Applying a time-inversion at time $T > 0$ to the adjoint system (23) and using Lemmas 15 and 16 gives:

$$\begin{aligned}
\tfrac{d}{dt}\left(\mathscr{T}_T\left\{E_\sigma^\top\right\} *_c \mathscr{T}_T\{p\}\right) =& \mathscr{T}_T\left\{A_\sigma^\top\right\} *_c \mathscr{T}_T\{p\} + \mathscr{T}_T\left\{C_\sigma^\top\right\} *_c \mathscr{T}_T\{u_a\} \\
&+ \mathscr{T}_T\left\{G[\cdot]^\top\right\} *_c \mathscr{T}_T\{p\}, \\
\mathscr{T}_T\{y_a\} =& \mathscr{T}_T\left\{B_\sigma^\top\right\} *_c \mathscr{T}_T\{p\}.
\end{aligned}$$

As E_σ is piecewise constant it holds $\mathscr{T}_T\{E_\sigma\} = E_{\overline{\sigma}}$ for the time-inverted switching signal $\overline{\sigma}$ with

$$\boxed{\overline{\sigma}(t) := \sigma(T-t)} \quad \forall t \in \mathbb{R}. \tag{26}$$

The same holds true for A_σ, B_σ, C_σ. Hence the time-inversion at $T > 0$ of (23) is given by

$$\begin{aligned}
\tfrac{d}{dt}\left(E_{\overline{\sigma}}^\top z\right) &= A_{\overline{\sigma}}^\top z + C_{\overline{\sigma}}^\top u_d + G[T-\cdot]^\top z, \\
y_d &= B_{\overline{\sigma}}^\top z
\end{aligned} \tag{27}$$

with $(u_d, z, y_d) = (\mathscr{T}_T\{u_a\}, \mathscr{T}_T\{p\}, \mathscr{T}_T\{y_a\})$. Here we used that G does not contain derivatives of Dirac impulses.

Definition 17 For a switched DAE with impacts (13) with suitable switching signal σ the *dual system with inversion time* $T > 0$ (or, short, T-dual) is defined by (27) with inverted switching signal $\overline{\sigma}$ given by (26). Its behavior is denoted by $\mathcal{B}_\sigma^{T\text{-dual}}$. ◁

Note that the inverted switching signal does not have the form (11). In particular, the system switches to mode $i-1$ at time $s_i := T - t_i$ (cf. forthcoming Figure 1).

The derivation of the dual system shows

$$(u_a, p, y_a) \in \mathcal{B}_\sigma^{\text{adj}} \Leftrightarrow (u_d, z, y_d) = (\mathscr{T}_T\{u_a\}, \mathscr{T}_T\{p\}, \mathscr{T}_T\{y_a\}) \in \mathcal{B}_\sigma^{T\text{-dual}}.$$

Remark 18 The T-dual (27) can be written in the form of a switched DAE with impulses:

$$\boxed{\begin{aligned} E_{\overline{\sigma}}^\top \tfrac{d}{ds} z &= A_{\overline{\sigma}}^\top z + C_{\overline{\sigma}}^\top u_d + \left(G[T-\cdot]^\top - \tfrac{d}{ds} E_{\overline{\sigma}}^\top\right) z, \\ y_d &= B_{\overline{\sigma}}^\top z. \end{aligned}} \tag{28}$$

As $E_{\overline{\sigma}}$ is piecewise constant it holds $\frac{d}{ds} E_{\overline{\sigma}}^\top = \sum_i (E_{i-1} - E_i)^\top \delta_{t_i}$. The reversed switching signal is (by definition) a suitable switching signal only if it is constant in the past. This is only the case if the original switching signal σ is constant on (T, ∞), i.e. all jumps of σ (and hence $\overline{\sigma}$) are contained in the interval $[0, T]$. In that case, we have

$$\boxed{\left(\mathcal{B}_\sigma^{T\text{-dual}}\right)^{T\text{-dual}} = \mathcal{B}_\sigma,}$$

i.e. the desired duality property D1 (see Introduction) holds. Note that in general the dual of a switched DAE (without impacts) is a switched DAE *with impacts*. Hence the enlargement of the considered system class was in fact necessary. ◁

The T-dual (28) depends on the time $T > 0$ chosen for time-inversion. In the sequel we will assume that the switching signal σ is constant on $[T, \infty)$. Since our forthcoming definitions of controllability / reachability / observability / determinability are with respect to a finite interval $[0, T]$ anyway, this is not a restriction of generality. Furthermore, it guarantees that the T-dual is again a switched DAE with impacts in the sense of Definition 7. We assume additionally that 0 and T are not switching times. This will be necessary for the duality result, see Remark 31. In particular, we only need to consider finitely many switching times $t_1, \ldots, t_m$ within the interval $(0, T)$.

5 System theoretic properties

In this section we will recall the system theoretic properties controllability, observability and determinability as given in [15, 21, 25, 26, 27] for switched DAEs and introduce a notion of reachability. We define and characterize these concepts for switched DAEs with impacts. Compared to the references, some changes in the notation were necessary - due to the effect of the impacts on jumps and impulses and also for a more convenient derivation of the duality.

5.1 Definitions

Definition 19 A switched DAE with impacts (13) is called

- *controllable on* $[0,T]$, $T>0$, iff it holds

$$\forall \omega, \hat{\omega} \in \mathcal{B}_\sigma \ \exists \tilde{\omega} \in \mathcal{B}_\sigma : \omega_{(-\infty,0)} = \tilde{\omega}_{(-\infty,0)},\ \hat{\omega}_{(T,\infty)} = \tilde{\omega}_{(T,\infty)};$$

- *reachable on* $[0,T]$, $T>0$, iff it holds

$$\forall \omega \in \mathcal{B}_\sigma,\ \hat{\omega} \in \mathcal{B}_{\sigma(T^+)} \ \exists \tilde{\omega} \in \mathcal{B}_\sigma : \omega_{(-\infty,0)} = \tilde{\omega}_{(-\infty,0)} \text{ and } \hat{\omega}_{(T,\infty)} = \tilde{\omega}_{(T,\infty)};$$

- *observable on* $[0,T]$, $T>0$, iff it holds

$$\forall\,(u,x,y)\,,(\hat{u},\hat{x},\hat{y}) \in \mathcal{B}_\sigma : u=\hat{u} \wedge y_{[0,T]} = \hat{y}_{[0,T]} \quad \Rightarrow \quad x=\hat{x};$$

- *determinable on* $[0,T]$, $T>0$, iff it holds

$$\forall\,(u,x,y)\,,(\hat{u},\hat{x},\hat{y}) \in \mathcal{B}_\sigma :\ u=\hat{u} \wedge y_{[0,T]} = \hat{y}_{[0,T]} \quad \Rightarrow \quad x_{(T,\infty)} = \hat{x}_{(T,\infty)}.$$

The difference of controllability and reachability is that for the latter all feasible solutions $\hat{\omega}$ of the last mode are taken into account while the first only considers feasible solutions of the switched system.

To see the difference between observability and determinability, note that the solution might contain singular jumps. Hence it might be possible to reconstruct the state after a certain time from input and output, but not the whole state trajectory.

The system theoretic properties can be simplified to zero-controllability, zero-reachability, etc:

Lemma 20 *A switched DAE with impacts* (13) *is*

- *controllable on* $[0,T]$, *iff* $\forall \omega \in \mathcal{B}_\sigma \ \exists \tilde{\omega} \in \mathcal{B}_\sigma$*:*

$$\omega_{(-\infty,0)} = \tilde{\omega}_{(-\infty,0)},\ 0_{(T,\infty)} = \tilde{\omega}_{(T,\infty)}.$$

- *reachable on* $[0,T]$ *iff* $\forall \hat{\omega} \in \mathcal{B}_{\sigma(T^+)} \ \exists \tilde{\omega} \in \mathcal{B}_\sigma$*:*

$$0_{(-\infty,0)} = \tilde{\omega}_{(-\infty,0)},\ \hat{\omega}_{(T,\infty)} = \tilde{\omega}_{(T,\infty)}.$$

- *observable on* $[0,T]$ *iff it holds* $\forall\,(u,x,y) \in \mathcal{B}_\sigma$*:*

$$u=0 \wedge y_{[0,T]} = 0 \ \Rightarrow x=0.$$

- *determinable on* $[0,T]$ *iff it holds* $\forall\,(u,x,y) \in \mathcal{B}_\sigma$*:*

$$u=0 \wedge y_{[0,T]} = 0 \ \Rightarrow x_{(T,\infty)} = 0.$$

Lemma 20 implies

$$\begin{aligned}(13) \text{ is observable } &\Rightarrow (13) \text{ is determinable},\\ (13) \text{ is reachable } &\Rightarrow (13) \text{ is controllable}\end{aligned}$$

The reverse does not hold true as the following example shows.

Example 21 All solutions of the switched DAE

$$\left(\mathbb{1}_{(-\infty,t_1)} + \mathbb{1}_{[t_2,\infty)}\right)\dot{x} = \mathbb{1}_{[t_1,t_2)}x + 0u,\ y = 0x$$

have the form $x = \mathbb{1}_{(-\infty,t_1)}c$ for some $c \in \mathbb{R}$. In particular, x is zero for $t \geq t_1$. Hence the system is (trivially) determinable and controllable on $[0,T]$, $T > t_1$. However, it is not observable as the output y is always zero. It is not reachable on $[0,T]$, $T > t_1$, as $\overline{\mathcal{V}_2^*} \neq \{0\}$ but $x_{[t_2,\infty)} = 0$ for any solution. ◁

For duality one usually considers only controllability and observability. However, they are not dual for switched DAEs as the Example 22 shows. This can be interpreted as a problem with the time-inversion of the dual system, which does not have any effect for unswitched ODE systems.

