Observer Design based on Constant-Input Observability for DAEs

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For differential-algebraic equations (DAEs) an observability notion is considered which assumes the input to be unknown and constant. Based on this, an observer design is proposed.

1 Introduction

Classical observability assumes the system's input to be known. For systems with unknown inputs, [5] and [3] proposed to assume the inputs to be governed by a known dynamic and add this dynamic to the system. We will generalize this approach in the simplest case, i.e. for constant inputs, to DAEs.

Constant-input detectability can be defined similar to constant-input observability. An observer for constant-input detectable systems (with time-varying input) will be considered. In the last section, the notion of constant-input observability will be applied to a power network model. It turns out that the property solely depends on the underlying graph and can be described by the Laplacian.

Notation 1.1 For a matrix $M \in \mathbb{R}^{n\times n}$ and two index sets $V, W \subseteq \{1,\dots,n\}$ denote by $M_{V,W} \in \mathbb{R}^{|V|\times|W|}$ the submatrix of M consisting of the rows corresponding to V and the columns corresponding to W. Analogously, the subvector $x_V \in \mathbb{R}^{|V|}$ of $x \in \mathbb{R}^n$ is defined.

2 Constant-input observability

We consider linear differential-algebraic equations of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

with $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{q\times p}$. In the remainder, the time-dependence will not be stated explicitly. We assume *regularity* of the matrix pencil (E, A), i.e. $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$. The *behavior* of (1) is given by

$$\mathcal{B} := \{\ (u,x,y) \in \mathcal{C}^\infty \ \mid\ (u,x,y) \text{ solves (1) }\}.$$

Definition 2.1 The system (1) is *observable* iff it holds for all $(u_1,x_1,y_1), (u_2,x_2,y_2) \in \mathcal{B}$:

$$u_1 = u_2 \ \wedge\ y_1 = y_2 \quad \Rightarrow \quad x_1 = x_2.$$

The system (1) is *constant-input observable* iff it holds for all $(u_1,x_1,y_1), (u_2,x_2,y_2) \in \mathcal{B}$ with u_1, u_2 constant:

$$y_1 = y_2 \quad \Rightarrow \quad x_1 = x_2 \wedge u_1 = u_2.$$

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Constant-input observability can be characterized using the *augmented system*:

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}}_{=:\bar{E}} \frac{d}{dt}\begin{pmatrix} x \\ u \end{pmatrix} = \underbrace{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}}_{=:\bar{A}} \begin{pmatrix} x \\ u \end{pmatrix}, \tag{2a}$$

$$y = \underbrace{\begin{bmatrix} C & D \end{bmatrix}}_{=:\bar{C}} \begin{pmatrix} x \\ u \end{pmatrix}. \tag{2b}$$

The augmented system (2) is regular iff (1) is regular. Furthermore, both systems have the same index.

Lemma 2.2 ([4]) *The system* (1) *is constant-input observable iff it is observable and it holds*

$$\ker \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \{0\}. \tag{3}$$

For $\ker \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] = \{0\}$, constant-input observability is weaker than strong observability (as defined in [2]):

Example 2.3 The system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

is constant-input observable, but not strong observable. Indeed, $x(0) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $u(t) = \mathrm{e}^t$ give $y = 0$.

3 Constant-input observer

Based on constant-input detectability we will define an observer and apply it to systems with nonconstant input.

Definition 3.1 The system (1) is *constant-input detectable* iff for all $(u_1,x_1,y), (u_2,x_2,y) \in \mathcal{B}$ with u_1, u_2 constant it holds:

$$u_1 = u_2 \ \wedge\ \lim_{t\to\infty} x_1(t) - x_2(t) = 0.$$

It is equivalent to detectability of (2) together with the blockmatrix condition (3).

Let the regular DAE (1) be constant-input detectable. Constructing an observer for the then detectable augmented system (2) gives

$$\bar{E}\frac{d}{dt}\begin{pmatrix} \hat{x} \\ \hat{u} \end{pmatrix} = \bar{A}\begin{pmatrix} \hat{x} \\ \hat{u} \end{pmatrix} + L\,(y - \hat{y}), \tag{4a}$$

$$\hat{y} = \bar{C}\begin{pmatrix} \hat{x} \\ \hat{u} \end{pmatrix} \tag{4b}$$

with observer gain L. If (1) has a time-varying input u, the error $e_{xu} := \binom{x-\hat{x}}{u-\hat{u}}$ of the observer is described by

$$\bar{E}\dot{e}_{xu} = \left(\bar{A} - L\bar{C}\right) e_{xu} + \begin{bmatrix} 0 \\ I \end{bmatrix} \dot{u}. \tag{5}$$

