Switch-observer for switched linear systems

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Abstract— **To determine the switching signal and the state of a switched linear system, one usually requires mode observability. This property requires that all individual modes are observable and that the modes are clearly distinguishable. In theory, mode detection allows to determine the active mode in an arbitrarily short time.**

If one enlarges the observation to an interval that contains a switch, both assumptions (observability of each mode and clearly distinct dynamics) can be relaxed. In [2] we formalized this observability notion, which we called *switch observability*. This concept is of particular interest for fault identification.

Based on switch-observability, we propose an observer. This observer combines the information obtained before and after a switching instant to determine both the state and the switching signal. It is analyzed and illustrated in an example.

I. Introduction

Switched systems can be used to model active switching or component failures of a physical system, e.g. the line outage of a power network. Fault identification is then a problem related to determining the switching signal and possibly also the state of the system. See [5] for a related problem formulation. Mode detection allows to determine the switching signal, but requires observability of each individual mode and clearly distinct dynamics. For fault identification, one can relax the requirement as we only need to determine the switching signal *if a switch occurs*. In [2] we formalized this as *switch observability*: The combined information before and after the switch has to suffice to determine the switching signal and the state.

The observer design for known switching signals has been considered e.g. in [8]. For other works on switching signal and state observation, see [1], [6]. In [1] the problem of determining state and switching signal is separated: At first the switching signal is estimated and, having this, the state is estimated. The procedure relies on observability of all modes. In [6], an observer is proposed under the assumption that a common Lyapunov function exists. Again, the problem of mode and state estimation is considered separately. In contrast to this, we want to allocate state information over two modes to arrive at both a state and a switching signal estimation.

II. Switch-observability

A *switching signal* is a piecewise constant, right-continuous function $\sigma : \mathbb{R} \to \mathcal{P} := \{1, \dots, N\}$, $N \in \mathbb{N}$, with locally finitely many discontinuities. The discontinuities of σ are also called *switching times*:

$$T_\sigma := \{ \, t_S \mid t_S \text{ is a discontinuity of } \sigma \, \} .$$

We assume that all switches occur for $t > 0$, i.e. $T_\sigma \subset \mathbb{R}_{>0}$. Consider switched ODEs of the form

$$\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma u, \quad x(0) = x_0, \\ y &= C_\sigma x + D_\sigma u, \end{aligned} \tag{1}$$

with switching signal σ, system matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times q}$, $C_i \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{p\times q}$ for all $i \in \mathcal{P}$ and smooth input $u : \mathbb{R} \to \mathbb{R}^q$. Denote solution and output of (1) by $x_{(x_0,\sigma,u)}$ and $y_{(x_0,\sigma,u)}$, respectively. We denote the unswitched ODE as

$$\Sigma : \quad \begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx + Du. \end{aligned} \tag{2}$$

Observability for known switching signals has been considered in [8]. As for nonlinear systems, the observability of state (and switching signal) can depend on the input. In the sequel, we consider a strong observability notion, i.e. we require observability for all inputs. Alternative approaches are requiring the existence of an input giving observability or requiring observability for generic inputs.

In [2], two formulations of switch-observability were introduced. The simpler version requires

(A1) u analytic,
(A2) $\ker B_i \cap \ker B_j \cap \ker (D_i - D_j) = \{0\} \; \forall i \neq j$.

In this work, we will stick to these assumptions. Note that (A2) can be omitted and (A1) can be replaced by merely smooth u if we consider equivalence classes of switching signals (see [2]). The proposed observer works also in this case. Actually, the observer works even for less regular inputs as smoothness is in particular necessary to determine switching times.

Definition 1: The system (1) is called *strongly* (x, σ_1)*-observable*, or *switch observable* iff for all smooth u and all $x_0 \neq 0$, σ non-constant, $\tilde{x}_0$ and $\tilde{\sigma}$ it holds

$$(x_0 \neq \tilde{x}_0 \, \vee \, \sigma \not\equiv \tilde{\sigma}) \; \Rightarrow \; y_{(x_0,\sigma,u)} \not\equiv y_{(\tilde{x}_0,\tilde{\sigma},u)}.$$

Without the assumption "σ non-constant", we arrive at strong (x, σ)-observability, or mode observability. This classical concept requires observability of each mode and is strictly stronger than switch-observability. A weaker notion is that of switching time observability: For t_S a switching time of σ, the output $y_{(x_0,\sigma,u)}$ has to be distinct from $y_{(\tilde{x}_0,\tilde{\sigma},u)}$ for $\tilde{\sigma}$ constant in a neighborhood of t_S. For details, see [2]. Switching-time observability corresponds to fault detection, switch observability to fault identification.

