Switch Observability for Homogeneous Switched DAEs ⋆

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Abstract: We introduce the notions of switching time observability and switch observability for homogeneous switched differential-algebraic equations (DAEs). In contrast to mode detection, they do not require observability of the individual modes and are thus more suitable for fault detection and identification. Based on results in (Küsters and Trenn, 2017) for switched ordinary differential equations (ODEs), we characterize these notions for homogeneous switched DAEs and propose an observer for switch observable systems.

Keywords: Differential-algebraic equations, Switched systems, Mode detection, Observer, Fault detection

1. INTRODUCTION

Many physical systems, e.g. electrical circuits and power grids, can be modeled as differential-algebraic equations (DAEs). Switching phenomena can occur in these systems due to active switching as well as component failures. Mode observability, i.e. the possibility to recover initial state and switching signal from the output, allows for detecting such failures. This concept has been widely studied for switched ordinary differential equations (ODEs), see e.g. (Babaali and Pappas, 2005; Elhamifar et al., 2009; Lou and Si, 2009; Vidal et al., 2003).

Since for mode observable systems it is in particular possible to recover the state for constant switching signals, each mode necessarily has to be observable. This might be too restrictive in the context of fault detection, where each mode describes a faulty or non-faulty variant of the system. Instead of mode observability, it would be sufficient to compute the state and the switching signal *if an error occurs.* This idea is formalized in the notion of switch observability, $(x, [\sigma_1])$-observability for short. In (Küsters and Trenn, 2017) it has been analyzed for switched ODEs.

This work deals with generalizing the aforementioned concept to homogeneous switched DAEs. The solution of a switched DAE might have jumps and even Dirac impulses at the switch, which makes it necessary to use the solution concept from (Trenn, 2012). It also leads to some differences in the analysis of the observability notions compared to switched ODEs. Some of these differences occur also for observability of switched DAEs with known switching signal, see (Petreczky et al., 2015).

We start this work with an example of an electrical circuit. After that, a short revision of switched DAEs is given in Section 2 and the observability properties are defined in Section 3. They are characterized in Section 4, mostly by rank-conditions on Kalman observability matrices. Finally, an observer based on switch observability is proposed in Section 5.

Example 1. (Electrical circuit). Consider an electrical circuit consisting of one capacitor, one switch and two inductors connected as in Figure 1. The output of this system is given by the voltage drop v_{L_2} at the second inductor. Modeling the system for both open and closed switch with the same variables makes it necessary to use differential-algebraic equations. With the state variable $x = [i_{L_1}\ i_S\ v_C\ v_{L_2}]^\top$ we arrive at the system

$$\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} x, \qquad y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x \tag{1}$$

for the closed switch and

$$\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x, \qquad y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x \tag{2}$$

for the open switch.

Clearly, (2) is not observable as v_{L_2} is always zero for consistent initial values. Neither is (1) observable as the state $[i_0\ -i_0\ 0\ 0]^\top$ is an equilibrium with $y \equiv 0$.

If the state for the open circuit is nonzero and we close the switch, the output becomes nonzero and the switch can be noted by this change in y. If we start with a non-equilibrium state in (1) and open the switch, v_{L_2} jumps to zero, which also makes the switch observable.

It remains to show that the switch can also be noted if we start in the nonzero equilibrium $[i_0\ -i_0\ 0\ 0]^\top$ of (1). Note that we have $y = 0$ before and after the switch. The equation $L_2 \frac{\mathrm{d}}{\mathrm{d}t} i_S = v_{L_2}$ holds both for open and closed

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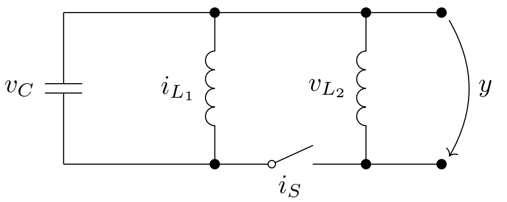

Fig. 1. Electrical circuit from Example 1.

switch and thus also during the switch. As i_S jumps from $i_0 \neq 0$ to zero, v_{L_2} contains the derivative of a jump, i.e. a Dirac impulse. This impulse also occurs in the output and makes the switch detectable.

A more detailed analysis reveals that in case of a switch we can also conclude the state and whether the switch was opened or closed.

2. SWITCHED DAES

A *switching signal* is a piecewise constant, right-continuous function $\sigma : \mathbb{R} \to \mathcal{P} := \{1, \dots, P\}$, $P \in \mathbb{N}$, with locally finitely many discontinuities. The discontinuities of σ are also called *switching times*:

$$T_\sigma := \{\ t \mid t \text{ is a discontinuity of } \sigma\ \}.$$

We assume σ to be constant on $(-\infty, 0]$, i.e. $T_\sigma \subset \mathbb{R}_{>0}$. Consider the system

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0, \tag{3a}$$

$$y = C_\sigma x \tag{3b}$$

with switching signal σ as above, E_i, $A_i \in \mathbb{R}^{n \times n}$, $C_i \in \mathbb{R}^{p \times n}$ and (E_i, A_i) regular for all $i \in \mathcal{P}$. The initial value x_0 is assumed to be consistent. By $x_{(x_0,\sigma)}$ and $y_{(x_0,\sigma)}$ we denote the solution and the output of (3), respectively.