For an ODE, the observer gain can be chosen such that (5) is asymptotically stable. Hence the error e_{xu} is bounded if $\dot{u}$ is bounded and it converges to zero if $\dot{u}$ does so. For the detectable, regular DAE (2) an observer of the form (4) can be constructed [1]. If the system (2) is impulse-observable (see [1]), we can achieve an index-1 observer, which guarantees that bounded $\dot{u}$ implies bounded e_{xu} and $\dot{u} \to 0$ implies $e_{xu} \to 0$. Else the observer might have a higher index and we would have to bound $\dot{u}, \ldots, u^{(n)}$ to get e_{xu} bounded. (2) is impulse-observable iff (1) is impulse-observable, i.e. iff

$$\operatorname{rank} \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + \operatorname{rank} E.$$

4 Application to power networks

A classical model for a power network is the swing equation [6]. It describes a transmission grid consisting of synchronous generators, loads and transmission lines. The generators are described by ODEs (6a). The loads are modeled as ODEs (6b) or algebraic constraints (6c). The transmission lines are represented by the load flow equations (6d), a simplification of the power flow equations: $f_i(\theta)$ describes the load flow from node i to the neighboring nodes. The overall system is then

$$\begin{aligned} M_i\ddot{\theta}_i + D_i\dot{\theta}_i &= P_i - f_i(\theta), && i \in g, && \text{(6a)} \\ D_i\dot{\theta}_i &= P_i - f_i(\theta), && i \in dl, && \text{(6b)} \\ 0 &= P_i - f_i(\theta), && i \in cl, && \text{(6c)} \\ f_i(\theta) &= \sum_{j \neq i} Y_{i,j} \sin\left(\theta_i - \theta_j\right), && i \in N. && \text{(6d)} \end{aligned}$$

where

- θ_i is the voltage phase angle at node i,
- M_i is the moment of inertia at generator node i,
- D_i is a damping constant at node i,
- P_i is the external power infeed/extraction at node i,
- $Y_{i,j}$ is the admittance between node i and node j,
- $cl, dl, g \subseteq \{1, \ldots, n\} =: N$ are the index sets of all constant load nodes, dynamic load nodes and generator nodes, respectively.

A linearization of (6) gives

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{7}$$

with

$$E := \begin{bmatrix} 0 & I & 0 & 0 \\ M_{g,g} & 0 & 0 & 0 \\ 0 & 0 & D_{dl,dl} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; B := \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$A := -\begin{bmatrix} -I & 0 & 0 & 0 \\ D_{g,g} & L_{g,g} & L_{g,dl} & L_{g,cl} \\ 0 & L_{dl,g} & L_{dl,dl} & L_{dl,cl} \\ 0 & L_{cl,g} & L_{cl,dl} & L_{cl,cl} \end{bmatrix}$$

and

$$x = \begin{pmatrix} \omega \\ \theta_g \\ \theta_{dl} \\ \theta_{cl} \end{pmatrix}, \quad u = \begin{pmatrix} P_g \\ P_{dl} \\ P_{cl} \end{pmatrix}.$$

Here, $\omega = \dot{\theta}_g$ is the angular velocity at the generators and $L_{i,j}$ is defined by $L_{i,j} := -Y_{i,j}$ for $i \neq j$ and $L_{i,i} = \sum_{j \neq i} Y_{i,j}$. L is a Laplace matrix. The DAE (7) is regular (and of index one) iff $\det(L_{cl,cl}) \neq 0$, i.e. if there is no connected component of constant load nodes.

A natural choice for the output y of (7) is to assume that for a certain subset $S \subset N$ of nodes both state and input are directly available in the output, i.e. measured, while for $S^c = N \setminus S$ no output information is available. This means the output can be written as

$$y = \begin{bmatrix} 0 & I_{S,N} \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I_{S,N} \end{bmatrix} u. \tag{8}$$

Theorem 4.1 ([4]) *The regular system* (7)*,* (8) *is constant-input observable iff*

$$\ker L_{S,S^c} = \{0\}. \tag{9}$$

(9) is also equivalent to constant-input detectability and strong observability.

Constant-input observability of (7), (8) depends solely on L, i.e. on the topology of the underlying graph. Direct consequences of this result are that at least every second node has to be in S and any node not in S has to be directly connected to a node in S.

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