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Remark 1 (Notations): For the characterization of strong (x, σ_1)-observability, we need the following notations/concepts:

- Let $\mathcal{O}_i^{[\nu]}$ denote the *Kalman observability matrix* for mode i with ν row blocks, i.e.
$$\mathcal{O}_i^{[\nu]} = \begin{bmatrix} C_i^\top & (C_iA_i)^\top & \cdots & (C_iA_i^{\nu-1})^\top \end{bmatrix}^\top .$$
- Let $\Gamma_i^{[\nu]}$ denote the *Hankel matrix* for mode i with ν row blocks and ν column blocks, i.e.
$$\Gamma_i^{[\nu]} = \begin{bmatrix} D_i & & & \\ C_iB_i & \ddots & & \\ \vdots & \ddots & \ddots & \\ C_iA_i^{\nu-2}B_i & \cdots & C_iB_i & D_i \end{bmatrix}.$$
- Let $u^{[k]}$ denote the vector of u and its first $k-1$ derivatives. Then we have for the solution of (1):
$$y_{(x_0,\sigma,u)}^{[\nu]}(t^\pm) = \mathcal{O}_{\sigma(t^\pm)}^{[\nu]} x_{(x_0,\sigma,u)}(t) + \Gamma_{\sigma(t^\pm)}^{[\nu]} u^{[\nu]}(t) \quad \forall t,$$
where $f(t^\pm)$ denotes the limit from above/below at t, i.e. $\lim_{s\searrow t} f(t)$ and $\lim_{s\nearrow t} f(t)$.
- For two modes i, j let $\Sigma_{i,j}$ denote the *augmented system* given by
$$\Sigma_{i,j}: \quad \begin{aligned} \dot{\xi} &= \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \xi + \begin{bmatrix} B_i \\ B_j \end{bmatrix} u, \\ y_{\Delta_{i,j}} &= \begin{bmatrix} C_i & -C_j \end{bmatrix} \xi + (D_i - D_j)\, u. \end{aligned}$$
- The set of controllable weakly unobservable states of an unswitched system (2) is
$$\mathcal{R}(\Sigma) := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{aligned} &\exists\, u(\cdot) \text{ smooth}, T > 0 : \\ &y_{(x_0,u)} \equiv 0 \\ &\wedge\, x_{(x_0,u)}(T) = 0 \end{aligned} \right\},$$
see [9]. For the augmented system $\Sigma_{i,j}$, $\mathcal{R}(\Sigma_{i,j})$ describes the initial values x_0, $\tilde{x}_0$ that can be simultaneously steered to zero for the same output, i.e. $x_{(x_0,i,u)}(T) = x_{(\tilde{x}_0,j,u)}(T)$ and $y_{(x_0,i,u)} \equiv y_{(\tilde{x}_0,j,u)}$ for some input u.

These notations enable us to formulate a characterization of strong (x, σ_1)-observability:

Theorem 1 (See [2]): The switched system (1) is strongly (x, σ_1)-observable if and only if it satisfies
$$\mathcal{R}(\Sigma_{i,j}) = \{0\} \tag{3}$$
for all i, $j \in \mathcal{P}$, $i \neq j$ and
$$\operatorname{rank} \begin{bmatrix} \mathcal{O}_i^{[4n]} & \mathcal{O}_p^{[4n]} & \Gamma_i^{[4n]} - \Gamma_p^{[4n]} \\ \mathcal{O}_j^{[4n]} & \mathcal{O}_q^{[4n]} & \Gamma_j^{[4n]} - \Gamma_q^{[4n]} \end{bmatrix} = 2n + \operatorname{rank} \begin{bmatrix} \Gamma_i^{[4n]} - \Gamma_p^{[4n]} \\ \Gamma_j^{[4n]} - \Gamma_q^{[4n]} \end{bmatrix} \tag{4}$$
for all i, j, p, $q \in \mathcal{P}$ with $i \neq j$, $p \neq q$ and $(i, j) \neq (p, q)$.

(4) implies that we can distinguish the switching signals $\sigma \neq \tilde{\sigma}$ given by
$$\sigma(t) = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S, \end{cases} \qquad \tilde{\sigma}(t) = \begin{cases} p, & t < t_S, \\ q, & t \geq t_S, \end{cases}$$
if at least one of the corresponding solutions $x_{(x_0,\sigma,u)}$ and $x_{(\tilde{x}_0,\tilde{\sigma},u)}$ is nonzero at the switching time t_S. Condition (3) deals with the case that the state is zero at the switching time: It implies that the input-output behavior steering a state to zero is uniquely related to some mode. Due to $\mathcal{R}(\Sigma_{i,p}) = \{0\}$, $x_{(x_0,\sigma,u)}(t_S) = x_{(\tilde{x}_0,\tilde{\sigma},u)}(t_S) = 0$ and $y_{(x_0,\sigma,u)} \equiv y_{(\tilde{x}_0,\tilde{\sigma},u)}$ implies $i = p$ or $x_{(x_0,\sigma,u)}(t) = x_{(\tilde{x}_0,\tilde{\sigma},)}(t) = 0$ for $t \leq t_S$. $\mathcal{R}(\Sigma_{j,q}) = \{0\}$ gives an analogous result for $t \geq t_S$.

III. Switch-observer

Based on *knowledge of the switching times* and *strong (x, σ_1)-observability*, we construct an observer for both state and switching signal. The observer is described for the single switch case, but can also be generalized to systems with multiple switches.

