Impulses in structured nonlinear switched DAEs

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***Abstract*— Switched nonlinear differential algebraic equations (DAEs) occur in mathematical modeling of sudden transients in various physical phenomenons. Hence, it is important to investigate them with respect to the nature of their solutions. The few existing solvability results for switched nonlinear DAEs exclude Dirac impulses by definition; however, in many cases this is too restrictive. For example, in water distribution networks the water hammer effect can only be studied when allowing Dirac impulses in a nonlinear switched DAE description. We investigate existence and uniqueness of solutions with impulses for a general class of nonlinear switched DAEs, where we exploit a certain sparse structure of the nonlinearity.**

I. Introduction

We consider nonlinear switched differential algebraic equations (DAEs) of the form

$$E_\sigma \dot{x} = A_\sigma x + g_\sigma(x) + f. \tag{1}$$

where $E_p, A_p \in \mathbb{R}^{n\times n}$, $g_p : \mathbb{R}^n \to \mathbb{R}^n$ for $p \in \{1, \cdots, \mathbf{P}\}$, $\mathbf{P} \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a (time-dependent) inhomogeneity and $\sigma : \mathbb{R} \to \{1, \cdots, \mathbf{P}\}$ is a piecewise constant switching signal, which is assumed to be right continuous and to have locally finitely many jumps.

In particular each subsystem is nonlinear DAE of the form

$$E\dot{x} = Ax + g(x) + f \tag{2}$$

Equations of such kind occur for example when modeling (nonlinear) electrical systems [1], mechanical system [2], [3]. Moreover, while modeling hydraulic transients in water distribution systems in the framework of switched DAEs [4], we observed that each subsystem turns out to be a nonlinear DAE of the form (2), and modeling of transients (e.g, changing valve or pump settings etc.) results in a switched nonlinear DAE of the form (1). This is our main motivation for studying the solution theory of switched nonlinear DAEs, but we are certain that our results will also be applicable in other areas.

The existing solution theory available for switched nonlinear DAEs in [5] excludes the presence of Dirac impulses by definition; however, when studying e.g. the water hammer effect in water distribution networks these impulses are crucial.

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Our key contribution is based on an observation in [4] that for the special case studied therein the family of nonlinearities g_p, $p \in \{1, \cdots, \mathbf{P}\}$ share a certain sparse structure, which can be used to define solutions with Dirac impulses even in the presence of nonlinear expressions. We generalize this idea and formulate the general sparsity assumption (G_p) to first define what a distributional solution for nonlinear switched DAEs actually is (Definition 6 in Section III-A). Our main result (Theorem 8) then provides sufficient conditions for the existence and uniqueness of solution of a so-called initial trajectory problem which then can be used to conclude existence and uniqueness for the switched nonlinear DAE (1).

II. Preliminaries

A. Regular matrix pairs

Definition 1: A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is called regular, if the polynomial $\det(sE - A)$ is not the zero polynomial.

The following characterization of regularity goes back to Weierstrass [6].

Proposition 2: A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if and only if there exist invertible transformation matrices $S, T \in \mathbb{R}^{n\times n}$ which put (E, A) into quasi Weierstrass form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{3}$$

where $N \in \mathbb{R}^{n_2 \times n_2}$, with $0 \le n_2 \le n$ is a nilpotent matrix, $J \in \mathbb{R}^{n_1\times n_1}$ with $n_1 := n - n_2$ is some matrix and I stands for an identity matrix of appropriate size.

We call (3) *quasi*-Weierstrass form (QWF) because we do not assume that J and N are in Jordan canonical form. In [7] (see also [8]) it was shown how to use the Wong-sequences [9] in order to obtain the transformation matrices S and T yielding a QWF. In fact, the Wong sequences are defined as follows

$$\mathcal{V}_0 := \mathbb{R}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \quad \mathcal{V}^* := \bigcup_{i\in\mathbb{N}} \mathcal{V}_i,$$
$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \quad \mathcal{W}^* := \bigcap_{i\in\mathbb{N}} \mathcal{W}_i.$$

Clearly, the Wong sequences are nested and are strictly decreasing/increasing until they become stationary and it can be shown that stationary occurs after exactly the same number of steps, i.e. there exists $i^* \in \{0, 1, \ldots, n\}$ such that

$$\mathcal{V}^* = \mathcal{V}_{i^*} \quad \text{and} \quad \mathcal{W}^* = \mathcal{W}_{i^*}.$$

For any choice of full column rank matrices $V \in \mathbb{R}^{n\times n_1}$ and $W \in \mathbb{R}^{n\times n_2}$ with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$ let

$$T = [V, W] \quad \text{and} \quad S = [EV, AW]^{-1}$$

which indeed transform the regular matrix pair (E, A) into QWF (3).

Based on the QWF (3) one can define the following "projectors".