Example 22 The system $\dot{x} = -\delta_{t_1}x + 0u,\ y = 0x$ has solutions of the form $x = c\mathbb{1}_{(-\infty,t_1)}$ for $c \in \mathbb{R}$. The system is controllable on $[0,T]$ $(T > t_1)$ as each solution x is zero on $[t_1,\infty)$. Its dual $\dot{z} = -\delta_{s_1}z + 0u_d,\ y_d = 0z$ is not observable as the output is zero and there are nonzero solutions $z = c\mathbb{1}_{(-\infty,s_1)}$, $c \in \mathbb{R}$. ◁

We can characterize the system theoretic properties with the following spaces:

Definition 23 Let $0 \leq s < t$ and define

$$\begin{aligned}
\mathcal{C}_\sigma^{[s,t]} &:= \left\{\ x_s \in \mathbb{R}^n \ \middle|\ \exists(u,x,y) \in \mathcal{B}_\sigma : x(s^-) = x_s \ \wedge\ x(t^+) = 0\ \right\},\\
\mathcal{C}_\sigma^{(s,t)} &:= \left\{\ x_s \in \mathbb{R}^n \ \middle|\ \exists(u,x,y) \in \mathcal{B}_\sigma : x(s^+) = x_s \ \wedge\ x(t^-) = 0\ \right\},\\
\mathcal{R}_\sigma^{[s,t]} &:= \left\{\ x_t \in \mathbb{R}^n \ \middle|\ \exists(u,x,y) \in \mathcal{B}_\sigma : x(s^-) = 0 \ \wedge\ x(t^+) = x_t\ \right\},\\
\mathcal{R}_\sigma^{(s,t)} &:= \left\{\ x_t \in \mathbb{R}^n \ \middle|\ \exists(u,x,y) \in \mathcal{B}_\sigma : x(s^+) = 0 \ \wedge\ x(t^-) = x_t\ \right\},\\
\mathcal{UO}_\sigma^{[s,t]} &:= \left\{\ x_s \in \mathbb{R}^n \ \middle|\ \exists(0,x,y) \in \mathcal{B}_\sigma : x(s^-) = x_s \ \wedge\ y_{[s,t]} = 0\ \right\},\\
\mathcal{UO}_\sigma^{(s,t)} &:= \left\{\ x_s \in \mathbb{R}^n \ \middle|\ \exists(0,x,y) \in \mathcal{B}_\sigma : x(s^+) = x_s \ \wedge\ y_{(s,t)} = 0\ \right\},\\
\mathcal{UD}_\sigma^{[s,t]} &:= \left\{\ x_t \in \mathbb{R}^n \ \middle|\ \exists(0,x,y) \in \mathcal{B}_\sigma : x(t^+) = x_t \ \wedge\ y_{[s,t]} = 0\ \right\},\\
\mathcal{UD}_\sigma^{(s,t)} &:= \left\{\ x_t \in \mathbb{R}^n \ \middle|\ \exists(0,x,y) \in \mathcal{B}_\sigma : x(t^+) = x_t \ \wedge\ y_{(s,t)} = 0\ \right\}.
\end{aligned}$$

These spaces are called *controllable, reachable, unobservable* and *undeterminable space*, respectively. ◁

A switched DAE with impacts is controllable on $[0,T]$ iff $\mathcal{C}_\sigma^{[0,T]} = \overline{\mathcal{V}_{\sigma(0^-)}^*}$, reachable on $[0,T]$ iff $\mathcal{R}_\sigma^{[0,T]} = \overline{\mathcal{V}_{\sigma(T^+)}^*}$, observable on $[0,T]$ iff $\mathcal{UO}_\sigma^{[0,T]} = \{0\}$ and determinable on $[0,T]$ iff $\mathcal{UD}_\sigma^{[0,T]} = \{0\}$.

Controllable, reachable, unobservable and undeterminable space were not only defined for the interval $[0,T]$ but also for the open interval $(0,T)$. The first notion fits to the definition of the system theoretic properties while the second helps to interpret the derivation of the system theoretic properties of switched DAEs. The spaces are related as follows:

Lemma 24 *Consider the switched DAE with impacts* (13) *with switching signal σ. Let $\mathcal{A} \in \{\mathcal{C}, \mathcal{R}, \mathcal{UO}, \mathcal{UD}\}$. If s,t are* not switching times *of σ then*

$$\mathcal{A}_\sigma^{[s,t]} = \mathcal{A}_\sigma^{(s,t)}.$$

Proof It is sufficient to consider smooth control functions [15, Remark 2.12] if s and t are not switching times. Hence x is also smooth at s and t. □

Example 25 (Time-inversion) Controllability and observability can be interpreted as properties of the states at time $t=0$, reachability and determinability as properties of the states at time $t=T$. This does however not mean that there is a relation between these properties if the switching signal is inverted. In fact, the system

$$\begin{aligned}(E_0,A_0,B_0,C_0) &= \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1&0\end{bmatrix}\right)\\ \text{and}\ \ (E_1,A_1,B_1,C_1) &= \left(\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0&0\end{bmatrix}\right).\end{aligned}$$

with $\sigma_1 := \mathbb{1}_{[t_1,\infty)}$ is reachable, controllable, not observable and not determinable. The same system with inverted switching signal $\overline{\sigma}_1 = \mathbb{1}_{(-\infty,T-t_1)}$ is not reachable, not controllable, observable and determinable. ◁

For a switched DAE with impacts (13) define for each mode i the Kalman matrices (here $[A/B] := [A^\top, B^\top]^\top$)

$$\begin{aligned}K_i^{\text{diff}} &:= \left[B_i^{\text{diff}}, A_i^{\text{diff}}B_i^{\text{diff}}, \dots, \left(A_i^{\text{diff}}\right)^{n-1} B_i^{\text{diff}}\right],\\ K_i^{\text{imp}} &:= \left[B_i^{\text{imp}}, E_i^{\text{imp}}B_i^{\text{imp}}, \dots, \left(E_i^{\text{imp}}\right)^{n-1} B_i^{\text{imp}}\right],\\ O_i^{\text{diff}} &:= \left[C_i^{\text{diff}}/C_i^{\text{diff}}A_i^{\text{diff}}/\dots/C_i^{\text{diff}}\left(A_i^{\text{diff}}\right)^{n-1}\right],\\ O_i^{\text{imp}} &:= \left[C_i^{\text{imp}}/C_i^{\text{imp}}E_i^{\text{imp}}/\dots/C_i^{\text{imp}}\left(E_i^{\text{imp}}\right)^{n-1}\right],\end{aligned}$$

for the differential and the impulsive part, respectively. The *jump matrix* is defined as $H_i := E_i + G_{t(i)}$, where $t(i)$ is the time mode i is entered (i.e. $t(i) = t_i$ for systems with switching signals of the form (11) and $t(i) = T - t_{i+1}$ for the corresponding reversed switching signal). Finally, define

$$\begin{aligned}\mathcal{C}_i &:= \operatorname{im} K_i^{\text{diff}} \oplus \operatorname{im} K_i^{\text{imp}},\\ \mathcal{U}_i &:= \ker O_i^{\text{diff}} \cap \ker O_i^{\text{imp}},\\ \mathcal{U}_i^{\text{H}} &:= \ker\left(O_i^{\text{diff}}\Pi_i^{\text{diff}}H_i\right) \cap \ker\left(O_i^{\text{imp}}\Pi_i^{\text{imp}}H_i\right).\end{aligned}$$

Defining H_i via $t(i)$ seems unusual, but it is necessary when dealing with the dual system as this system does not have a switching signal of the form (11). Using Remark 18 it holds $\widehat{H}_i = \widehat{G}_{s_{i+1}} + E_i^\top = G_{t_{i+1}}^\top - E_i^\top + E_{i+1}^\top + E_i^\top = H_{i+1}^\top$ (see Figure 1). The circumflex (̂) refers to the dual system.

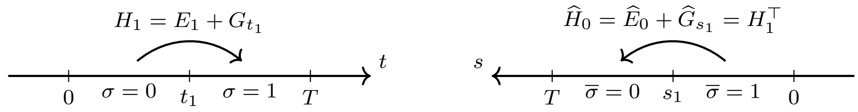

Fig. 1 Jump matrices H_1, $\widehat{H}_0$ for the switched DAE with impacts and its dual.

5.2 System theoretic properties and duality of unswitched DAEs

For a switched DAE with impacts (13) with constant $\sigma = 0$ the notions of controllability and reachability as well as the notions of observability and determinability coincide. It holds (c.f. [15] and [25])

$$\mathcal{C}_\sigma^{[0,T]} = \mathcal{R}_\sigma^{[0,T]} = \operatorname{im} K_0^{\text{diff}} + \operatorname{im} K_0^{\text{imp}},$$
$$\mathcal{UO}_\sigma^{[0,T]} = \mathcal{UD}_\sigma^{[0,T]} = \operatorname{im} \Pi_0 \cap \ker O_0^{\text{diff}},$$

hence controllability/reachability and observability/determinability are characterized by the conditions:

$$\operatorname{im} K_0^{\text{diff}} + \operatorname{im} K_0^{\text{imp}} = \overline{\mathcal{V}_0^*} \text{ and } \ker O_0^{\text{diff}} \cap \operatorname{im} \Pi_0 = \{0\},$$

respectively. These characterizations do not directly appear to be dual. However it is easily seen that only the differential part of the DAE is relevant: An equivalent condition for controllability/reachability is given by

$$\operatorname{im} K_0^{\text{diff}} + \ker \Pi_0 = \mathbb{R}^n.$$

Now duality for the unswitched case is apparent, because for the dual we have $\widehat{K}_0^{\text{diff}} = {O_0^{\text{diff}}}^\top$ and in general we have $(\operatorname{im} M)^\perp = \ker M^\top$ for some matrix M.