A matrix pair (E, A) is regular iff it can be transformed to quasi-Weierstraß form, i.e. iff there exist S, T invertible such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

with N nilpotent. Then we can define the *consistency projector* $\Pi := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}$ and the matrices

$$\Pi^{\text{diff}} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S, \quad A^{\text{diff}} := \Pi^{\text{diff}} A, \quad C^{\text{diff}} := C\Pi,$$

$$\Pi^{\text{imp}} := T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] S, \quad E^{\text{imp}} := \Pi^{\text{imp}} E, \quad C^{\text{imp}} := C\,(I - \Pi).$$

$\mathcal{V}^* := \operatorname{im} \Pi$ gives the *consistency space*. Set $\mathcal{W}^* := \ker \Pi$.

Regularity of (E, A) is equivalent to $E\dot{x} = Ax$ having a unique classical solution for each consistent initial value $x(t_0) = x_0$. It is also equivalent to having a solution in the space of piecewise smooth distributions, $\mathbb{D}_{pw\mathcal{C}^\infty}$, for every initial condition $x(t_0^-) = x_0$ (Trenn, 2009). A nonconsistent initial value $x(t_0^-)$ causes a jump in the state and possibly also Dirac impulses and their derivatives. $\mathbb{D}_{pw\mathcal{C}^\infty}$ is a subspace of the distributions that contains piecewise smooth functions and their derivatives. In particular, Dirac impulses are included. It allows for restrictions to intervals and has a (noncommutative) multiplication. Both are not possible for distributions in general. The solution of $E\dot{x} = Ax$ with (possibly inconsistent) $x(t_0^-) = x_0$ can be expressed as

$$x(t) = \mathrm{e}^{A^{\text{diff}}(t - t_0)} \Pi x_0, \quad x[t_0] = -\sum_{i=0}^{n} \left(E^{\text{imp}}\right)^{i+1} \delta_{t_0}^{(i)} x_0,$$

where δ_{t_0} denotes the Dirac impulse concentrated at t_0, $\delta_{t_0}^{(i)}$ its i-th derivative and $x[\cdot]$ the impulsive part of the solution. For $t \neq t_0$ we have $x[t] = 0$. In particular, the solution $x_{(x_0,\sigma)}$ of (3) is not a classical function, but a distribution. It can, however, be written as the sum of a piecewise smooth function and an impulsive part at the switching instants.

The matrices $\mathcal{O}^{\text{diff}}$ and $\mathcal{O}^{\text{imp}}$ are defined as

$$\mathcal{O}^{\text{diff}} := \begin{bmatrix} C^{\text{diff}} \\ C^{\text{diff}} A^{\text{diff}} \\ \vdots \\ C^{\text{diff}} \left(A^{\text{diff}}\right)^{2n-1} \end{bmatrix}, \quad \mathcal{O}^{\text{imp}} := \begin{bmatrix} C^{\text{imp}} E^{\text{imp}} \\ C^{\text{imp}} \left(E^{\text{imp}}\right)^2 \\ \vdots \\ C^{\text{imp}} \left(E^{\text{imp}}\right)^{n-1} \end{bmatrix}.$$

$\mathcal{O}^{\text{diff}}$ is the Kalman observability matrix of the ODE-part. $\mathcal{O}^{\text{imp}}$ can be used to describe impulses caused by switches.

Remark 2. Using the Cayley-Hamilton theorem, one can show that only n rowblocks are required in the Kalman-observability matrix of an unswitched linear system. For our purpose, at most $2n$ rowblocks are needed for $\mathcal{O}^{\text{diff}}$ as an analysis of the augmented systems will show. For $\mathcal{O}^{\text{imp}}$, fewer rowblocks are sufficient as the matrix E^{imp} is nilpotent.

There are several observability notions for unswitched DAEs (Berger et al., 2017). Relevant here is behavioral observability:

Definition 3. The system (3) is called observable iff it holds $x_1 = x_2$ for all classical solutions (x_1, y_1), (x_2, y_2) of (3) with $y_1 = y_2$.

This is equivalent to zero observability, i.e. $y = 0$ implying $x = 0$. It can be characterized by $\ker \mathcal{O}^{\text{diff}} \cap \mathcal{V}^* = \{0\}$, i.e. it depends only on the differentiable part of the DAE. The condition is equivalent to $\operatorname{rank} \mathcal{O}^{\text{diff}} = \dim \mathcal{V}^*$ as it holds $\mathcal{O}^{\text{diff}} = \mathcal{O}^{\text{diff}} \Pi$.

Remark 4. (Notations). Let $y^{[\nu]}$ denote the vector of y and its derivatives up to order $\nu - 1$. For $y[t_S] = \sum_{i=0}^{n-2} \alpha_i \delta_{t_S}^{(i)}$ let $\overline{y[t_S]}$ be given as $\overline{y[t_S]} = \begin{bmatrix} \alpha_0^\top & \cdots & \alpha_{n-2}^\top \end{bmatrix}^\top$. This means

$$y_{(x_0,\sigma)}^{[2n]}(t_S^+) = \mathcal{O}^{\text{diff}}_{\sigma(t_S^+)} x_{(x_0,\sigma)}(t_S^-)$$

$$\text{and} \quad \overline{y_{(x_0,\sigma)}[t_S]} = -\mathcal{O}^{\text{imp}}_{\sigma(t_S^+)} x_{(x_0,\sigma)}(t_S^-).$$

3. OBSERVABILITY NOTIONS

Before defining observability of state and switching signal, we first have to observe that some trivial case has to be excluded: If the initial value x_0 is zero, the state $x_{(x_0,\sigma)}$ and thus also the output $y_{(x_0,\sigma)}$ stays zero, independent of the switching signal. Hence one cannot determine σ in this case. Another, closely related, problem is that if the state jumps to zero at a switch, no later switch can be observed and we can at most determine the switching signal until the zero-jump of the state.