Definition 3: Consider a regular matrix pair (E, A) with QWF (3). Then the consistency projector is

$$\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \tag{4a}$$

the differential projector is

$$\Pi^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \tag{4b}$$

and the impulse projector is

$$\Pi^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S, \tag{4c}$$

with block sizes corresponding to the QWF. Furthermore let the flow and impulse matrix be given by

$$A^{\text{diff}} := \Pi^{\text{diff}} A = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$
$$E^{\text{imp}} := \Pi^{\text{imp}} E = T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}.$$

Note that only the consistency projector is a projector in the usual sense (i.e., it is an idempotent matrix). Furthermore, it can be shown [10] that the projectors (and hence the flow and impulse matrix) don't depend on the specific choice of the matrices T and S for obtaining the QWF.

The importance of the flow/impulse matrix and the projectors will become clear in Section II-C.

B. Distributions

The presence of inconsistent initial values (or switching) makes it necessary to consider distributional solutions containing in particular Dirac impulses. We therefore briefly recall the necessary basic facts about distributions in the following.

The space of distributions $\mathbb{D}$ consists of all continuous linear maps (functionals) from the space of test functions $\mathcal{C}_0^\infty$ into the real numbers, where $\mathcal{C}_0^\infty$ denotes the space of smooth functions with compact support equipped with a suitable topology. Distributions are also called generalized functions because any locally integrable function $f : \mathbb{R} \to \mathbb{R}$ induces a distribution via

$$f_{\mathbb{D}}(\varphi) := \int_{\mathbb{R}} f\varphi.$$

Every distribution $D \in \mathbb{D}$ can be differentiated via

$$D'(\varphi) := -D(\varphi')$$

and for every differentiable $f : \mathbb{R} \to \mathbb{R}$ it holds

$$(f_{\mathbb{D}})' = (f')_{\mathbb{D}}.$$

The most famous distribution which is not induced by a function is the Dirac impulse which can be defined as the distributional derivative of the Heaviside step function $\mathbb{1}_{[0,\infty)}$, i.e.

$$\delta := (\mathbb{1}_{[0,\infty)\mathbb{D}})'.$$

As shown in [10] the whole space of distributions $\mathbb{D}$ is not a suitable solution space for switched DAEs and it is necessary to introduce a appropriate subspace, namely the space of piecewise-smooth distributions given by

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}},\ T \subseteq \mathbb{R} \text{ is discrete} \\ \forall \tau \in T : D_\tau \in \operatorname{span}\{\delta_\tau, \delta'_\tau, \delta''_\tau, \ldots\} \end{array} \right\},$$

where $\mathcal{C}^\infty_{\text{pw}}$ is the space of piecewise-smooth functions and δ_τ is the Dirac impulse located at $\tau \in T$.

In contrast to general distributions, a piecewise-smooth distribution $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau$ can be evaluated at any $t \in \mathbb{R}$ in the following three different ways:

$$D(t^+) := f(t^+),\ D(t^-) := f(t^-),\ D[t] := \begin{cases} D_t, & t \in T \\ 0, & t \notin T, \end{cases}$$

where $f(t^\pm)$ denotes the left/right limit of the piecewise-smooth function f at $t \in \mathbb{R}$. Furthermore the restriction of a piecewise-smooth distribution $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau$ to any interval $\mathfrak{I} \subseteq \mathbb{R}$ is well defined by

$$D_{\mathfrak{I}} := (f_{\mathfrak{I}})_{\mathbb{D}} + \sum_{\tau\in T\cap\mathfrak{I}} D_\tau$$

where $f_{\mathfrak{I}}(t) = f(t)$ if $t \in \mathfrak{I}$ and $f(t) = 0$ otherwise.

C. Initial trajectory problems and switched DAEs

Theorem 4 ([10], [11]): Let $x^0 \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $f \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$ and (E, A) be a regular matrix pair. Then the linear initial trajectory problem (ITP)

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax + f)_{[0,\infty)} \end{aligned} \tag{5}$$

has a unique solution $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$. If f is induced by a piecewise-smooth function the unique solution x satisfies, for $t \in (0, \infty)$,

$$\begin{aligned} x(t^+) =& e^{A^{\text{diff}} t} \Pi x^0(0^-) + \int_0^t e^{A^{\text{diff}}(t-s)} \Pi^{\text{diff}} f(s) \mathrm{d}s \\ & - \sum_{i=0}^{n-1} (E^{\text{imp}})^i \Pi^{\text{imp}} f^{(i)}(t^+) \end{aligned}$$

and

$$\begin{aligned} x[0] = & -\sum_{i=0}^{n-1} (E^{\text{imp}})^i x^0(0^-) \delta^{(i)} \\ & - \sum_{i=0}^{n-1} (E^{\text{imp}})^i \sum_{j=0}^{i} f^{(i-j)}(0^+) \delta^{(j)} \end{aligned} \tag{6}$$

where $\delta^{(i)}$ denotes the i^{th} derivative of the Dirac impulse δ. In particular, if $f=0$, then

$$x(0^+) = \Pi x(0^-).$$

By reapplying the ITP at each switching time we immediately have the following result for switched DAEs.

Corollary 5: The switched system

$$E_\sigma \dot{x} = A_\sigma x + f$$

with regular matrix pairs (E_p, A_p), $p \in \{1, \cdots, P\}, P \in \mathbb{N}$ has a unique solution for every $f \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ and every initial trajectory $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$. In particular, the jumps and Dirac impulses induced by the switches are uniquely determined.