The duality result for unswitched systems differs from those given in [9] and [10] as these papers use different definitions for controllability (and in case of [9] also a different definition of observability). The duality in [10] requires the technical assumption $\ker E \cap \ker C \subseteq \ker A$, which is not motivated there. This assumption is equivalent to $\operatorname{im} \hat{E} + \operatorname{im} \hat{B} \supseteq \operatorname{im} \hat{A}$ for the dual system, for which controllability is considered. This condition, however, implies that the augmented consistency space of the dual is the whole space; under this condition the controllability notions of [10] and our notion coincide.

5.3 Characterizations

In this section the system properties controllability, reachability, observability and determinability are characterized by their corresponding spaces. We start with the single switch case.

Lemma 26 (Single switch) *Consider the switched DAE with impacts* (13) *with switching signal* $\sigma_1 := \mathbb{1}_{[t_1,\infty)}$, $t_1 > 0$. *For* $T > t_1$ *the controllable/ reachable/ unobservable/ undeterminable space are given by*

$$\mathcal{C}_{\sigma_1}^{(0,T)} = \left(\mathcal{C}_0 + \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0}\left(\Pi_1^{\mathrm{diff}}H_1\right)^{-1}\mathcal{C}_1\right) \cap \overline{\mathcal{V}_0^*}, \tag{29}$$

$$\mathcal{R}_{\sigma_1}^{(0,T)} = \mathcal{C}_1 + \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1\mathcal{C}_0, \tag{30}$$

$$\mathcal{UO}_{\sigma_1}^{(0,T)} = \operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0}\mathcal{U}_1^{\mathrm{H}}, \tag{31}$$

$$\mathcal{UD}_{\sigma_1}^{(0,T)} = \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1\left(\operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathcal{U}_1^{\mathrm{H}}\right), \tag{32}$$

respectively.

Proof See Appendix C. □

The single switch result is now used to derive a recursive formula for the multi-switch case. For controllability, we have to go backwards in time, i.e. start with the last switch. To make use of the single switch result, we have to consider a switching signal whose switches are restricted to the interval (t_{m-1}, T) as otherwise we would have to care about feasibility of (consistent) states. Using the restricted switching signal $\sigma_{>t_i}$ guarantees that any $x_i \in \overline{\mathcal{V}_i^*}$ is a feasible state at time t_i^+, i.e. there exists $(u,x,y) \in \mathcal{B}_{\sigma_{>t_i}}$ with $x(t_i^+) = x_i$.

For a switched DAE with impacts (13) and switchting signal (11) with switching times $0 < t_1 < \ldots < t_m < T$ this leads to the recursion

$$\begin{aligned}\mathcal{P}_m^m &:= \mathcal{C}_m,\\ \mathcal{P}_i^m &:= \mathcal{C}_i + \mathrm{e}^{-A_i^{\mathrm{diff}}\tau_i}\left(\Pi_{i+1}^{\mathrm{diff}}H_{i+1}\right)^{-1}\mathcal{P}_{i+1}^m \quad \text{for } i = m-1,\ldots,0.\end{aligned} \tag{33}$$

For reachability, the recursion goes forward in time. Hence a restriction of (the switches of) the switching signal is not required:

$$\begin{aligned}\mathcal{Q}_0^0 &:= \mathcal{C}_0,\\ \mathcal{Q}_0^i &:= \mathcal{C}_i + \mathrm{e}^{A_i^{\mathrm{diff}}\tau_i}\Pi_i^{\mathrm{diff}}H_i\mathcal{Q}_0^{i-1} \quad \text{for } i = 1,\ldots,m.\end{aligned} \tag{34}$$

The single switch result on observability motivates the definition of the local unobservable space at t_i^- for (13) and switching signal (11):

$$\widetilde{\mathcal{M}}_i = \operatorname{im}\Pi_{i-1} \cap \ker O_{i-1}^{\mathrm{diff}} \cap \mathcal{U}_i^{\mathrm{H}}. \tag{35}$$

With this, a recursion for the unobservable space can be given as

$$\begin{aligned}\widetilde{\mathcal{M}}_m^m &:= \mathrm{e}^{-A_{m-1}^{\mathrm{diff}}\tau_{m-1}}\widetilde{\mathcal{M}}_m,\\ \widetilde{\mathcal{M}}_i^m &:= \mathrm{e}^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}}\left(\widetilde{\mathcal{M}}_i \cap \left(\Pi_i^{\mathrm{diff}}H_i\right)^{-1}\widetilde{\mathcal{M}}_{i+1}^m\right) \quad \text{for } i = m-1,\ldots,1.\end{aligned} \tag{36}$$

A recursion for the undeterminable space is given by

$$\begin{aligned} \widetilde{\mathcal{N}}_1^1 &:= \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1} \varPi_1^{\mathrm{diff}} H_1 \widetilde{\mathcal{M}}_1, \\ \widetilde{\mathcal{N}}_1^i &:= \mathrm{e}^{A_i^{\mathrm{diff}}\tau_i} \varPi_i^{\mathrm{diff}} H_i \left(\widetilde{\mathcal{M}}_i \cap \widetilde{\mathcal{N}}_1^{i-1}\right) \quad \text{for } i = 2, \dots, m. \end{aligned} \tag{37}$$

Theorem 27 (General switching signal) *Consider the switched DAE with impacts* (13) *with switching signal* (11) *and switching times* $0 < t_1 < \ldots < t_m < T$. *Then it holds*

$$\begin{aligned} \mathcal{C}_{\sigma_{>t_i}}^{(t_i,T)} &= \mathcal{P}_i^m \cap \overline{\mathcal{V}_i^*} && \textit{for } i = 0, \dots, m, \\ \mathcal{R}_{\sigma}^{(0,t_{i+1})} &= \mathcal{Q}_0^i && \textit{for } i = 0, \dots, m, \\ \mathcal{UO}_{\sigma_{>t_{i-1}}}^{(t_{i-1},T)} &= \widetilde{\mathcal{M}}_i^m && \textit{for } i = 1, \dots, m, \\ \mathcal{UD}_{\sigma}^{(0,t_{i+1})} &= \widetilde{\mathcal{N}}_1^i && \textit{for } i = 1, \dots, m. \end{aligned}$$

In particular, the system is

- *controllable on* $[0,T]$ *iff* $\overline{\mathcal{V}_0^*} \subseteq \mathcal{P}_0^m$,
- *reachable on* $[0,T]$ *iff* $\overline{\mathcal{V}_m^*} = \mathcal{Q}_0^m$,
- *observable on* $[0,T]$ *iff* $\{0\} = \widetilde{\mathcal{M}}_1^m$,
- *determinable on* $[0,T]$ *iff* $\{0\} = \widetilde{\mathcal{N}}_1^m$.

Proof See Appendix C. □

5.4 Normalization and explicit formulas

To give rather simple explicit formulas for the system theoretic properties we introduce the notion of a normalized system. This will also be helpful for proving duality.

Definition 28 A DAE (5) is called *normalized* iff $E = E^{\mathrm{diff}} + E^{\mathrm{imp}}$ and $A = A^{\mathrm{diff}} + A^{\mathrm{imp}}$. The switched system (13) is called normalized iff each mode is normalized. ◁

Normalizedness is equivalent to $\left(\varPi^{\mathrm{diff}} + \varPi^{\mathrm{imp}}\right)(\lambda E + A) = \lambda E + A$ for all $\lambda \in \mathbb{R}$. As (E, A) is regular, this gives $\varPi^{\mathrm{diff}} + \varPi^{\mathrm{imp}} = I$ or, in terms of S and T from the QWF, $TS = I$. Hence, for normalized DAEs it holds $\varPi^{\mathrm{diff}} = \varPi$, $\varPi^{\mathrm{imp}} = I - \varPi$, $\mathcal{U}_i^{\mathrm{H}} = H_i^{-1}\mathcal{U}_i$, $O_i^{\mathrm{diff}}\varPi_i^{\mathrm{diff}} = O_i^{\mathrm{diff}}$ and $O_i^{\mathrm{imp}}\varPi_i^{\mathrm{imp}} = O_i^{\mathrm{imp}}$.

It is easily seen that any DAE can be transformed into a normalized DAE via

$$\begin{aligned} (TS)\, E\dot{x} &= (TS)\, Ax + (TS)\, Bu, \\ y &= Cx \end{aligned} \tag{38}$$

where (S, T) transforms the matrix pair (E, A) to QWF. Note that for normalized DAEs the matrices E and A commute (the converse is not true in general) and the premultiplication to obtain (38) has some similarity with the often used trick from [7, Lem. 3.1.1] to obtain commutativity of E and A.