Excluding jumps to zero is not a reasonable approach as this would mean a restriction to ODE-dynamics. Thus it is necessary to generalize the concept of observability and to consider equivalence classes of switching signals. This approach has been introduced in (Kaba, 2014) for invertability of switched systems and has been applied to observ-

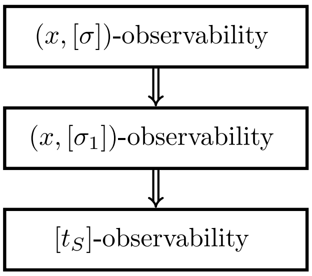

Fig. 2. Relation of the observability notions.

ability of inhomogeneous switched ODEs in (Küsters and Trenn, 2017).

Definition 5. For given x_0 the switching signals σ and $\widetilde{\sigma}$ are equivalent, denoted by $\sigma \overset{x_0}{\sim} \widetilde{\sigma}$, iff

- $x_{(x_0,\sigma)} = x_{(x_0,\tilde{\sigma})}$ and
- $\sigma(t) = \widetilde{\sigma}(t)$ except for intervals I with $\left(x_{(x_0,\sigma)}\right)_I = 0$.

The corresponding equivalence class is denoted by

$$[\sigma_{x_0}] := \left\{ \ \widetilde{\sigma} \ \middle| \ \widetilde{\sigma} \overset{x_0}{\sim} \sigma \ \right\}.$$

Furthermore, we define the *essential switching times*

$$T_{[\sigma_{x_0}]} := \bigcap_{\hat{\sigma} \overset{x_0}{\sim} \sigma} T_{\hat{\sigma}}.$$

Using equivalence classes of switching signals means that we are only interested in their values on the interval $\mathcal{I}_{(x_0,\sigma)} := \left\{ \ t \ \middle| \ x_{(x_0,\sigma)}(t^+) \neq 0 \vee x_{(x_0,\sigma)}[t] \neq 0 \ \right\}$.

The following observability notions are based on equivalence classes of switching signals:

Definition 6. The system (3) is called

- $[t_S]$-*observable* (switching time observable) if and only if for all σ, $\widetilde{\sigma}$, $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\widetilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ it holds
$$T_{[\sigma_{x_0}]} \neq T_{[\tilde{\sigma}_{\tilde{x}_0}]} \ \Rightarrow \ y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}.$$
- $[\sigma]$-*observable* (switching signal observable) if and only if for all σ, $\widetilde{\sigma}$, $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\widetilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ it holds
$$[\sigma_{x_0}] \neq [\widetilde{\sigma}_{\tilde{x}_0}] \ \Rightarrow \ y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}.$$
- $(x, [\sigma])$-*observable* if and only if for all σ, $\widetilde{\sigma}$, $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\widetilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ it holds
$$(x_0, [\sigma_{x_0}]) \neq (\widetilde{x}_0, [\widetilde{\sigma}_{\tilde{x}_0}]) \ \Rightarrow \ y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}. \tag{4}$$
- $(x, [\sigma_1])$-*observable* (switch observable) if and only if it holds (4) for all σ, $\widetilde{\sigma}$, $x_0 \in \mathcal{V}^*_{\sigma(0)}$, $\widetilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ with
$$1 \leq \min \left\{ \ |T_{\hat{\sigma}}| \ \middle| \ \hat{\sigma} \overset{x_0}{\sim} \sigma \ \right\}. \tag{5}$$

$(x, [\sigma])$-observability is closely related to mode observability and equivalent to $[\sigma]$-observability, see (Küsters and Trenn, 2017). The relation of $(x, [\sigma])$-, $(x, [\sigma_1])$- and $[t_S]$-observability is illustrated in Figure 2. These relations and counterexamples for their inversions have been shown for switched ODEs in (Küsters and Trenn, 2017).

4. OBSERVABILITY CHARACTERIZATIONS

In this section, we characterize the observability notions introduced in Definition 6 and relate them to the results for switched ODEs from (Küsters and Trenn, 2017).

4.1 $(x, [\sigma])$-observability

Observability of each mode, i.e. $\operatorname{rank} \mathcal{O}^{\text{diff}}_i = \dim \mathcal{V}^*_i$ for all $i \in \mathcal{P}$, is necessary for $(x, [\sigma])$-observability. Furthermore it is necessary to have

$$\operatorname{rank} \left[\mathcal{O}^{\text{diff}}_i \ \mathcal{O}^{\text{diff}}_j \right] = \dim \mathcal{V}^*_i + \dim \mathcal{V}^*_j \quad \forall i \neq j \tag{6}$$

as otherwise there would be $(x_0, \widetilde{x}_0) \neq (0, 0)$ and modes $i \neq j$ with $y_{(x_0,i)} = y_{(\tilde{x}_0,j)}$. (6) corresponds to and generalizes the criterion for (x, σ)-observability of homogeneous switched ODEs, which is

$$\operatorname{rank} [\mathcal{O}_i \ \mathcal{O}_j] = 2n \quad \forall i \neq j.$$

However, (6) is not sufficient for $(x, [\sigma])$-observability as impulses at a switch are relevant and make it necessary for $(x, [\sigma])$-observability to determine the ingoing mode:

Example 7. The system (3) with $\mathcal{P} = \{1, 2, 3\}$ and modes

$$\begin{aligned}
(E_1, A_1, C_1) &= \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right), \\
(E_2, A_2, C_2) &= \left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \right), \\
(E_3, A_3, C_3) &= \left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \right).
\end{aligned}$$

satisfies (6). Nevertheless, for $x_0 = [0 \ 1]^\top$ and

$$\sigma(t) := \begin{cases} 1, & t < 1, \\ 2, & t \geq 1, \end{cases} \qquad \widetilde{\sigma}(t) := \begin{cases} 1, & t < 1, \\ 3, & t \geq 1 \end{cases}$$

we have

$$x_{(x_0,\sigma)} = x_0 \mathbb{1}_{(-\infty,1)} - x_0 \delta, \quad x_{(x_0,\tilde{\sigma})} = x_0 \mathbb{1}_{(-\infty,1)},$$

and $y_{(x_0,\tilde{\sigma})} = y_{(x_0,\tilde{\sigma})} = x_0 \mathbb{1}_{(-\infty,1)}$. As the solutions $x_{(x_0,\sigma)}$, $x_{(x_0,\tilde{\sigma})}$ are not equal, the switching signals are not equivalent and thus the system is not $(x, [\sigma])$-observable.

The problem of Example 7 is that, although the state jumps to zero, there are impulses at the switch that make it necessary to distinguish the ingoing modes. To prevent this issue, we need that for any modes i, j, $k \in \mathcal{P}$ pairwise different and any consistent initial state $x_1 \in \mathcal{V}^*_i \setminus \{0\}$ with $x_1 \in \mathcal{W}^*_j \cap \mathcal{W}^*_k$, i.e. which jumps to zero in both modes j and k, the impulsive part of the outputs are different or there is no impulse in the state of any of these two solutions. This means we require

$$\mathcal{O}^{\text{imp}}_j x_1 = \mathcal{O}^{\text{imp}}_k x_1 \ \Rightarrow \ E^{\text{imp}}_j x_1 = E^{\text{imp}}_k x_1 = 0.$$

Thus we get the following lemma:

Lemma 8. The system (3) is $(x, [\sigma])$-observable if and only if it holds (6) and for all i, j, k pairwise different

$$\mathcal{V}^*_i \cap \mathcal{W}^*_j \cap \mathcal{W}^*_k \cap \ker \left(\mathcal{O}^{\text{imp}}_j - \mathcal{O}^{\text{imp}}_k \right) \subseteq \ker \begin{bmatrix} E^{\text{imp}}_j \\ E^{\text{imp}}_k \end{bmatrix}. \tag{7}$$

(7) ensures that jumps to zero with nonzero impulses can be distinguished.

Before proving Lemma 8 we relate (6) and (7) to the *augmented system* $\Sigma_{i,j}$, i, $j \in \mathcal{P}$:

$$\Sigma_{i,j}: \quad \begin{aligned} \begin{bmatrix} E_i & 0 \\ 0 & E_j \end{bmatrix} \dot{\xi} &= \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \xi, \\ y_\Delta &= [C_i \ -C_j] \, \xi. \end{aligned}$$

Regularity of $\Sigma_{i,j}$ follows by that of the underlying modes. (6) is equivalent to observability of each augmented system $\Sigma_{i,j}$, $i \neq j$. While (7) cannot be characterized by the augmented systems, a sufficient condition can be given: (7) holds for i, j, k pairwise different if $\Sigma_{i,j}$ is impulse observable for all $i \neq j$. (See (Berger et al., 2017) for impulse observability.)

(7) is also satisfied if all modes have nilpotence index one.

Proof of Lemma 8. The necessity of (6) and (7) has been shown above.

Let $\sigma, \widetilde{\sigma}, x_0 \in \mathcal{V}^*_{\sigma(0)}$ and $\widetilde{x}_0 \in \mathcal{V}^*_{\widetilde{\sigma}(0)}$ be given. If $[\sigma_{x_0}] = [\widetilde{\sigma}_{\tilde{x}_0}]$ and $x_0 \neq \widetilde{x}_0$, we have $y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}$ by $\sigma(0) = \widetilde{\sigma}(0)$ and $\operatorname{rank} \mathcal{O}^{\text{diff}}_{\sigma(0)} = \dim \mathcal{V}^*_{\sigma(0)}$.

For $[\sigma_{x_0}] \neq [\widetilde{\sigma}_{\tilde{x}_0}]$ and $(x_0, \widetilde{x}_0) \neq (0,0)$ there exists a time $\hat{t}$ with $\sigma(\hat{t}^+) \neq \widetilde{\sigma}(\hat{t}^+)$ and

$$x_{(x_0,\sigma)}(\hat{t}^+) \neq 0 \vee x_{(\tilde{x}_0,\tilde{\sigma})}(\hat{t}^+) \neq 0 \vee x_{(x_0,\sigma)}[\hat{t}] \neq 0 \vee x_{(\tilde{x}_0,\tilde{\sigma})}[\hat{t}] \neq 0.$$

If $x_{(x_0,\sigma)}(\hat{t}^+)$ or $x_{(\tilde{x}_0,\tilde{\sigma})}(\hat{t}^+)$ is nonzero, (6) implies $y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}$. If both are zero and an impulsive part is nonzero, (7) gives $y_{(x_0,\sigma)}[\hat{t}] \neq y_{(\tilde{x}_0,\tilde{\sigma})}[\hat{t}]$. □

4.2 $[t_S]$-*observability*

For switched ODEs, the criterion for t_S-observability is

$$\operatorname{rank}(\mathcal{O}_i - \mathcal{O}_j) = n \quad \forall i \neq j.$$

To derive a similar condition for (3), we consider a single switch signal σ with switching time t_S. Set $i := \sigma(t_S^-)$, $j := \sigma(t_S^+)$ and assume that there exists a $x_1 \in \mathcal{V}^*_i \setminus \{0\}$ with $\mathcal{O}^{\text{diff}}_i x_1 = \mathcal{O}^{\text{diff}}_j x_1$ and $\mathcal{O}^{\text{imp}}_j x_1 = 0$. The first means that the output is smooth for $x_{(x_0,\sigma)}(t_S^-) = x_1$ and the second implies that there is no impulse. Hence for $x_0 := \mathrm{e}^{-A^{\text{diff}}_i} x_1$ and $\widetilde{\sigma} = i$ we get $y_{(x_0,\sigma)} = y_{(x_0,\tilde{\sigma})}$, i.e. the system is not $[t_S]$-observable.