The main goal of this note is the generalization of that linear existence and uniqueness result to nonlinear switched DAEs of the form (1).

III. Main theoretical result

A. Solution concept

The first challenge in studying the nonlinear switched DAE (1)

$$E_\sigma \dot{x} = A_\sigma x + g_\sigma(x) + f$$

within a distributional solution framework is the nonlinear evaluation $g_\sigma(x)$ for distributional x. Due to the linear nature of the space of distributions it is not possible to have a general nonlinear evaluation of distributions without leaving the space of distributions. Our approach to overcome this problem is the assumption that the nonlinearity is sparse in some sense and that g_σ is independent of the possible impulsive parts of x. This is made precise in the following definition:

Definition 6: Consider a nonlinear switched DAE of the form (1) with $f \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$. We make the following sparsity assumption for all $p \in \{1, \ldots, \mathbf{P}\}$

(G$_p$) $\exists \overline{g}_p : \mathbb{R}^{m_p} \to \mathbb{R}^{n_p}$ $\exists \mathcal{M}_p \in \mathbb{R}^{m_p \times n}$ $\exists \mathcal{N}_p \in \mathbb{R}^{n \times n_p}$ $\forall \xi \in \mathbb{R}^n :$ $\boxed{g_p(\xi) = \mathcal{N}_p \overline{g}_p(\mathcal{M}_p \xi)}$.

Then $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ is a solution of (1), if

A1: $\mathcal{M}_\sigma x$ is impulse-free, i.e. $(\mathcal{M}_\sigma x)[t] = 0$ for all $t \in \mathbb{R}$ or, in other words, there exists a piecewise-smooth function $\overline{x} : \mathbb{R} \to \mathbb{R}^n$ such that $\mathcal{M}_\sigma x$ is induced by the piecewise-smooth function $\mathcal{M}_\sigma \overline{x}$;

A2: $\mathcal{N}_\sigma \overline{g}_\sigma(\mathcal{M}_\sigma \overline{x})$ is a piecewise-smooth function; and

A3: $E_\sigma \dot{x} = A_\sigma x + G_x + f$ holds as an equality within the space of piecewise-smooth distributions where G_x is the distribution induced by the piecewise-smooth function $\mathcal{N}_\sigma \overline{g}_\sigma(\mathcal{M}_\sigma \overline{x})$.

Remark 7: The choice of the matrices $\mathcal{M}_p$ and $\mathcal{N}_p$ in assumption (G_p) is not unique; in fact, it is always possible to chose $\mathcal{M}_p = \mathcal{N}_p = I$ and $\overline{g} = g$. However, this trivial choice will prohibit Dirac impulses in the solution and is therefore not suitable for our purpose. Furthermore, it is actually not correct to just say "x is a solution of (1)", because being a solution depends on the choice of $\mathcal{M}_p$ and $\mathcal{N}_p$. Consequently, we write "x is a solution if" and not "x is a solution if, and only if,", because a given x which does not satisfy conditions A1, A2, and A3 may satisfy them for different matrices $\mathcal{M}_p$ and $\mathcal{N}_p$ (the suitable choice may actually depend on x). Even if for a given x there does not exist matrices $\mathcal{M}_p$ and $\mathcal{N}_p$ such that A1, A2 and A3 holds, it may still be possible that with a suitably defined nonlinear distributional evaluation x could be seen as a solution of (1). Finding a suitable necessary condition for a distributional x being a solution of (1) is the topic of future research and we content ourselves here with a definition giving a sufficient condition for being a solution.

B. Motivation for solution concept: water networks

In our recent work [4] we have investigated a switched DAE model for water distribution networks with a special focus on the so called water hammer effect occurring in the simple water network as shown in Figure 1 [12], [13].

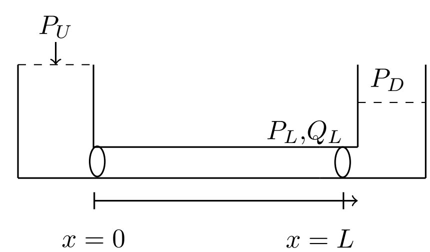

Fig. 1. Simple set up for water hammer with a valve at position $x=L$

The corresponding switched DAE model (1) with $x = (Q, P_0, P_L)^\top$ and

$$\sigma(t) = \begin{cases} 1, & t \in [0, t_S), \quad \text{valve open} \\ 2, & t \in [t_S, \infty) \quad \text{valve closed} \end{cases}$$

is given by

$$\left.\begin{aligned} E_1 = E_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ f &= \begin{cases} (0, -P_U, -P_D)^\top, & \text{on } [0, t_S), \\ (0, -P_U, 0)^\top, & \text{on } [t_S, \infty), \end{cases} \\ g_1(x) = g_2(x) = g(x) &= \begin{pmatrix} -c_2 Q|Q| \\ 0 \\ 0 \end{pmatrix}, \end{aligned}\right\} \qquad (7)$$

where $c_1 > 0$ and $c_2 > 0$ are some constants. Clearly, the nonlinearity $g_\sigma(x) = g(x)$ in (1) does not depend on all components of x and is also not present in all equations, hence we can write

$$g(x) = \mathcal{N} \overline{g}(\mathcal{M} x)$$

where

$$\mathcal{N} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathcal{M} = [1,0,0] \quad \overline{g}(Q) = -c_2 Q|Q|.$$

i.e. the sparsity condition (G_p) holds. In [4] we have utilized this special structure to show existence and uniqueness of solutions of the corresponding nonlinear switched DAE; however, we have not investigated the general case.