Theorem 29 (Explicit formulas) *A* normalized *system* (13) *with switching signal* (11) *and switching times* $0 < t_1 < \ldots < t_m < T$ *is*

– *controllable on* $[0, T]$ *iff*

$$\Pi_0^{-1}\left(\mathcal{C}_0 + \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0} H_1^{-1}\Pi_1^{-1}\left(\ldots\left(\mathcal{C}_{m-1} + \mathrm{e}^{-A_{m-1}^{\mathrm{diff}}\tau_{m-1}} H_m^{-1}\Pi_m^{-1}\mathcal{C}_m\right)\cdots\right)\right) = \mathbb{R}^n;$$

– *reachable on* $[0, T]$ *iff*

$$\ker \Pi_m + \mathcal{C}_m + \mathrm{e}^{A_m^{\mathrm{diff}}\tau_m} H_m \sum_{j=0}^{m-1}\left(\prod_{k=1}^{m-1-j} \mathrm{e}^{A_{m-k}^{\mathrm{diff}}\tau_{m-k}}\Pi_{m-k}H_{m-k}\right)\mathcal{C}_j = \mathbb{R}^n;$$

– *observable on* $[0, T]$ *iff*

$$\operatorname{im}\Pi_0 \cap \mathcal{U}_0 \cap \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0} H_1^{-1}\left(\bigcap_{j=1}^{m}\left(\prod_{k=1}^{j-1}\mathrm{e}^{-A_k^{\mathrm{diff}}\tau_k}\Pi_k^{-1}H_{k+1}^{-1}\right)\mathcal{U}_j\right) = \{0\};$$

– *determinable on* $[0, T]$ *iff*

$$\Pi_m\left(\mathcal{U}_m \cap \mathrm{e}^{A_m^{\mathrm{diff}}\tau_m} H_m \Pi_{m-1}\left(\ldots\left(\mathcal{U}_1 \cap \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1} H_1 \Pi_0 \mathcal{U}_0\right)\cdots\right)\right) = \{0\}.$$

6 Duality statement

When defining the dual system, there were two sources for non-uniqueness:

1. a multiplication of the systems equation (13) from the left with some S_σ, S_k invertible, and
2. the precise value of the inversion time $T > t_m$.

Neither of those has an influence on the system theoretic properties and thus on the duality result. Multiplying (13) from the left with some S_σ, S_k invertible, yields a state transformation ($\tilde{z} = S_\sigma^\top z$) of the dual system. In particular, this does not influence the system theoretic properties of the dual. Hence we can assume the system and thus also its dual to be normalized. To see that the system theoretic properties do not depend on the precise value of $T > t_m$, observe that the statements in Theorem 29 do not depend on τ_0 and τ_m. The terms $\mathrm{e}^{A_0^{\mathrm{diff}}\tau_0}$ and $\mathrm{e}^{A_m^{\mathrm{diff}}\tau_m}$ can be removed from the equations using the $\mathrm{e}^{A_i^{\mathrm{diff}}\tau_i}$-invariance of $\mathcal{C}_i$ and $\mathcal{U}_i$. Thus the system theoretic properties on $[0, T]$ and on $[0, T']$ are the same for $T, T' > t_m$. Also, the properties of the T-dual and the properties of the T'-dual are the same.

6.1 Normalization and the dual system

Using the above considerations we can assume the system to be normalized for the duality result. This simplifies the subsequent calculations as the following indicates: The matrices $\widehat{\Pi}, \widehat{\Pi}^{\text{diff}}, \widehat{E}^{\text{imp}}, \ldots$ of the dual system and the corresponding matrices of the original system are related via the transformation matrices S, T of the quasi-Weierstrass form (QWF, see Section 2.1):

$$\widehat{\Pi} = (TS)^\top \Pi^\top (TS)^{-\top}, \ \widehat{\Pi}^{\text{diff}} = \left(\Pi^{\text{diff}}\right)^\top, \ \widehat{E}^{\text{diff}} = (TS)^\top \left(E^{\text{diff}}\right)^\top (TS)^{-\top}.$$

This can be seen by straightforward calculations as $(T^\top, S^\top)$ transform the dual to QWF. For normalized systems, it holds $S = T^{-1}$. Thus the matrices and spaces of the dual system simplify to $\widehat{\Pi} = \Pi^\top$ and

$$\widehat{\mathcal{A}}^{\text{part}} = \left(\mathcal{A}^{\text{part}}\right)^\top \ \text{ for } \mathcal{A} \in \{E, A, B, C\} \ \text{ and part} \in \{\text{diff}, \text{imp}\}$$

where we used $\widehat{B} = C^\top$ and $\widehat{C} = B^\top$. Consequently, it holds for normalized systems

$$\widehat{\mathcal{C}} = \mathcal{U}^\perp \ \text{ and } \ \widehat{\mathcal{U}} = \mathcal{C}^\perp.$$

6.2 Duality statement

In contrast to switched ODEs with jumps [14] one cannot work directly with the recursions from Section 5. One reason is that the conditions for reachability ($\mathcal{R}_\sigma^{[0,T]} = \overline{\mathcal{V}^*_{\sigma(T^+)}}$) and observability ($\mathcal{UO}_\sigma^{[0,T]} = \{0\}$) are not complementary as $\overline{\mathcal{V}^*_{\sigma(T^+)}} \neq \mathbb{R}^n$ in general.

Theorem 30 (Duality of switched DAEs with impacts) *For a switched DAE with impacts* (13) *with switching signal* (11) *whose switching times are contained in* $(0, T)$ *and its* T*-dual* (28) *the duality statement* (2) *holds, i.e.* (13) *is observable (determinable) on* $[0, T]$ *if and only its* T*-dual* (28) *is reachable (controllable) on* $[0, T]$.

Proof We assume the system to be normalized as this does not have any influence on the system theoretic properties of the system or its dual. Note that for the dual system it holds $\widehat{H}_i = H_{i+1}^\top$.

By Theorem 29 a normalized switched DAE with impacts is reachable iff

$$\mathbb{R}^n = \ker \Pi_m + \mathcal{C}_m + \mathrm{e}^{A_m^{\text{diff}}\tau_m} H_m \sum_{j=0}^{m-1} \left(\prod_{k=1}^{m-1-j} \mathrm{e}^{A_{m-k}^{\text{diff}}\tau_{m-k}} \Pi_{m-k} H_{m-k} \right) \mathcal{C}_j.$$

Hence a dual system is reachable iff

$$\begin{aligned}\mathbb{R}^n &= \ker \widehat{\Pi}_0 + \widehat{\mathcal{C}}_0 + \mathrm{e}^{\widehat{A}_0^{\mathrm{diff}}\tau_0}\widehat{H}_0 \sum_{j=1}^{m}\left(\prod_{k=1}^{j-1} \mathrm{e}^{\widehat{A}_k^{\mathrm{diff}}\tau_k}\widehat{\Pi}_k\widehat{H}_k\right)\widehat{\mathcal{C}}_j \\ &= \ker \Pi_0^\top + \mathcal{U}_0^\perp + \mathrm{e}^{\left(A_0^{\mathrm{diff}}\right)^\top\tau_0} H_1^\top \sum_{j=1}^{m}\left(\prod_{k=1}^{j-1} \mathrm{e}^{\left(A_k^{\mathrm{diff}}\right)^\top\tau_k}\Pi_k^\top H_{k+1}^\top\right)\mathcal{U}_j^\perp \\ &= \left(\operatorname{im}\Pi_0 \cap \mathcal{U}_0 \cap \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0}H_1^{-1}\bigcap_{j=1}^{m}\left(\prod_{k=1}^{j-1}\mathrm{e}^{-A_k^{\mathrm{diff}}\tau_k}\Pi_k^{-1}H_{k+1}^{-1}\right)\mathcal{U}_j\right)^\perp.\end{aligned}$$

This is the observability condition of the original system (Theorem 29). Hence a switched DAE with impacts is observable iff its dual is reachable.

The same theorem gives that a normalized switched DAE with impacts is controllable iff

$$\begin{aligned}\mathbb{R}^n = \Pi_0^{-1}\Big(\mathcal{C}_0 + \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0}H_1^{-1}\Pi_1^{-1}\big(\dots(\mathcal{C}_{m-1} \\ + \mathrm{e}^{-A_{m-1}^{\mathrm{diff}}\tau_{m-1}}H_m^{-1}\Pi_m^{-1}\mathcal{C}_m\Big)\cdots\Big).\end{aligned}$$

Hence a dual system is controllable iff

$$\begin{aligned}\mathbb{R}^n &= \widehat{\Pi}_m^{-1}\left(\widehat{\mathcal{C}}_m + \mathrm{e}^{-\widehat{A}_m^{\mathrm{diff}}\tau_m}\widehat{H}_{m-1}^{-1}\widehat{\Pi}_{m-1}^{-1}\left(\dots\left(\widehat{\mathcal{C}}_1 + \mathrm{e}^{-\widehat{A}_1^{\mathrm{diff}}\tau_1}\widehat{H}_0^{-1}\widehat{\Pi}_0^{-1}\widehat{\mathcal{C}}_0\right)\cdots\right)\right. \\ &= \Pi_m^{-\top}\Big(\mathcal{U}_m^\perp + \mathrm{e}^{-\left(A_m^{\mathrm{diff}}\right)^\top\tau_m}H_m^{-\top}\Pi_{m-1}^{-\top}\big(\dots(\mathcal{U}_1^\perp \\ &\qquad + \mathrm{e}^{-\left(A_1^{\mathrm{diff}}\right)^\top\tau_1}H_1^{-\top}\Pi_0^{-\top}\mathcal{U}_0^\perp\Big)\cdots\Big) \\ &= \Big(\Pi_m\Big(\mathcal{U}_m \cap \mathrm{e}^{A_m^{\mathrm{diff}}\tau_m}H_m\Pi_{m-1}\Big(\dots\Big(\mathcal{U}_1 \cap \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}H_1\Pi_0\mathcal{U}_0\Big)\cdots\Big)^\perp\end{aligned}$$

This is the determinability condition of the original system (again Theorem 29). Hence a switched DAE with impacts is determinable iff its dual is controllable.

Applying these results to the dual of a switched DAE with impacts and the dual's dual, which is again the original system (Remark 18), gives that a switched DAE with impacts is controllable iff its dual is determinable and reachable iff its dual is observable. □

Remark 31 (Assumptions on the switching signal) We assumed the switching signal to have only finitely many switches and not to have a switch at $t = 0$. Neither of these assumptions is used in [15, 21, 25], but both are crucial here. The switching signal is assumed to be constant on $(-\infty, 0)$ to ensure that the feasibility set at $t = 0^-$ is $\overline{\mathcal{V}^*_{\sigma(0^-)}}$. A restriction on the switchings in $(-\infty, 0)$ is necessary to ensure that nontrivial solutions exist. To achieve the same for the dual system with the time-inverted switching signal $\overline{\sigma}$, the switching signal σ has to be constant after some time $T > 0$. Hence only finitely many switches are allowed.