Lemma 9. The system (3) is $[t_S]$-observable if and only if it holds for all $i \neq j$:

$$\operatorname{rank} \begin{bmatrix} \mathcal{O}^{\text{diff}}_i - \mathcal{O}^{\text{diff}}_j \Pi_i \\ \mathcal{O}^{\text{imp}}_j \Pi_i \end{bmatrix} = \dim \mathcal{V}^*_i. \tag{8}$$

Proof. Necessity is clear from the argumentation above. Now assume that $\sigma, \widetilde{\sigma}, x_0 \in \mathcal{V}^*_{\sigma(0)}, \widetilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ are given with $T_{[\sigma_{x_0}]} \neq T_{[\tilde{\sigma}_{\tilde{x}_0}]}$. W.l.o.g. assume $t_S \in T_{[\sigma_{x_0}]} \setminus T_{[\tilde{\sigma}_{\tilde{x}_0}]}$. $t_S \in T_{[\sigma_{x_0}]}$ implies that $x_1 := x_{(x_0,\sigma)}(t_S^-)$ is nonzero. Set $i := \sigma(t_S^-)$, $j := \sigma(t_S^+)$. Then (8) gives

$$\mathcal{O}^{\text{diff}}_i x_1 \neq \mathcal{O}^{\text{diff}}_j x_1 \text{ or } \mathcal{O}^{\text{imp}}_i x_1 \neq 0.$$

This means $y_{(x_0,\sigma)}$ is nonsmooth at t_S or has an impulse at this time. In particular $y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}$, as the latter is smooth and impulse-free at t_S. □

4.3 $(x, [\sigma_1])$-*observability*

Again we start by recalling the corresponding condition for switched ODEs. A homogeneous switched ODE is (x, σ_1)-observable iff it holds for all $i, j, p, q \in \mathcal{P}$ with $i \neq j, p \neq q$ and $(i,j) \neq (p,q)$:

$$\operatorname{rank} \begin{bmatrix} \mathcal{O}_i & \mathcal{O}_p \\ \mathcal{O}_j & \mathcal{O}_q \end{bmatrix} = 2n.$$

In contrast to the ODE-case, the considered mode pairs (i,j) and (p,q) with the assumptions above might be equivalent if $i = p$ and the state jumps impulse-freely to zero for the solutions corresponding to both mode pairs.

Lemma 10. The system (3) is $(x, [\sigma_1])$-observable if and only if it is $[t_S]$-observable and it holds for all $i, j, p, q \in \mathcal{P}$ with $i \neq j$, $p \neq q$ and $(i,j) \neq (p,q)$:

$$\operatorname{rank} \begin{bmatrix} \mathcal{O}^{\text{diff}}_i & \mathcal{O}^{\text{diff}}_p \\ \mathcal{O}^{\text{diff}}_j \Pi_i & \mathcal{O}^{\text{diff}}_q \Pi_p \\ \mathcal{O}^{\text{imp}}_j \Pi_i & \mathcal{O}^{\text{imp}}_q \Pi_p \end{bmatrix} = \dim \mathcal{V}^*_i + \dim \mathcal{V}^*_p - \dim \mathcal{M}_{i,j,p,q}, \tag{9}$$

where $\mathcal{M}_{i,j,p,q} := \begin{cases} \mathcal{V}^*_i \cap \ker E_j \cap \ker E_q, & i = p, \\ \{0\}, & i \neq p. \end{cases}$

Proof. Clearly, $[t_S]$-observability is necessary for $(x, [\sigma_1])$-observability.

Assume that (9) does not hold, i.e. there exist i, j, p, q with the conditions above and $x_1 \in \mathcal{V}^*_i$, $\widetilde{x}_1 \in \mathcal{V}^*_p$ with $(x_1, \widetilde{x}_1) \neq (0,0)$ and

$$\begin{bmatrix} \mathcal{O}^{\text{diff}}_i & \mathcal{O}^{\text{diff}}_p \\ \mathcal{O}^{\text{diff}}_j \Pi_i & \mathcal{O}^{\text{diff}}_q \Pi_p \\ \mathcal{O}^{\text{imp}}_j \Pi_i & \mathcal{O}^{\text{imp}}_q \Pi_p \end{bmatrix} \begin{bmatrix} x_1 \\ \widetilde{x}_1 \end{bmatrix} = 0.$$

Define $x_0 := \mathrm{e}^{-A^{\text{diff}}_i} x_1$, $\widetilde{x}_0 := \mathrm{e}^{-A^{\text{diff}}_p} \widetilde{x}_1$ and

$$\sigma(t) := \begin{cases} i, & t < 1, \\ j, & t \geq 1, \end{cases} \quad \widetilde{\sigma}(t) := \begin{cases} p, & t < 1, \\ q, & t \geq 1. \end{cases}$$

Then it holds $y_{(x_0,\sigma)} = y_{(\tilde{x}_0,\tilde{\sigma})}$.

If $i \neq p$ the switching signals are not equivalent.