C. Existence and uniqueness of solutions

Similar as in the linear case we will establish an existence and uniqueness result for nonlinear ITPs first:

Theorem 8: For $\omega \in (0,\infty]$, consider the local nonlinear ITP

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\omega)} &= (Ax + g(x) + f)_{[0,\omega)} \end{aligned} \tag{8}$$

with initial trajectory $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$. We make the following assumptions:

(R): (E,A) is regular.

(F): The inhomogeneity f is induced by a piecewise-smooth function $\overline{f}:\mathbb{R}\to\mathbb{R}^n$, i.e. $f = \overline{f}_{\mathbb{D}}$.

(S): $g:\mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and piecewise-smooth.

(G): $\exists \overline{g}:\mathbb{R}^{m_g}\to\mathbb{R}^{n_g}$ $\exists \mathcal{M}\in\mathbb{R}^{m_g\times n}$ $\exists \mathcal{N}\in\mathbb{R}^{n\times n_g}$ $\forall \xi \in \mathbb{R}^n:$ $\boxed{g(\xi) = \mathcal{N}\overline{g}(\mathcal{M}\xi)}$.

(M): $\mathcal{M}E^{\mathrm{imp}} = 0$.

(N): $\operatorname{im}\mathcal{N} \subseteq \operatorname{im} E$.

If all these assumptions are satisfied, then there exists $\omega > 0$ such that the local nonlinear ITP (8), has a unique solution $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ (in an analogue sense of Definition 6) on $(-\infty,\omega)$.

The proof of this theorem is based on the following lemma.

Lemma 9 (Modified QWF): Assume the QWF of a regular matrix pair (E,A) has the special form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 & 0\\ 0 & 0 & 0\\ 0 & N_1 & N_2\end{bmatrix}, \begin{bmatrix} J & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\right) \tag{9}$$

where $[N_1,N_2]$ has full column rank and N_2 is nilpotent. Write $T = [T^v, T_1^w, T_2^w]$ and $S^\top = [S^{v\top}, S_1^{w\top}, S_2^{w\top}]$ corresponding to the block sizes of (9). Then for any $\mathcal{M}$ and $\mathcal{N}$ as in assumption (G) the following equivalences hold

$$\begin{aligned} \mathcal{M}E^{\mathrm{imp}} = 0 &\iff \mathcal{M}T_2^w = 0,\\ \operatorname{im}\mathcal{N} \subseteq \operatorname{im} E &\iff S_1^w \mathcal{N} = 0. \end{aligned}$$

Proof: The first equivalence is shown as follows

$$\begin{aligned} \mathcal{M}E^{\mathrm{imp}} &= \mathcal{M}\cdot T \left[\begin{smallmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & N_1 & N_2\end{smallmatrix}\right] T^{-1} = 0\\ \iff& \mathcal{M}[T^v,\ T_1^w,\ T_2^w]\left[\begin{smallmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & N_1 & N_2\end{smallmatrix}\right] = 0\\ \iff& \mathcal{M}[0,\ T_2^w N_1,\ T_2^w N_2] = 0\\ \iff& \mathcal{M}T_2^w[N_1,\ N_2] = 0\\ \overset{*}{\iff}& \mathcal{M}T_2^w = 0. \end{aligned}$$

where equivalence $*$ is a consequence from the full row rank of $[N_1,N_2]$. In order to derive the second equivalence, we have to observe first that

$$\operatorname{im} S_1^{w\top} = \ker E^\top \text{ or, equivalently, } \ker S_1^w = \operatorname{im} E$$

hence the second equivalence follows from

$$\begin{aligned} & S_1^w\mathcal{N} = 0,\\ \iff& \operatorname{im}\mathcal{N} \subseteq \ker S_1^w\\ \iff& \operatorname{im}\mathcal{N} \subseteq \operatorname{im} E. \end{aligned}$$

■

Proof of Theorem 8. The proof proceeds in several steps.

Step 1: We construct S and T such that (9) holds.