The switching time $t = 0$ is excluded in this work as it does not give rise to the duality statement: The system $\dot{x} = 0, y = \mathbb{1}_{(-\infty,0)}x$ is not determinable, but its dual is controllable (and reachable) via instantaneous control (see [15, Lemma 2.11]). One can also find a switched DAE that is not reachable, but whose dual is observable (see [13, Example 7.3.1]). ◁

Remark 32 (Other notions of the system theoretic properties for switched systems) So far we fixed a switching signal and considered the system theoretic properties of the system for the given switching signal σ. Other notions for the system theoretic properties of a switched system are that the relevant property hold for all switching signals or that there exists a switching signal such that the property holds. For observability, this is called *strong observability* and *controlled observability* [21], respectively. Strong observability (strong controllability, etc.) is equivalent to observability (controllability, etc.) of each mode. Hence strong observability and strong determinability as well as strong reachability and strong controllability coincide. Furthermore, the duality follows directly form the duality for unswitched DAEs [6]. Duality for the first notion can be derived from Theorem 30. Assume that the system (13) is contr./reach./obsv./det. for a given switching signal σ. Then there exists a switching signal that has only switches in $(0, T)$ for some $T > 0$ and that yields the same property. By Theorem 30 the dual system is then det./obsv./reach./contr. for the switching signal $\overline{\sigma}$. ◁

Remark 33 Due to the results for switched DAEs we conjecture that a suitable adjointness condition for (smoothly) time-varying DAEs should be

$$\tfrac{d}{dt}\left(p^\top E x\right) - y_a^\top u + u_a^\top y = 0, \qquad \triangleleft$$

which the authors have not found in the literature so far.

7 Conclusion

We have established a duality result between controllability/reachability and determinability/observability for switched DAEs. It turned out that the problem of duality cannot be solved within the class of switched DAEs as the dual of a switched DAE does not belong to this system class. Thus we considered the more general class of switched DAEs with impacts, for which characterizations of controllability, reachability, observability and determinability are derived. These are then the basis for our duality result.

It is well known that the dual of a system plays an fundamental role in optimal control and it is therefore a canonical topic for future research whether the duality presented here is a fruitful basis for studying optimal control for switched DAEs.

Another line of future research is an relaxation on the assumption that the switching signal is given. The observability / determinability notions would then include the ability to detect the current mode and the controllability /

reachability notions may allow for the switching signal to be an additional input. Our approach will presumably not carry over directly, because in our framework we could still exploit the linearity of the (time-varying) dynamical system, which will not be possible when the switching signal is not known.

A Some basics on linear algebra

Lemma 34 *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping and $\mathcal{S}_1$, $\mathcal{S}_2$ be subspaces of $\mathbb{R}^n$. Then it holds*

1. $\left(A^{-1}\mathcal{S}\right)^\perp = A^\top \mathcal{S}^\perp$ *and* $(A\mathcal{S})^\perp = A^{-\top}\mathcal{S}^\perp$;
2. $(\ker A)^\perp = \operatorname{im} A^\top$ *and* $(\operatorname{im} A)^\perp = \ker A^\top$;
3. $(\mathcal{S}_1 + \mathcal{S}_2)^\perp = \mathcal{S}_1^\perp \cap \mathcal{S}_2^\perp$ *and* $(\mathcal{S}_1 \cap \mathcal{S}_2)^\perp = \mathcal{S}_1^\perp + \mathcal{S}_2^\perp$;
4. $A\left(A^{-1}\mathcal{S}_1 \cap \mathcal{S}_2\right) = \mathcal{S}_1 \cap A\mathcal{S}_2$;
5. $A^{-1}\left(A\mathcal{S}_1 + \mathcal{S}_2\right) = \mathcal{S}_1 + A^{-1}\mathcal{S}_2$.

Proof 1. [4, Lemma 4.1],
2. [3, Property A.3.4],
3. The first statement is shown in [16, Lemma 4.6], the second follows by computing the orthogonal complement,
4. "$\subseteq$": $A\left(A^{-1}\mathcal{S}_1 \cap \mathcal{S}_2\right) \subseteq AA^{-1}\mathcal{S}_1 \cap A\mathcal{S}_2 = \mathcal{S}_1 \cap \operatorname{im} A \cap A\mathcal{S}_2 = \mathcal{S}_1 \cap A\mathcal{S}_2$.
   "$\supseteq$": Let $x \in \mathcal{S}_1 \cap A\mathcal{S}_2$, i.e $x \in \mathcal{S}_1$ and $\exists y \in \mathcal{S}_2 : x = Ay$. y fulfills $Ay \in \mathcal{S}_1$, hence $y \in A^{-1}\mathcal{S}_1$ and $y \in A^{-1}\mathcal{S}_1 \cap \mathcal{S}_2$. Finally, $x = Ay \in A\left(A^{-1}\mathcal{S}_1 \cap \mathcal{S}_2\right)$.
5. "$\supseteq$": $\mathcal{S}_1 + A^{-1}\mathcal{S}_2 = \mathcal{S}_1 + \ker A + A^{-1}\mathcal{S}_2 = A^{-1}A\mathcal{S}_1 + A^{-1}\mathcal{S}_2 \subseteq A^{-1}\left(A\mathcal{S}_1 + \mathcal{S}_2\right)$.
   "$\subseteq$": Let $x \in A^{-1}\left(A\mathcal{S}_1 + \mathcal{S}_2\right)$, hence there exist $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$ such that $Ax = As_1 + s_2$. Therefore $A(x - s_1) = s_2$, i.e. $x - s_1 \in A^{-1}\mathcal{S}_2$. This gives $x \in \mathcal{S}_1 + A^{-1}\mathcal{S}_2$.

□

Lemma 35 *Let $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ be a projector and $\mathcal{S}$ be a subspace of $\mathbb{R}^n$. Then it holds*

1. $\mathcal{S} + \ker \Pi = \Pi\mathcal{S} + \ker \Pi$ *and* $\mathcal{S} \cap \operatorname{im} \Pi = \Pi^{-1}\mathcal{S} \cap \operatorname{im} \Pi$;
2. $\operatorname{im} \Pi \subseteq \mathcal{S} \Leftrightarrow \Pi^{-1}\mathcal{S} = \mathbb{R}^n$;
3. *for* $\ker \Pi \subseteq \mathcal{S}$: $\operatorname{im} \Pi \cap \mathcal{S} = \Pi\mathcal{S}$.

Proof 1. The second statement follows from the first by computing the orthogonal complement using Lemma 34.1 and renaming $\overline{\mathcal{S}} = \mathcal{S}^\perp$ and $\overline{\Pi} = \Pi^\top$. Consider the first statement: "$\subseteq$": Let $x \in \mathcal{S} + \ker \Pi$, i.e. $\exists s \in \mathcal{S}, y \in \ker \Pi$: $x = s + y = \Pi s + ((I - \Pi)s + y) \in \Pi\mathcal{S} + \ker \Pi$. "$\supseteq$": Let $x \in \Pi\mathcal{S} + \ker \Pi$ i.e. $\exists s \in \mathcal{S}, y \in \ker \Pi$: $x = \Pi s + y = s + ((\Pi - I)s + y) \in \mathcal{S} + \ker \Pi$.
2. Let $\operatorname{im} \Pi \subseteq \mathcal{S}$. Then it holds $\mathbb{R}^n = \Pi^{-1}\left(\operatorname{im} \Pi\right) \subseteq \Pi^{-1}\mathcal{S}$.
   For the other inclusion it holds $\operatorname{im} \Pi = \Pi\mathbb{R}^n = \Pi\left(\Pi^{-1}\mathcal{S}\right) \subseteq \mathcal{S}$ because of $\Pi\left(\Pi^{-1}\mathcal{S}\right) \subseteq \mathcal{S}$.
3. Let $x \in \operatorname{im} \Pi \cap \mathcal{S}$. Hence $\Pi x = x$ and thus $x \in \Pi\mathcal{S}$.
   Let $x \in \Pi\mathcal{S}$, i.e. there exists $s \in \mathcal{S}$ with $x = \Pi s$. It is $s = \Pi s + (I - \Pi)s$. As $(I - \Pi_i)s \in \ker \Pi \subseteq \mathcal{S}$ it follows $x \in \mathcal{S}$ and hence $x \in \operatorname{im} \Pi \cap \mathcal{S}$.

□

B Proofs of Section 4.2

Proof of Lemma 15 Write a, b as $a = \alpha_{\mathbb{D}} + \sum_{t \in \Gamma^a} a[t]$ and $b = \beta_{\mathbb{D}} + \sum_{t \in \Gamma^b} b[t]$. The product of $\sum_{t \in \Gamma^a} a[t]$ and $\sum_{t \in \Gamma^b} b[t]$ (or their time-inversions) is zero for both causal and anticausal multiplication. The product of $\alpha_{\mathbb{D}}$ and $\beta_{\mathbb{D}}$ is the same for both kinds of multiplication, furthermore (25) yields $\mathscr{T}_T\left\{\alpha_{\mathbb{D}} *_c \beta_{\mathbb{D}}\right\} = \mathscr{T}_T\left\{\alpha_{\mathbb{D}}\right\} *_c \mathscr{T}_T\left\{\beta_{\mathbb{D}}\right\}$.