For $i = p$, the set of initial states $x_0 = \tilde{x}_0$ that give equivalent switching signals is described by $\mathrm{e}^{-A^{\text{diff}}_i} \mathcal{M}_{i,j,p,q}$. Thus, if (9) is violated, there exist $(x_0, \tilde{x}_0) \neq (0,0)$ not giving equivalent switching signals, but identical output. Hence $(x, [\sigma_1])$-observability is violated.

Now assume that the system is $[t_S]$-observable and satisfies (9). Let σ, $\widetilde{\sigma}$, $x_0 \in \mathcal{V}^*_{\sigma(0)}$ and $\widetilde{x}_0 \in \mathcal{V}^*_{\tilde{\sigma}(0)}$ be given with (5). Due to t_S-observability we can assume $T_{[\sigma_{x_0}]} = T_{[\tilde{\sigma}_{\tilde{x}_0}]}$ as otherwise we would have $y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}$. As $[t_S]$-observability implies

$$\operatorname{rank} \begin{bmatrix} \mathcal{O}^{\text{diff}}_i \\ \mathcal{O}^{\text{diff}}_j \Pi_i \\ \mathcal{O}^{\text{imp}}_j \Pi_i \end{bmatrix} = \dim \mathcal{V}^*_i, \quad \forall i \neq j, \tag{10}$$

$[\sigma_{x_0}] = [\widetilde{\sigma}_{\tilde{x}_0}]$ and $x_0 \neq \widetilde{x}_0$ would yield $y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}$.

Now let $[\sigma_{x_0}] \neq [\widetilde{\sigma}_{\tilde{x}_0}]$. This already implies $(x_0, \widetilde{x}_0) \neq (0,0)$. There exists a common switching time t_S with $\sigma(t_S^-) \neq \widetilde{\sigma}(t_S^-)$ or $\sigma(t_S^+) \neq \widetilde{\sigma}(t_S^+)$. Define $i := \sigma(t_S^-)$, $j := \sigma(t_S^+)$, $p := \widetilde{\sigma}(t_S^-)$ and $q := \widetilde{\sigma}(t_S^+)$. Then the conditions of (9) are satisfied and we can conclude $y_{(x_0,\sigma)} \neq y_{(\tilde{x}_0,\tilde{\sigma})}$ if the switching signals are not equivalent. □

The following example shows that (9) is indeed not sufficient for switching time observability:

Example 11. Consider the switched DAE (3) with $\mathcal{P} = \{1,2\}$ and modes (E_1, A_1, C_1), (E_2, A_2, C_2) given by

$$\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) \text{ and } \left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1/2 & 0 \\ 1/2 & 0 \end{smallmatrix}\right]\right)$$

respectively. The second mode has the consistency projector $\Pi_2 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $A^{\text{diff}}_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The system is not $[t_S]$-observable as we have $\operatorname{rank}\left(\mathcal{O}^{\text{diff}}_1 - \mathcal{O}^{\text{diff}}_2 \Pi_1\right) = 1 < 2$. However, (9) is satisfied:

$$\operatorname{rank}\begin{bmatrix}\mathcal{O}_1^{\text{diff}} & \mathcal{O}_2^{\text{diff}}\\ \mathcal{O}_2^{\text{diff}}\Pi_1 & \mathcal{O}_1^{\text{diff}}\Pi_2\\ \mathcal{O}_2^{\text{imp}}\Pi_1 & \mathcal{O}_1^{\text{imp}}\Pi_2\end{bmatrix} = \operatorname{rank}\begin{bmatrix}C_1 & C_2^{\text{diff}}\\ C_2^{\text{diff}} & C_1\Pi_2\\ 0 & 0\end{bmatrix} = 3.$$

Example 12. The system given in Example 1 is $[t_S]$- and $(x,[\sigma_1])$-observable, but not $(x,[\sigma])$-observable.

5. OBSERVER

We will now construct an observer for systems that are $(x,[\sigma_1])$-observable. The main idea is to collect information (dynamics before the switch, impulses at the switch and dynamics after the switch) and later combine this knowledge to obtain the switching signal and the state.

Consider a switched DAE (3) on $[0,T]$ with σ having exactly one switching time $t_S \in (0,T)$. If the system is $(x,[\sigma_1])$-observable, both switching signal and state are uniquely defined by $y^{[2n]}(t_S^-)$, $y^{[2n]}(t_S^+)$ and $y[t_S]$ as the solutions $i=\sigma(t_S^-)$, $j=\sigma(t_S^+)$ and $x_1 = x_{(x_0,\sigma)}(t_S^-)$ of

$$\begin{bmatrix}0\\ y^{[2n]}(t_S^-)\\ y^{[2n]}(t_S^+)\\ \overline{y[t_S]}\end{bmatrix} = \begin{bmatrix}I-\Pi_i\\ \mathcal{O}_i^{\text{diff}}\\ \mathcal{O}_j^{\text{diff}}\\ -\mathcal{O}_j^{\text{imp}}\end{bmatrix} x_1.$$

("Uniquely" up to equivalence classes of switching signals.)

Instead of using the exact derivatives of the outputs, we utilize Luenberger-observers for both time intervals $(0,t_S)$ and (t_S,T). Note that, as the individual modes of a $(x,[\sigma_1])$-observable system do not have to be observable, the Luenberger-observers might converge for several modes. In the algorithm, we store all these modes as candidates.

In (Tanwani and Trenn, 2017) an observer was given for switched DAEs with known switching signal. We could use this observer for all possible mode sequences, as only the correct one returns a feasible solution. The computations can be simplified heavily as observers before and after the switch can be used independently. Hence we only have to combine the results of the individual observers and check these combinations.