Let $\widehat{\mathcal{V}}^*$ and $\widehat{\mathcal{W}}^*$ the Wong limits of the transposed matrix pair $(E^\top,A^\top)$ and let $n_1 := \dim \widehat{\mathcal{V}}^*$, $n_2^1 := \dim\ker E^\top$ and $n_2^2 := \dim\widehat{\mathcal{W}}^* - n_2^1$. Since by construction $\ker E^\top = \widehat{\mathcal{W}}_1 \subseteq \widehat{\mathcal{W}}^*$ we can choose full column rank matrices $\widehat{V}$ and $\widehat{W} = [\widehat{W}_1, \widehat{W}_2]$ such that

$$\operatorname{im}\widehat{V} = \widehat{\mathcal{V}}^*,\ \operatorname{im}\widehat{W} = \widehat{\mathcal{W}}^*,\ \operatorname{im}\widehat{W}_1 = \ker E^\top.$$

With

$$S := [\widehat{V}, \widehat{W}_1, \widehat{W}_2]^\top, \quad T := [E^\top\widehat{V}, A^\top\widehat{W}_1, A^\top\widehat{W}_2]^{-\top}$$

it follows that (SET,SAT) is the transpose of the QWF of $(E^\top,A^\top)$ and hence a QWF itself. Furthermore, by construction $\widehat{W}_1 E = 0$, which shows that (SET,SAT) has the form (9) and it remains to be shown that $[N_1,N_2]$ has full row rank. Assume the contrary, then there exists a vector $v \in \mathbb{R}^{n_2^2}\setminus\{0\}$ with $v^\top[N_1,N_2] = 0$ and, therefore,

$$0 = [0,0,v^\top]\left[\begin{smallmatrix} I & 0 & 0\\ 0 & 0 & 0\\ 0 & N_1 & N_2\end{smallmatrix}\right] = [0,0,v^\top]SET,$$

which is equivalent to $0 = [0,0,v^\top][\widehat{V},\widehat{W}_1,\widehat{W}_2]^\top E = 0$. Hence $v^\top\widehat{W}_2^\top E = 0$, or equivalently, $E^\top\widehat{W}_2 v = 0$ which implies that

$$\{0\} \neq \operatorname{im}\widehat{W}_2 \cap \ker E^\top = \operatorname{im}\widehat{W}_2 \cap \operatorname{im}\widehat{W}_1.$$

This contradicts full rank of $\widehat{W} = [\widehat{W}_1,\widehat{W}_2]$ and Step 1 is complete.

Step 2: We rewrite the nonlinear DAE in coordinates corresponding to the QWF (9).

Let $\left(\begin{smallmatrix} v\\ w_1\\ w_2\end{smallmatrix}\right) := T^{-1}x$ then $E\dot{x} = Ax + g(x) + f$ is equivalent to

$$SET\left(\begin{smallmatrix} v\\ w_1\\ w_2\end{smallmatrix}\right) = SAT\left(\begin{smallmatrix} v\\ w_1\\ w_2\end{smallmatrix}\right) + Sg\left(T\left(\begin{smallmatrix} v\\ w_1\\ w_2\end{smallmatrix}\right)\right) + Sf$$

Choosing $S^\top = [S^{v\top}, S_1^{w\top}, S_2^{w\top}]$ and $T = [T^v, T_1^w, T_2^w]$ as in Step 1, the ITP (8) is therefore equivalent to

$$\begin{aligned} v_{(-\infty,0)} &= v^0_{(-\infty,0)}\\ \dot{v}_{[0,\infty)} &= (Jv + \mathcal{N}^v\overline{g}(\mathcal{M}x) + f^v)_{[0,\omega)} \end{aligned} \tag{10a}$$

$$\begin{aligned} w_{1(-\infty,0)} &= w^0_{1(-\infty,0)}\\ 0 &= (w_1 + \mathcal{N}_1^w\overline{g}(\mathcal{M}x) + f_1^w)_{[0,\omega)} \end{aligned} \tag{10b}$$

$$\begin{aligned} w_{2(-\infty,0)} &= w^0_{2(-\infty,0)}\\ (N_1\dot{w}_1 + N_2\dot{w}_2)_{[0,\infty)} &= (w_2 + \mathcal{N}_2^w\overline{g}(\mathcal{M}x) + f_2^w)_{[0,\omega)} \end{aligned} \tag{10c}$$

where $\begin{pmatrix} v^0 \\ w_1^0 \\ w_2^0 \end{pmatrix} := T^{-1}x^0$, $\begin{pmatrix} f^v \\ f_1^w \\ f_2^w \end{pmatrix} = Sf$ and $\begin{bmatrix} \mathcal{N}^v \\ \mathcal{N}_1^w \\ \mathcal{N}_2^w \end{bmatrix} = S\mathcal{N}$.

Step 3: Existence and uniqueness of solutions.