Using the linearity of $\mathscr{T}_T$ it is sufficient to consider the product of a piecewise-smooth function and a Dirac impulse (or derivatives of a Dirac impulse):

$$\begin{aligned}\mathscr{T}_T\{\alpha *_c \delta_t\} &= \mathscr{T}_T\left\{\alpha(t^+)\delta_t\right\} = \alpha(t^+)\delta_{T-t} = \mathscr{T}_T\{\alpha\}((T-t)^-)\delta_{T-t}\\ &= \mathscr{T}_T\{\alpha\} *_{ac} \delta_{T-t} = \mathscr{T}_T\{\alpha\} *_{ac} \mathscr{T}_T\{\delta_t\}\end{aligned}$$

and analogously $\mathscr{T}_T\{\delta_t *_c a\} = \mathscr{T}_T\{\delta_t\} *_{ac} \mathscr{T}_T\{a\}$. Applying the differentiation rule of the multiplication gives inductively for $D = \delta_t, \delta_t^{(1)}, \delta_t^{(2)}, \ldots$:

$$\begin{aligned}\mathscr{T}_T\left\{D' *_c \alpha\right\} &= \mathscr{T}_T\left\{(D *_c \alpha)' - D *_c \alpha'\right\}\\ &\overset{(24)}{=} -(\mathscr{T}_T\{D *_c \alpha\})' - \mathscr{T}_T\left\{D *_c \alpha'\right\}\\ &\overset{\text{Ind.}}{=} -(\mathscr{T}_T\{D\} *_{ac} \mathscr{T}_T\{\alpha\})' - \mathscr{T}_T\{D\} *_{ac} \mathscr{T}_T\left\{\alpha'\right\}\\ &\overset{(24)}{=} -(\mathscr{T}_T\{D\} *_{ac} \mathscr{T}_T\{\alpha\})' + \mathscr{T}_T\{D\} *_{ac} (\mathscr{T}_T\{\alpha\})'\\ &= -\left((\mathscr{T}_T\{D\})' *_{ac} \mathscr{T}_T\{\alpha\}\right)\\ &\overset{(24)}{=} \mathscr{T}_T\left\{D'\right\} *_{ac} \mathscr{T}_T\{\alpha\}.\end{aligned}$$

and analogously for $\mathscr{T}_T\{\alpha *_c D'\}$. Hence the first statement is shown. The second statement follows by applying $\tilde{a} = \mathscr{T}_T\{a\}$ and $\tilde{b} = \mathscr{T}_T\{b\}$ to the first statement and using the involution property $D = \mathscr{T}_T\{\mathscr{T}_T\{D\}\}$.

Proof of Lemma 16 As in the proof of Lemma 15 we observe that it suffices to consider the product of a piecewise-smooth function α and a Dirac impulse:

$$\alpha *_c \delta_t = \alpha(t^+)\delta_t = \delta_t *_{ac} \alpha \text{ and } \alpha *_{ac} \delta_t = \alpha(t^-)\delta_t = \delta_t *_c \alpha.$$

Hence the entries of the matrices $(A *_c B)^\top$ and $B^\top *_{ac} A^\top$ are identical.

C Proofs of Section 5.3

While the proof for controllability can easily be deduced from the one for switched DAEs without impacts given in [15], the same is not true for observability and determinability. The proofs in [21, 25] use properties of jump and impulse of a switched DAE which do not hold true any more when impacts are added to the system. Hence these proofs are given here together with the proof for reachability, which has not been considered before for switched DAEs.

Proof of Lemma 26 Controllability: The proof is analogous to the one for switched DAEs without impacts given in [15].

Reachability, "$\subseteq$": Let $x_T \in \mathcal{R}_{\sigma_1}^{(0,T)}$, i.e. there exists $(u,x,y) \in \mathcal{B}_{\sigma_1}$ with $x(0^+) = 0$ and $x(T^-) = x_T$. We assume u to be zero on $[t_1, t_1+\varepsilon)$ for some $\varepsilon \in (0, \tau_1)$ ([15, Lemma 3.3]). Define $\overline{u} := u_{(-\infty,t_1)}$, $\hat{u} = u_{[t_1,\infty)}$ and corresponding solutions $\overline{x}, \hat{x}$ with zero initial condition. Clearly, $x = \overline{x} + \hat{x}$. It holds $\overline{x}(t_1^-) \in \mathcal{C}_0$ and therefore $\overline{x}(T^-) \in \mathrm{e}^{A_1^{\text{diff}}\tau_1} \Pi_1^{\text{diff}} H_1 \mathcal{C}_0$. For $\hat{x}$ it holds $\hat{x}(t_1^-) = 0$ and hence $\hat{x}(T^-) \in \mathcal{C}_1$. This gives $x(T^-) = \overline{x}(T^-) + \hat{x}(T^-) \in \mathrm{e}^{A_1^{\text{diff}}\tau_1} \Pi_1^{\text{diff}} H_1 \mathcal{C}_0 + \mathcal{C}_1$.

Reachability, "$\supseteq$": Let $x_T \in \mathcal{C}_1 + \mathrm{e}^{A_1^{\text{diff}}\tau_1} \Pi_1^{\text{diff}} H_1 \mathcal{C}_0$. Hence there exists $x_1 \in \mathcal{C}_0$ such that $x_T - \mathrm{e}^{A_1^{\text{diff}}\tau_1} \Pi_1^{\text{diff}} H_1 x_1 \in \mathcal{C}_1$. Define $\overline{u}$ on $(0, t_1)$ such that $(\overline{u}, \overline{x}, \overline{y}) \in \mathcal{B}_{\sigma_1}$ with zero initial condition and $\overline{x}(t_1^-) = x_1$ and define $\hat{u}$ on $(t_1+\varepsilon, T)$ for some $\varepsilon \in (0, \tau_1)$ such that $(\hat{u}, \hat{x}, \hat{y}) \in \mathcal{B}_{\sigma_1}$ with zero initial condition and $\hat{x}(T^-) = x_T - \mathrm{e}^{A_1^{\text{diff}}\tau_1} \Pi_1^{\text{diff}} H_1 x_1$. Note that $\overline{u}$ is zero outside $(0,t_1)$ and $\hat{u}$ is zero outside $(t_1+\varepsilon,T)$. It holds for $(u,x,y) := (\overline{u}+\hat{u},\overline{x}+\hat{x},\overline{y}+\hat{y}) \in \mathcal{B}_{\sigma_1}$: $x(0^+) = \overline{x}(0^+)+\hat{x}(0^+) = 0$ and

$$x(T^-) = \overline{x}(T^-)+\hat{x}(T^-) = \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1x_1 + x_T - \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1x_1 = x_T.$$

Hence $\mathcal{C}_1 + \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1\mathcal{C}_0 \subseteq \mathcal{R}_{\sigma_1}^{(0,T)}$.

Observability, "$\subseteq$": Let $x_0 \in \mathcal{UO}_{\sigma_1}^{(0,T)}$, i.e. there exists $(0,x,y) \in \mathcal{B}_\sigma$ with $x(0^-) = x_0$ and $y_{(0,T)} = 0$. This gives

1. $y_{(0,t_1)} = 0$, which is equivalent to $x(0^-) \in \ker O_0^{\mathrm{diff}} \cap \mathcal{V}_0^*$;
2. $y_{(t_1,T)} = 0$, which is equivalent to $x(t_1^+) \in \ker O_1^{\mathrm{diff}} \cap \mathcal{V}_1^*$ and thus

$$\begin{aligned} x(t_1^-) &\overset{(14a)}{\in} \left(\Pi_1^{\mathrm{diff}}H_1\right)^{-1}\{x(t_1^+)\} \subseteq \left(\Pi_1^{\mathrm{diff}}H_1\right)^{-1}\left(\ker O_1^{\mathrm{diff}} \cap \mathcal{V}_1^*\right) \\ &\subseteq \left(\Pi_1^{\mathrm{diff}}H_1\right)^{-1}\ker O_1^{\mathrm{diff}} = \ker\left(O_1^{\mathrm{diff}}\Pi_1^{\mathrm{diff}}H_1\right); \end{aligned}$$

3. $y[t_1] = 0$, which is by (14b) equivalent to $x(t_1^-) \in \ker\left(O_1^{\mathrm{imp}}\Pi_1^{\mathrm{imp}}H_1\right)$.

Using $x(t_1^-) = \mathrm{e}^{A_0^{\mathrm{diff}}\tau_0}x(0^-)$ for the input-free solution x gives the desired inclusion

$$\mathcal{UO}_{\sigma_1}^{(0,T)} \subseteq \operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0}\left(\ker\left(O_1^{\mathrm{diff}}\Pi_1^{\mathrm{diff}}H_1\right) \cap \ker\left(O_1^{\mathrm{imp}}\Pi_1^{\mathrm{imp}}H_1\right)\right).$$

Observability, "$\supseteq$": Let $x_0 \in \operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0}\mathcal{U}_1^{\mathrm{H}}$. Then there exists a solution $(0,x,y) \in \mathcal{B}_{\sigma_1}$ with $x(0^-) = x_0$ as $x_0 \in \operatorname{im}\Pi_0$ is consistent. By the derivations above it follows $y_{(0,T)} = 0$. Thus $x_0 \in \mathcal{UO}_{\sigma_1}^{(0,T)}$.