To collect and combine information from the different time intervals and the switching instant, we define for all modes i matrices Z_i^{cons}, Z_i^{diff} and Z_i^{imp} with orthonormal columns such that their images are given by

$$(\mathcal{V}_i^*)^\perp,\ \left(\ker\mathcal{O}_i^{\text{diff}}\right)^\perp \quad\text{and}\quad \left(\ker\mathcal{O}_i^{\text{imp}}\right)^\perp,$$

respectively. $(x,[\sigma_1])$-observability gives

$$\operatorname{im}\Pi_i\cap\ker\mathcal{O}_i^{\text{diff}}\cap\ker\mathcal{O}_j^{\text{imp}}\cap\ker\mathcal{O}_j^{\text{diff}} = \{0\}\quad \forall i\neq j,$$

i.e. observability for known, nonconstant σ, see (Petreczky et al., 2015). Thus for $i\neq j$ there exists a matrix $U_{i,j}$ with

$$\left[Z_i^{\text{cons}}\ Z_i^{\text{diff}}\ Z_j^{\text{imp}}\ Z_j^{\text{diff}}\right]U_{i,j} = I. \tag{11}$$

We will now compute estimates of the individual components $\left(Z_i^{\text{cons}}\right)^\top x$, $\left(Z_i^{\text{diff}}\right)^\top x$, $\left(Z_j^{\text{imp}}\right)^\top x$ and $\left(Z_j^{\text{diff}}\right)^\top x$ and then combine these values to an estimate of x.

- Consistency of $x(t)$ with mode i on $t\in[0,t_S)$ gives $x(t)\in\mathcal{V}_i^*$, i.e. $\left(Z_i^{\text{cons}}\right)^\top x(t) = 0$ for $t\in[0,t_S)$.
- $z_i^{\text{diff}} := \left(Z_i^{\text{diff}}\right)^\top x$ describes the "observable part" of the state on the interval $[0,t_S)$. It is governed by the observable system
$$\dot z_i^{\text{diff}} = \left(\left(Z_i^{\text{diff}}\right)^\top A_i^{\text{diff}} Z_i^{\text{diff}}\right) z_i^{\text{diff}}, \tag{12a}$$
$$y_z = \left(C_i^{\text{diff}} Z_i^{\text{diff}}\right) z_i^{\text{diff}}, \tag{12b}$$
see (Tanwani and Trenn, 2017). Hence we can use a Luenberger-observer to estimate $Z_i^- := z_i^{\text{diff}}(t_S^-)$.
- Similarly, a Luenberger-observer can be used to estimate $z_j^{\text{diff}}(T)$. Backward-propagation gives $z_j^+ := \mathrm{e}^{-S_j^{\text{diff}}(T-t_S)} z_j^{\text{diff}}(T)$, for which it holds
$$z_j^+ = \left(Z_j^{\text{diff}}\right)^\top x(t_S^+) = \left(Z_j^{\text{diff}}\right)^\top x(t_S^-).$$
- To get $z_j^{\text{imp}} := \left(Z_j^{\text{imp}}\right)^\top x(t_S^-)$ from $y[t_S]$ we need to choose a matrix U_j^{imp} such that
$$-\left(\mathcal{O}_j^{\text{imp}}\right)^\top U_j^{\text{imp}} = Z_j^{\text{imp}}.$$
As it holds $\overline{y[t_S]} = -\mathcal{O}_j^{\text{imp}} x(t_S^-)$, one gets
$$z_j^{\text{imp}} = -\left(U_j^{\text{imp}}\right)^\top \mathcal{O}_j^{\text{imp}} x(t_S^-) = \left(Z_j^{\text{imp}}\right)^\top \overline{y[t_S]}.$$

The pre-switch state $x(t_S^-)$ can then be obtained as

$$x(t_S^-) = U_{i,j}^\top \left[0\ \left(z_i^-\right)^\top\ \left(z_j^{\text{imp}}\right)^\top\ \left(z_j^+\right)^\top\right]^\top.$$

Algorithm 1 now uses all possible modes $\hat\imath$, $\hat\jmath$ and computes candidate sets $\mathcal{P}^-$, $\mathcal{P}^+$, $\mathcal{P}^{\text{imp}}$ for the modes fitting to the dynamics before, the dynamics after and the impulses at the switch, respectively. Afterwards, the estimated state for all candidates of mode pairs is computed and the corresponding output is compared to the actual output. A $(x,[\sigma_1])$-observable system returns only the mode pair (i,j) of the correct switching signal (and those for equivalent switching signals) as well as the corresponding state prior to the switch.

The switch observer will now be applied to an academic example.

Example 13. We define the system

$$(E_1,A_1,C_1) = \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}5&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&3\end{bmatrix}, \begin{bmatrix}3&2&1&0\end{bmatrix}\right),$$
$$(E_2,A_2,C_2) = \left(\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&1&3&1\end{bmatrix}\right),$$
$$(E_3,A_3,C_3) = \left(\begin{bmatrix}0&1&1&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&2\end{bmatrix}, \begin{bmatrix}3&3&5&1\end{bmatrix}\right),$$
$$(E_4,A_4,C_4) = \left(\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}2&2&2&2\end{bmatrix}\right),$$

The solution for $x_0 = [0\ 0\ 1\ 0]^\top$ and $\sigma(t) = \begin{cases}1, & t<1,\\ 3, & t\geq 1,\end{cases}$ is given by

$$x_{(x_0,\sigma)} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\mathbb{1}_{(-\infty,1)} - \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\delta_1 + \mathrm{e}^{t-1}\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}\mathbb{1}_{[1,\infty)}$$

with output $y_{(x_0,\sigma)} = \mathbb{1}_{(-\infty,1)} - 5\delta_1 + 2\mathrm{e}^t\mathbb{1}_{[1,\infty)}$.