Assumption (N) together with Lemma 9 yields that $\mathcal{N}_1^w = 0$, hence the ITP (10b) simplifies to

$$\begin{aligned} w_{1(-\infty,0)} &= w^0_{1(-\infty,0)} \\ 0 &= (w_1 + f_1^w)_{[0,\omega)} \end{aligned}$$

which clearly has the unique solution

$$w_1 = w^0_{1(-\infty,0)} - f^w_{1\,[0,\omega)}.$$

Note that w_1 is a piecewise-smooth function (and not a distribution) on $[0,\omega)$. We can plug this solution into (10a) and take into account assumption (M) together with Lemma 9 to obtain

$$\begin{aligned} v_{(-\infty,0)} &= v^0_{(-\infty,0)} \\ \dot{v}_{[0,\infty)} &= h(\cdot,v)_{[0,\omega)} \end{aligned}$$

where

$$h(t,v) = Jv + \mathcal{N}^v \overline{g}(\mathcal{M}T_1^w v + \mathcal{M}T_1^w w_1(t)) + f_1^w(t),$$

i.e. (10a) is a usual ODE where h is smooth in v (in particular, locally Lipschitz) and piecewise-smooth in t (in particular, measurable), hence classical ODE solution theory guarantees existence and uniqueness of a (local) solution v. Note that v is a piecewise-smooth and absolutely continuous function on $[0,\omega)$. Finally, we see that (10c) can be written as

$$\begin{aligned} w_{2(-\infty,0)} &= w^0_{2(-\infty,0)} \\ (N_2\dot{w}_2)_{[0,\infty)} &= \left(w_2 + \widetilde{f_2^w}\right)_{[0,\omega)} \end{aligned}$$

where

$$\widetilde{f_2^w} = f_2^w - N_1\dot{w}_1 + \mathcal{N}_2^w \overline{g}(\mathcal{M}T^v v + \mathcal{M}T_1^w w_1).$$

Hence (10c) becomes a usual nilpotent DAE ITP with (possibly distributional) inhomogeneity $\widetilde{f_2^w}$ and has a unique (distributional) solution on $(-\infty,\omega)$. ∎

Remarks 10: We want to discuss in the following how the assumptions of Theorem 8 may be relaxed.

(R) In linear DAEs the regularity assumption on (E,A) is necessary and sufficient for existence and uniqueness of solutions. Since we do not make strong assumptions on the nonlinearity g it is not excluded that g still contains a linear component. In the extreme case $g(x) = Mx$ for some matrix M, the regularity of the matrix pair $(E, A+M)$ is more or less independent of the regularity of (E,A). Hence in a nonlinear setup without further restrictions on g regularity of (E,A) is neither necessary nor sufficient for existence and uniqueness of solutions; however it allows us to use a certain coordinate transformation which reveals certain structural aspects.

(F) Requiring that the inhomogeneity does not contain Dirac impulses is important to ensure that the solution w_1 of (10b) does not contain Dirac impulses, because w_1 is plugged into the nonlinearity. Furthermore, classical solvability of the nonlinear ODE (10a) is only guaranteed for non-impulsive inhomogeneities. However, in the context of impulsive systems one may allow Dirac-impulses in f^v (but not derivatives of Dirac impulses) then the solution exhibit jumps. In (10c) the presence of Dirac impulses (and its derivatives) in f_2^w isn't a problem at all.

(S) Local Lipschitz continuity is needed to have existence and uniqueness of solutions of the nonlinear ODE (10a). Additionally piecewise-smoothness is assumed to ensure that condition A2 in the solution Definition 6 is satisfied.

(M) The intuition behind this assumption is that due to (6) the impulsive parts in the solution x of a (linear) DAE in response to an inconsistent initial value is in the image of E^{imp}. Hence if $\mathcal{M}E^{\text{imp}} = 0$ then the nonlinearity satisfying (G) doesn't "see" the possible Dirac impulses in x and can therefore be evaluated even for distributional x. A convenient consequence of (M) is the ability to solve (10a) and (10b) independently of (10c).

(N) This assumption was used in the proof to show that (10b) has a unique solution which then can be plugged into (10a) as an inhomogeneity. If (M) holds, one can significantly relax (N) by just requiring that the nonlinear algebraic equation

$$0 = w_1 + \mathcal{N}_1^w \overline{g}(\mathcal{M}T^v v + \mathcal{M}T_1^w w_1) + f_1^w \qquad (11)$$

is uniquely solvable for w_1 in terms of v and f_1^w or in other words the combined DAE (10a), (10b) (which due to (M) is independent of w_2) has index one. The problem with this index one assumption is that it is depending on $\overline{g}$ and may be hard to check in the original coordinates.

In the context of switched DAEs we are usually interested in global solutions, i.e. in order to apply Theorem 8 to (1) we need to make an additional assumption to exclude the occurrence of finite escape time. From the equivalent representation of each ITP in the form (10) it becomes clear that the only source for finite escape time is the nonlinearity in (10a). Therefore, it is sufficient to make the following assumption for each $p \in \Sigma$:

(∞_p) All solutions $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$ of the ITP (5) corresponding to mode p do not exhibit finite escape time, i.e. $\omega = \infty$.

Provided all assumptions of Theorem 8 are satisfied, a sufficient condition for existence of global solutions is *global* Lipschitz continuity of the nonlinear term g. However, in water networks the nonlinearity is quadratic and hence not globally Lipschitz (in that case the nonlinearities are friction terms and hence have a stabilizing effect and do not produce finite escape time). In general, it is difficult to formulate non-conservative conditions ensuring global solutions.