Determinability, "$\subseteq$": By (39) we know that for an input-free solution $(0,x,y) \in \mathcal{B}_{\sigma_1}$ with $y_{(0,T)} = 0$ it holds $x(t_1^-) \in \widetilde{\mathcal{M}}_1 = \operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathcal{U}_1^{\mathrm{H}}$. Because of $u = 0$ this gives

$$x(T^-) = \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1x(t_1^-) \in \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1\left(\operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathcal{U}_1^{\mathrm{H}}\right).$$

Determinability, "$\supseteq$": Let $0 \neq x_T \in \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}\Pi_1^{\mathrm{diff}}H_1\left(\operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathcal{U}_1^{\mathrm{H}}\right)$. Using (14a) there exists an input-free solution x with $(0,x,y) \in \mathcal{B}_{\sigma_1}$ and $0 \neq x(t_1^-) \in \operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathcal{U}_1^{\mathrm{H}}$ (as $\operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathrm{e}^{-A_0^{\mathrm{diff}}\tau_0}\mathcal{U}_1^{\mathrm{H}} \subseteq \mathcal{V}_0^*$ is a set of consistent initial values and nonempty by assumption). Using (39) this gives $y_{(0,T)} = 0$, i.e. $x_T \in \mathcal{UD}_{\sigma_1}^{(0,T)}$. □

The proof of (31) gives for $\widetilde{\mathcal{M}}_1 = \operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathcal{U}_1^{\mathrm{H}}$:

$$\left\{ x(t_1^-) \;\middle|\; (0,x,y) \in \mathcal{B}_{\sigma_1} \text{ with } y_{(0,T)} = 0 \right\} = \widetilde{\mathcal{M}}_1. \tag{39}$$

Note that one gets the same space $\widetilde{\mathcal{M}}_1$ if one assumes only $y_{(t_1-\varepsilon_1,t_1+\varepsilon_2)} = 0$ for $\varepsilon_1,\varepsilon_2 > 0$. The restricted switching signal $\sigma_{>t_{i-1}}$ has only one switch on the open interval $(0,t_{i+1})$. Hence we get for $\widetilde{\mathcal{M}}_i$:

$$\left\{ x(t_i^-) \;\middle|\; (0,x,y) \in \mathcal{B}_{\sigma_{>t_{i-1}}} \text{ with } y_{(t_{i-1},t_{i+1})} \right\} = \widetilde{\mathcal{M}}_i. \tag{40}$$

Proof of Theorem 27 We start by proving the recursions. For controllability, the proof can be carried out analogously to [15]. The formulas are shown by induction. The induction start ($i = 1$ for reachability and determinability, $i = m$ for observability) is precisely the single switch case (Lemma 26). For reachability, $i = 0$ corresponds to an unswitched system.

Reachability: Analogously to (30) it holds for $i \geq 1$

$$\mathcal{R}_\sigma^{(0,t_{i+1})} = \mathcal{C}_i + \mathrm{e}^{A_i^{\mathrm{diff}}\tau_i}\Pi_i^{\mathrm{diff}}H_i\mathcal{R}_\sigma^{(0,t_i)}.$$

Hence it holds by induction

$$\mathcal{R}_\sigma^{(0,t_{i+1})} = \mathcal{Q}_0^i \quad \text{for } i = 0, 1, \ldots, m.$$

Observability: Assume the statement holds for i: Let $x_{i-2} \in \mathcal{UO}_{\sigma > t_{i-2}}^{(t_{i-2},T)}$. Hence there exists $(0, x, y) \in \mathcal{B}_{\sigma > t_{i-2}}$ with $x(t_{i-2}^+) = x_{i-2}$ and $y_{(t_{i-2},T)} = 0$. Thus it holds

$$\begin{aligned} y_{(t_{i-1},T)} = 0 \quad &\Rightarrow \quad x(t_{i-1}^+) \in \mathcal{UO}_{\sigma > t_{i-1}}^{(t_{i-1},T)} \overset{\text{Ind.}}{=} \widetilde{\mathcal{M}}_i^m, \\ y_{(t_{i-2},t_i)} = 0 \quad &\overset{(40)}{\Rightarrow} \qquad\qquad x(t_{i-1}^-) \in \widetilde{\mathcal{M}}_{i-1} \end{aligned}$$

and therewith $x(t_{i-1}^-) \in \widetilde{\mathcal{M}}_{i-1} \cap \left(\Pi_{i-1}^{\text{diff}} H_{i-1}\right)^{-1} \widetilde{\mathcal{M}}_i^m$. This implies

$$x(t_{i-2}^+) = \mathrm{e}^{-A_{i-2}^{\text{diff}} \tau_{i-2}} x(t_{i-1}^-) \in \mathrm{e}^{-A_{i-2}\tau_{i-2}} \left(\widetilde{\mathcal{M}}_{i-1} \cap \left(\Pi_{i-1}^{\text{diff}} H_{i-1}\right)^{-1} \widetilde{\mathcal{M}}_i^m \right) \overset{\text{Def.}}{=} \widetilde{\mathcal{M}}_{i-1}^m.$$

For the other direction let $x_{i-2} \in \widetilde{\mathcal{M}}_{i-1}^m$. As $\widetilde{\mathcal{M}}_{i-1}^m$ is a subset of $\overline{\mathcal{V}_{i-2}^*}$ there exists a solution $(0, x, y) \in \mathcal{B}_{\sigma > t_{i-2}}$ with $x(t_{i-2}^+) = x_{i-2}$. It holds $x(t_{i-1}^-) \in \widetilde{\mathcal{M}}_{i-1}$, hence by (40) we get $y_{(t_{i-2},t_i)} = 0$. $x(t_{i-1}^+) \in \widetilde{\mathcal{M}}_i^m$ gives by induction $y_{(t_{i-1},T)} = 0$. Hence $y_{(t_{i-2},T)} = 0$ and thus $x_{i-2} \in \mathcal{UO}_{\sigma > t_{i-2}}^{(t_{i-2},T)}$.

Determinability: For the induction step $i - 1 \to i$ let $x_{i+1} \in \mathcal{UD}_\sigma^{(0,t_{i+1})}$. Hence there exists $(0, x, y) \in \mathcal{B}_\sigma$ with $x(t_{i+1}^-) = x_{i+1}$ and $y_{(0,t_{i+1})} = 0$. Thus it holds

$$\begin{aligned} y_{(0,t_i)} = 0 \quad &\Rightarrow \quad x(t_i^-) \in \mathcal{UD}_\sigma^{(0,t_i)} \overset{\text{Ind.}}{=} \widetilde{\mathcal{N}}_1^{i-1}, \\ y_{(t_{i-1},t_{i+1})} = 0 \quad &\overset{(40)}{\Rightarrow} \qquad\qquad x(t_i^-) \in \widetilde{\mathcal{M}}_i. \end{aligned}$$

All in all we obtain

$$x_{i+1} = x(t_{i+1}^-) = \mathrm{e}^{A_i^{\text{diff}} \tau_i} x(t_i^+) \in \mathrm{e}^{A_i^{\text{diff}} \tau_i} \Pi_i^{\text{diff}} H_i \left(\widetilde{\mathcal{M}}_i \cap \widetilde{\mathcal{N}}_1^{i-1} \right) \overset{\text{Def.}}{=} \widetilde{\mathcal{N}}_1^i.$$

For the other inclusion let $x_{i+1} \in \widetilde{\mathcal{N}}_1^i \overset{\text{Def.}}{=} \mathrm{e}^{A_i^{\text{diff}} \tau_i} \Pi_i^{\text{diff}} H_i \left(\widetilde{\mathcal{M}}_i \cap \widetilde{\mathcal{N}}_1^{i-1} \right)$. Thus there exists $x_i \in \widetilde{\mathcal{M}}_i \cap \widetilde{\mathcal{N}}_1^{i-1}$ with $x_{i+1} = \mathrm{e}^{A_i^{\text{diff}} \tau_i} \Pi_i^{\text{diff}} H_i x_i$. By the induction assumption it holds $\widetilde{\mathcal{N}}_1^{i-1} = \mathcal{UD}_\sigma^{(0,t_i)}$, hence there exists $(0, x, y) \in \mathcal{B}_\sigma$ with $x(t_i^-) = x_i$ and $y_{(0,t_i)} = 0$. $x(t_i^-) \in \widetilde{\mathcal{M}}_i$ gives $y_{(t_{i-1},t_{i+1})} = 0$ by (40). Hence $y_{(0,t_{i+1})} = 0$ and thus $x_{i+1} = x(t_{i+1}^-) \in \mathcal{UD}_\sigma^{(0,t_{i+1})}$.

Characterization of system propteries: The system is reachable on $[0, T]$ iff it holds $\mathcal{R}_\sigma^{[0,T]} = \overline{\mathcal{V}_m^*}$. By Lemma 24 this is equivalent to $\mathcal{R}_\sigma^{(0,T)} = \overline{\mathcal{V}_m^*}$ and the claim follows from $\mathcal{R}_\sigma^{(0,T)} = \mathcal{Q}_0^m$.

The same argument can be used for observability, determinability and controllability. For the latter, note that $\mathcal{P}_0^m \cap \overline{\mathcal{V}_0^*} = \overline{\mathcal{V}_0^*}$ is equivalent to $\overline{\mathcal{V}_0^*} \subseteq \mathcal{P}_0^m$. ☐

Proof of Theorem 29 Controllability: By Theorem 27 the system is controllable on $[0, T]$ iff $\overline{\mathcal{V}_0^*} \subseteq \mathcal{P}_0^m$. As in the proof of the theorem we use $\operatorname{im} K_0^{\text{imp}} \subseteq \mathcal{C}_0 \subseteq \mathcal{P}_0^m$ and $\overline{\mathcal{V}_0^*} = \operatorname{im} \Pi_0 \oplus \operatorname{im} K_0^{\text{imp}}$ to obtain as an equivalent criterion $\operatorname{im} \Pi_0 \subseteq \mathcal{P}_0^m$. By Lemma 35.2 this is equivalent to

$$\Pi_0^{-1} \mathcal{P}_0^m = \mathbb{R}^n.$$

Using the recursion formula for $\mathcal{P}_i^m$ we can write $\mathcal{P}_0^m$ explicitly as

$$\mathcal{C}_0 + \mathrm{e}^{-A_0^{\text{diff}} \tau_0} \left(\Pi_1^{\text{diff}} H_1 \right)^{-1} \left(\ldots \left(\mathcal{C}_{m-1} + \mathrm{e}^{-A_{m-1}^{\text{diff}} \tau_{m-1}} \left(\Pi_m^{\text{diff}} H_m \right)^{-1} \mathcal{C}_m \right) \cdots \right).$$

The statement follows then by the fact that it holds $\Pi^{\mathrm{diff}} = \Pi$ for normalized systems.