We get $\mathcal{P}^- = \{1,2\}$, $\mathcal{P}^+ = \{2,3\}$ and $\mathcal{P}^{\text{imp}} = \{2,3,4\}$ as the output on $(-\infty,1)$ - a nonzero constant value -

Algorithm 1: Switch observer.

Data: t_S, T, $\mathcal{P}$, y

Result: M, $\xi_{\hat{i},\hat{j}}$ for $(\hat{i},\hat{j}) \in M$

$\mathcal{P}^- \leftarrow \emptyset$, $\mathcal{P}^{\text{imp}} \leftarrow \emptyset$, $\mathcal{P}^+ \leftarrow \emptyset$, $M \leftarrow \emptyset$;

for $\hat{i} \in \mathcal{P}$ **do**

- Compute $Z_{\hat{i}}^{\text{cons}}$, $Z_{\hat{i}}^{\text{diff}}$, $Z_{\hat{i}}^{\text{imp}}$ and $U_{\hat{i}}^{\text{imp}}$;
- Construct Luenberger observer for (12);
- **if** *Observer converges on* $[0,t_S)$ **then**
  - $\mathcal{P}^- \leftarrow \mathcal{P}^- \cup \{\hat{i}\}$; $\hat{z}_{\hat{i}}^- \leftarrow \hat{z}_{\hat{i}}(t_S^-)$;
- **if** *Observer converges on* (t_S,T) **then**
  - $\mathcal{P}^+ \leftarrow \mathcal{P}^+ \cup \{\hat{i}\}$; $\hat{z}_{\hat{i}}^+ \leftarrow \mathrm{e}^{-S_{\hat{i}}^{\text{diff}}(T-\tau)}\hat{z}_{\hat{i}}(T)$;
- **if** $y[t_S] \in \operatorname{im}\mathcal{O}_{\hat{i}}^{\text{imp}}$ **then**
  - $\mathcal{P}^{\text{imp}} \leftarrow \mathcal{P}^{\text{imp}} \cup \{\hat{i}\}$; $z_{\hat{i}}^{\text{imp}} \leftarrow \left(U_{\hat{i}}^{\text{imp}}\right)^\top \overline{y[t_S]}$;

for $\hat{i} \in \mathcal{P}^-$, $\hat{j} \in \mathcal{P}^{\text{imp}} \cap \mathcal{P}^+$, $\hat{i} \neq \hat{j}$ **do**

- Construct $U_{\hat{i},\hat{j}}$ with (11);
- $\xi_{\hat{i},\hat{j}} \leftarrow U_{\hat{i},\hat{j}}^\top \left[0 \ \left(z_{\hat{i}}^-\right)^\top \ \left(z_{\hat{j}}^{\text{imp}}\right)^\top \ \left(z_{\hat{j}}^+\right)^\top\right]^\top$;
- Solve (3) on $[0,T)$ with $x(t_S^-) = \xi_{\hat{i},\hat{j}}$, $\sigma(t) = \begin{cases} \tilde{i}, & t < t_S, \\ \tilde{j}, & t \geq t_S. \end{cases}$ Denote the output by $\hat{y}_{\hat{i},\hat{j}}$;
- **if** $y \approx \hat{y}_{\hat{i},\hat{j}}$ **then**
  - $M \leftarrow M \cup \{(\hat{i},\hat{j})\}$;

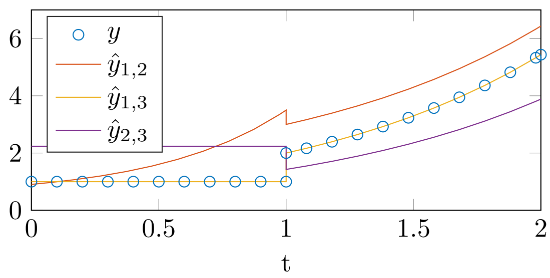

Fig. 3. *Smooth part* of the output and its estimates in Example 13.

can be produced by modes 1 and 2, the output on $(1,\infty)$ can be produced by modes 2, 3 and the impulse at the switch might come from mode 2, 3 or 4. This leads to $\sigma(t_S^-) \in \{1,2\}$ and $\sigma(t_S^+) \in \mathcal{P}^+ \cap \mathcal{P}^{\text{imp}} = \{2,3\}$. In the second part of Algorithm 1, the mode pairs $(1,2)$, $(1,3)$ and $(2,3)$ have to be considered. One can easily see from Figure 3 or the impulsive parts $\hat{y}_{1,2}[1] = -1.5\delta_1$, $\hat{y}_{1,3}[1] = -3\delta_1$, $\hat{y}_{2,3}[1] \approx -2.43\delta_1$ that Algorithm 1 will return only the correct mode pair $(1,3)$ along with the state.

The observer can be adapted to systems with more than one switch.

6. CONCLUSION

We characterized $[t_S]$- and $(x,[\sigma_1])$-observability for homogeneous switched DAEs and proposed an observer based on the latter notion. In a final example we pointed out the usefulness of this observer: Local information of the output is not sufficient for determining the current mode. Neither the dynamics before the switch, nor the dynamics after the switch or the impulse at the switch could be uniquely related to a mode. Only by combining the measurements at all three instances and using knowledge of how the state jumps to a consistent value at the switch we were able to determine the correct mode sequence and therewith also the state.

A future research topic is the $(x,[\sigma_1])$-observability of inhomogeneous switched DAEs.

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