In the following, we will denote with (R_p), (S_p), (M_p), (N_p) the corresponding conditions (R), (S), (M), (N) for mode $p \in \Sigma$. We can now formulate our main existence and uniqueness result for solutions of switched nonlinear DAEs of the form (1) as a corollary of Theorem 8.

Corollary 11: Consider the switched DAE (1) satisfying conditions (R_p), (F), (S_p), (G_p), (M_p), (N_p), (∞_p) for each mode $p \in \Sigma$. Then for any initial trajectory $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ on $(-\infty, 0)$, there exists a unique distributional solution $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ of (1) (in the sense of Definition 6).

Remark 12: The assumption (∞_p) is usually too strong because it suffices that the local solution of mode i on $[t_i, t_i + \omega_i)$ covers the (usually finite) interval $[t_i, t_{i+1}]$. Furthermore, not all initial values for mode i have to be considered, only the consistent ones from the previous mode. The advantage of condition (∞_p) is the independence of the switching signal, i.e. existence and uniqueness of solutions can be guaranteed for arbitrary switching signals.

IV. Examples

A. Application to the example in Section III-B

We consider again the motivational example of a simple water network given by (7).

It is easily seen that (E_1, A_1), (E_2, A_2) are regular, i.e. (R_p) holds. According to the construction of Step 1 of the proof of Theorem 8 we can calculate

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & -c_1 & c_1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_1^{imp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{c_1} \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2^{imp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{c_1} & 0 & 0 \end{bmatrix}$$

With $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}$, $g_1 = g_2 = g$ as in Section III-B we immediately see that (S_p), (G_p), (M_p) and (N_p) holds. Under the assumption that the reservoir pressure only changes smoothly in time (or are just constant), then the inhomogeneity f is piecewise-smooth, i.e. (F) holds. With some simple arguments it can be shown that all solutions of the corresponding ITP are global, hence Corollary 11 ensures existence and uniqueness of distributional solutions of the switched DAE modeling the simple water network in Figure 1. Indeed, Dirac impulses occur when switching from mode 1 (valve open) to mode 2 (valve closed).

The solution on $[t_s, \infty)$ then reads as:

$$x = (0, P_D, P_U - \frac{1}{c_1} Q(t_s^-)\delta_{t_s}).$$

B. Academic example with nontrivial nonlinearity

The above example of the simple water network as well as other example based on water networks usually have very simple nonlinearities and the choice of $\mathcal{M}$ and $\mathcal{N}$ is rather obvious (because they just contain rows and columns of the identity and zero matrix). In the following we would like to analyze the application of Theorem 8 to a nonlinear DAE with a more interesting nonlinearity. Therefore consider the ITP (8) with

$$E = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & c_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 & -c_1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -c_1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$g(x) = \begin{pmatrix} c_2(x_1+x_2+x_4)^2 + c_3 x_1^2 \\ c_4(x_4-x_3)^3 \\ c_5 x_6^4 - c_6 x_1^2 \\ c_7(x_4-x_6)^3 \\ c_8 x_1^2 \\ 0 \\ 0 \end{pmatrix}, \quad f = 0.$$

It is easily verified that conditions (R), (F), (S) are satisfies and we can calculate S, T, and E^{imp} as follows:

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ -\alpha & 1-\alpha & 1 & -1 & -2 & -\alpha & \alpha \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & -\alpha & 1-\alpha & 0 & -\alpha \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & c_1+1 & c_1+1 \\ 0 & 1 & -1 & 0 & 0 & c_1+1 & c_1 \\ 0 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -c_1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$E^{\mathrm{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha & -\alpha & 0 & -\alpha & 0 & \alpha & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & -\alpha & 0 \end{bmatrix} \quad \text{with} \quad \alpha = \frac{1}{c_1}.$$

To satisfy condition (G) we can choose:

$$\mathcal{M} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} c_2 & 0 & 0 & 0 & c_3 \\ 0 & c_4 & 0 & 0 & 0 \\ 0 & 0 & c_5 & 0 & -c_6 \\ 0 & 0 & 0 & c_7 & 0 \\ 0 & 0 & 0 & 0 & c_8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{g}(\xi) = \begin{pmatrix} (\xi_1)^2 \\ (\xi_2)^3 \\ (\xi_3)^4 \\ (\xi_4)^3 \\ (\xi_5)^2 \end{pmatrix} \quad \text{with} \quad \xi = \mathcal{M}x = \begin{pmatrix} x_1+x_2+x_4 \\ x_4-x_3 \\ x_6 \\ x_4-x_6 \\ x_1 \end{pmatrix},$$

for which $g(x) = \mathcal{N}\bar{g}(\mathcal{M}x)$ holds. With this choice it is easily checked that (M) and (N) hold. Altogether, the assumptions of Theorem 8 hold and we can conclude that for any initial trajectory there is a unique distributional solution of the nonlinear ITP (8).

C. Water network with nonlinear pump

Instantaneous starting (mode 1) and stopping (mode 2) of a pump in a water network can cause hydraulic transients, and may cause ruptures in the pump casings [14]. We now present these transients as simple example of a water network shown in Figure 2 with both pump settings.