Reachability: The recursion formula for $\mathcal{Q}_0^i$ yields:

$$\mathcal{Q}_0^m = \sum_{j=0}^{i} \left(\prod_{k=0}^{i-1-j} \mathrm{e}^{A_{i-k}^{\mathrm{diff}} \tau_{i-k}} \Pi_{i-k}^{\mathrm{diff}} H_{i-k} \right) \mathcal{C}_j.$$

By Theorem 27 this is the reachable set $\mathcal{R}_\sigma^{(0,T)}$. It can be rewritten as

$$\left(\operatorname{im} K_m^{\mathrm{diff}} + \sum_{j=0}^{m-1} \mathrm{e}^{A_m^{\mathrm{diff}} \tau_m} \Pi_m^{\mathrm{diff}} H_m \left(\prod_{k=1}^{m-1-j} \mathrm{e}^{A_{m-k}^{\mathrm{diff}} \tau_{m-k}} \Pi_{m-k}^{\mathrm{diff}} H_{m-k} \right) \mathcal{C}_j \right) \oplus \operatorname{im} K_m^{\mathrm{imp}}.$$

Normalization gives $\Pi^{\mathrm{diff}} = \Pi$. Using the commutativity of $\mathrm{e}^{A_m^{\mathrm{diff}} \tau_m}$ and Π_m as well as $\operatorname{im} K_m^{\mathrm{diff}} = \Pi \mathcal{C}_m$ gives

$$\mathcal{R}_\sigma^{(0,T)} = \Pi_m \left(\mathcal{C}_m + \mathrm{e}^{A_m^{\mathrm{diff}} \tau_m} H_m \sum_{j=0}^{m-1} \left(\prod_{k=1}^{m-1-j} \mathrm{e}^{A_{m-k}^{\mathrm{diff}} \tau_{m-k}} \Pi_{m-k} H_{m-k} \right) \mathcal{C}_j \right) + \operatorname{im} K_m^{\mathrm{imp}}.$$

The system is reachable iff $\mathcal{R}_\sigma^{(0,T)} = \overline{\mathcal{V}_m^*}$, which is equivalent to

$$\Pi_m \left(\mathcal{C}_m + \mathrm{e}^{A_m^{\mathrm{diff}} \tau_m} H_m \sum_{j=0}^{m-1} \left(\prod_{k=1}^{m-1-j} \mathrm{e}^{A_{m-k}^{\mathrm{diff}} \tau_{m-k}} \Pi_{m-k} H_{m-k} \right) \mathcal{C}_j \right) + \ker \Pi_m = \mathbb{R}^n.$$

By Lemma 35.1 this is equivalent to the claim.

Observability: Show by induction that it holds for the normalized system

$$\widetilde{\mathcal{M}}_i^m = \operatorname{im} \Pi_{i-1} \cap \ker O_{i-1}^{\mathrm{diff}} \cap \mathrm{e}^{-A_{i-1}^{\mathrm{diff}} \tau_{i-1}} H_i^{-1} \left(\bigcap_{j=i}^{m} \left(\prod_{k=i}^{j-1} \mathrm{e}^{-A_k^{\mathrm{diff}} \tau_k} \Pi_k^{-1} H_{k+1}^{-1} \right) \mathcal{U}_j \right) \tag{41}$$

for $i = m, \ldots, 1$.

For $i = m$ this holds true by the $\mathrm{e}^{-A_{i-1}^{\mathrm{diff}} \tau_{i-1}}$-invariance of $\operatorname{im} \Pi_{i-1}$ and $\ker O_{i-1}^{\mathrm{diff}}$.

By the same argument we get for the induction step $i+1 \to i$:

$$\widetilde{\mathcal{M}}_i^m = \operatorname{im} \Pi_{i-1} \cap \ker O_{i-1}^{\mathrm{diff}} \cap \mathrm{e}^{-A_{i-1}^{\mathrm{diff}} \tau_{i-1}} \left(\mathcal{U}_i^{\mathrm{H}} \cap \left(\Pi_i^{\mathrm{diff}} H_i \right)^{-1} \widetilde{\mathcal{M}}_{i+1}^m \right).$$

For normalized systems it holds $\Pi_i^{\mathrm{diff}} = \Pi_i$ and $\mathcal{U}_i^{\mathrm{H}} = H_i^{-1} \mathcal{U}_i$. This yields:

$$\widetilde{\mathcal{M}}_i^m = \operatorname{im} \Pi_{i-1} \cap \ker O_{i-1}^{\mathrm{diff}} \cap \mathrm{e}^{-A_{i-1}^{\mathrm{diff}} \tau_{i-1}} H_i^{-1} \left(\mathcal{U}_i \cap \Pi_i^{-1} \widetilde{\mathcal{M}}_{i+1}^m . \right) \tag{42}$$

Observe that it holds $\Pi_i^{-1} \operatorname{im} \Pi_i = \mathbb{R}^n$ and $\Pi_i^{-1} \ker O_i^{\mathrm{diff}} \supseteq \mathcal{U}_i$. Hence we get from the induction assumption

$$\begin{aligned} \mathcal{U}_i \cap \Pi_i^{-1} \widetilde{\mathcal{M}}_{i+1}^m &= U_i \cap \mathrm{e}^{-A_i^{\mathrm{diff}} \tau_i} \Pi_i^{-1} H_{i+1}^{-1} \left(\bigcap_{j=i+1}^{m} \left(\prod_{k=i+1}^{j-1} \mathrm{e}^{-A_k^{\mathrm{diff}} \tau_k} \Pi_k^{-1} H_{k+1}^{-1} \right) \mathcal{U}_j \right) \\ &= \bigcap_{j=i}^{m} \left(\prod_{k=i}^{j-1} \mathrm{e}^{-A_k^{\mathrm{diff}} \tau_k} \Pi_k^{-1} H_{k+1}^{-1} \right) \mathcal{U}_j. \end{aligned}$$

Inserting this into equation (42) yields the induction step. Finally, applying $\operatorname{im} \Pi_{i-1} \cap \ker O_{i-1}^{\mathrm{diff}} = \operatorname{im} \Pi_{i-1} \cap \mathcal{U}_{i-1}$ to (41) for $i = 1$ gives the desired result.

Determinability: We show by induction

$$\widetilde{\mathcal{N}}_1^i = \Pi_i \left(\mathcal{U}_i \cap \mathrm{e}^{A_i^{\mathrm{diff}}\tau_i} H_i \Pi_{i-1} \left(\ldots \left(\mathcal{U}_1 \cap \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1} H_1 \Pi_0 \mathcal{U}_0\right) \cdots \right)\right).$$

For $i = 1$ it holds

$$\widetilde{\mathcal{N}}_1^1 = \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1} \Pi_1^{\mathrm{diff}} H_1 \left(\operatorname{im} \Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \mathcal{U}_1^{\mathrm{H}}\right).$$

As the system is normalized it follows

$$\widetilde{\mathcal{N}}_1^1 = \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1} \Pi_1 H_1 \left(\operatorname{im} \Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap H_1^{-1} \mathcal{U}_1\right).$$

Applying Lemma 34.4 gives

$$\widetilde{\mathcal{N}}_1^1 = \mathrm{e}^{A_1^{\mathrm{diff}}\tau_1} \Pi_1 \left(H_1 \left(\operatorname{im} \Pi_0 \cap \ker O_0^{\mathrm{diff}}\right) \cap \mathcal{U}_1\right).$$

Using $\operatorname{im} \Pi_0 \cap \ker O_0^{\mathrm{diff}} = \Pi_0 \ker O_0^{\mathrm{diff}}$ (Lemma 35.3) and the $\mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}$-invariance of $\mathcal{U}_1$ as well as the commutativity of Π_1 and $\mathrm{e}^{A_1^{\mathrm{diff}}\tau_1}$ gives the claim for $i = 1$.

By $\widetilde{\mathcal{N}}_1^i \subseteq \mathcal{V}_i^*$ and $\widetilde{\mathcal{N}}_1^i \subseteq \mathrm{e}^{A_i^{\mathrm{diff}}\tau_i} \Pi_i^{\mathrm{diff}} H_i \mathcal{U}_i^{\mathrm{H}} \subseteq \mathrm{e}^{A_i^{\mathrm{diff}}\tau_i} \ker O_i^{\mathrm{diff}} = \ker O_i^{\mathrm{diff}}$ it follows for $\widetilde{\mathcal{M}}_{i+1}$:

$$\widetilde{\mathcal{M}}_{i+1} \cap \widetilde{\mathcal{N}}_1^i = \mathcal{U}_{i+1}^{\mathrm{H}} \cap \widetilde{\mathcal{N}}_1^i.$$

Hence it holds for the induction step $i \to i+1$:

$$\widetilde{\mathcal{N}}_1^{i+1} = \mathrm{e}^{A_{i+1}^{\mathrm{diff}}\tau_{i+1}} \Pi_{i+1}^{\mathrm{diff}} H_{i+1} \left(\mathcal{U}_{i+1}^{\mathrm{H}} \cap \widetilde{\mathcal{N}}_1^i\right).$$

The same arguments as for the induction start yield

$$\widetilde{\mathcal{N}}_1^{i+1} = \Pi_{i+1} \left(\mathcal{U}_{i+1} \cap \mathrm{e}^{A_{i+1}^{\mathrm{diff}}\tau_{i+1}} H_{i+1} \widetilde{\mathcal{N}}_1^i\right).$$

Therefore, the statement follows by Theorem 27. □

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