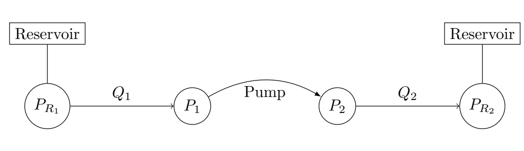

Fig. 2. Water network−1

The pump is modeled via the following nonlinear expression, see e.g. [15] or [16, Sec. 3.2]:

$$\text{Mode-1:} \quad P_2 - P_1 = aQ_1^d, \quad \text{Mode-2:} \quad Q_1 = 0,$$

for some physical constants $a, d > 0$. It will turn out that for this example the condition (N_p) is not satisfied for mode 1; nevertheless, Remark 10 (N) can be utilized to conclude existence and uniqueness of solutions. The DAE model of Figure 2, with mode 1 and mode 2 described by (E_1, A_1, f, g_1) and (E_2, A_2, f, g_2), respectively is an ITP of the form (8) with $x = (Q_1, Q_2, P_1, P_2)^\top$ and

$$E_2 = E_1 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \qquad f = \begin{pmatrix} P_{R_1} \\ -P_{R_2} \\ 0 \\ 0 \end{pmatrix},$$

$$A_1 = \begin{bmatrix} 0&0&-c_1&0\\0&0&0&c_1\\0&0&-1&1\\1&-1&0&0 \end{bmatrix}, \qquad g_1(x) = \begin{pmatrix} -c_2Q_1|Q_1| \\ -c_2Q_2|Q_2| \\ -aQ_1^d \\ 0 \end{pmatrix},$$

$$A_2 = \begin{bmatrix} 0&0&-c_1&0\\0&0&0&c_1\\1&0&0&0\\1&-1&0&0 \end{bmatrix}, \qquad g_2(x) = \begin{pmatrix} -c_2Q_1|Q_1| \\ -c_2Q_2|Q_2| \\ 0 \\ 0 \end{pmatrix}.$$

It is not difficult to verify regularity of (E_1, A_1) , (E_2, A_2), hence (R_p) holds. In particular, we can calculate

$$E_1^{\text{imp}} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\-b\alpha&b\alpha&0&0\\-b\alpha&b\alpha&0&0 \end{bmatrix}, \quad E_2^{\text{imp}} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\-\alpha&0&0&0\\0&\alpha&0&0 \end{bmatrix}.$$

For smooth (in particular, for constant) reservoir pressures assumption (F) holds and clearly (S_p) also holds. With

$$\mathcal{M}_1 = \mathcal{M}_2 = \left[\begin{smallmatrix} 1&0&0&0\\0&1&0&0 \end{smallmatrix}\right],$$
$$\mathcal{N}_1 = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\\0&0&0 \end{bmatrix}, \quad \mathcal{N}_2 = \begin{bmatrix} 1&0\\0&1\\0&0\\0&0 \end{bmatrix},$$

it is easily verified that (G_p) as well as (M_p) hold. However, $\operatorname{im}\mathcal{N}_1 \not\subseteq \operatorname{im} E_1$, hence ($N_p$) does not hold and Theorem 8 is not directly applicable. In order to utilize Remark 10 (N) we first have to transform the system into the special QWF (9) via

$$T_1 = \begin{bmatrix} -b&0&b&0\\-b&0&-b&0\\0&-b&0&b\alpha\\0&b&0&b\alpha \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1&1&c_1&0\\0&0&1&0\\0&0&0&1\\1&-1&0&0 \end{bmatrix},$$
$$T_2 = \begin{bmatrix} 1&0&0&0\\1&-1&0&0\\0&0&\alpha&0\\0&0&0&-\alpha \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix},$$
$$\text{with} \quad \alpha = \frac{1}{c_1}, \quad b = 0.5.$$

In the new coordinates $\left(\begin{smallmatrix} v \\ w_1 \\ w_2 \end{smallmatrix}\right) = T^{-1}x$ the equation (11) reads as

$$0 = w_1 + a(-bv)^d.$$

which is solvable for w_1 in terms of v. Hence all assumptions of Theorem 8 holds, which conclude that for any initial trajectory there is a unique local distributional solution to ITP corresponds to mode 1 and mode 2.

To extend the solution to switched nonlinear DAEs, all solution to ITP must be global. For that assumption (∞_p) should hold. In this example we can not make a general statement as (∞_p) did not hold for all values of d in pump expression in new coordinates. However, for example for $d = 3$ assumption (∞_p) holds. Hence, with Corollary 11 we can conclude existence and uniqueness of distributional solutions of the nonlinear switched DAE model of the water network as in Figure 2 for arbitrary switching signals.

V. Conclusion

We have studied switched nonlinear DAEs with respect to existence and uniqueness of solution in the presence of impulses. A theorem with sufficient conditions for the existence of local solution of ITP is presented. Moreover, its extension to switched nonlinear DAEs is presented which is possible under the assumption that no finite escape time occurs between the switches. We provide some simple water network example where this solution framework is applicable. We suspect, that this solution framework is indeed applicable for all water networks under mild topological assumptions; this is a topic of ongoing research.

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