In the sequel, we assume that the switching times are known. It is reasonable to separate the fault identification problem from that of fault detection as a much simpler procedure is possible in this case. While fault detection is in general more involved, it can be rather simple in some applications, e.g. if switches cause jumps in the output. In general, the switching times of a strongly switching time observable system are described by
$$\left\{ t \;\middle|\; y^{[2n]}(t^-) \neq y^{[2n]}(t^+) \right\}.$$
for a nonzero state $x_{(x_0,\sigma,u)}(t_S)$ at the switching instant. For $x_{(x_0,\sigma,u)}(t_S) = 0$, one might have to consider the change in the output dynamic in a neighborhood of the switching time.

Assume that the system (1) is strongly (x, σ_1)-observable and that there is exactly one switching time $t_S \in (0, T)$. If $x_{(x_0,\sigma,u)}(t_S) \neq 0$, the switching signal is given by the unique solution $\big(\sigma(t_S^-), \sigma(t_S^+)\big) = (i, j)$ of
$$\operatorname{rank} \begin{bmatrix} y^{[4n]}(t_S^-) & \mathcal{O}_i^{[4n]} & \Gamma_i^{[4n]} \\ y^{[4n]}(t_S^+) & \mathcal{O}_j^{[4n]} & \Gamma_j^{[4n]} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathcal{O}_i^{[4n]} & \Gamma_i^{[4n]} \\ \mathcal{O}_j^{[4n]} & \Gamma_j^{[4n]} \end{bmatrix}.$$
Then $x_{(x_0,\sigma,u)}(t_S) = x_1$ can be determined as the solution of
$$\begin{bmatrix} y^{[4n]}(t_S^-) \\ y^{[4n]}(t_S^+) \end{bmatrix} = \begin{bmatrix} \mathcal{O}_i^{[4n]} \\ \mathcal{O}_j^{[4n]} \end{bmatrix} x_1 + \begin{bmatrix} \Gamma_i^{[4n]} \\ \Gamma_j^{[4n]} \end{bmatrix} u^{[4n]}.$$
For $x_{(x_0,\sigma,u)}(t_S) = 0$ the above procedure might not work. In this case, we can uniquely reconstruct i and j by the input-output data on $[0, t_S]$ and $[t_S, T]$, respectively. This means we compare classical observers for each mode. In this case, i and j are computed independently of each other.

Computing the output's derivatives is clearly disadvantageous even in the presence of small errors. Hence we consider another approach, based on classical observers such as the Luenberger observer [4].

In [8], an observer is proposed for switched systems with known switching signal. A naive approach for the unknown switching signal setup would be to use an observer as in [8] for each possible mode sequence, giving in total $N(N-1)$ known-signal observers, of which only one will behave reasonably. It turns out that one can carry out the (partial) state estimations on the pre- and post-switch interval independently. Therefore we need only N classical observers, one for each mode. The partial state results from pre- and post-switch interval are then used to determine the correct mode sequence and the state. Before describing the switch-observer, we consider the observability problem with known switching signal:

Known switching signal: Assume that the system (1) has the known switching signal

$$\sigma(t) = \begin{cases} i, & t < t_S, \\ j, & t \geq t_S. \end{cases}$$

Strong (x, σ_1)-observability implies observability for known switching signals (in the sense of [8]), i.e. it implies

$$\ker \mathcal{O}_i \cap \ker \mathcal{O}_j = \{0\} \tag{5}$$

for all $i \neq j$. For $i \in \mathcal{P}$ let Z_i be a matrix whose columns form an orthonormal basis of

$$\operatorname{im} \left(\mathcal{O}_i^{[n]}\right)^\top = \left(\ker \mathcal{O}_i^{[n]}\right)^\perp.$$

$z_i := Z_i^\top x$ describes the observable part of x if the system is in mode i. In this case, z_i satisfies

$$\begin{aligned} \dot{z}_i &= Z_i^\top A_i Z_i z + Z_i^\top B_i u, \\ y &= C_i Z_i z_i + D_i u. \end{aligned} \tag{6}$$

By $\mathcal{O}_i^{\text{o.p.}}$ we denote the Kalman matrix corresponding to (6), i.e. $\mathcal{O}_i^{\text{o.p.}} = \mathcal{O}_i Z_i$. Analogously, $z_j := Z_j^\top x$ describes the observable part of x in mode j, i.e. the observable part of x on $[t_S, T]$. As (6) is observable, we can use a Luenberger-observer to determine z_i (and z_j). (5) implies that there exists a matrix $U_{i,j}$ satisfying

$$\begin{bmatrix} Z_i & Z_j \end{bmatrix} U_{i,j} = I. \tag{7}$$

With this, we compute the state x by the partial state information as

$$x(t_S) = U_{i,j}^\top \begin{bmatrix} z_i(t_S) \\ z_j(t_S) \end{bmatrix}.$$

As announced, we do not repeat this process $N(N-1)$ times for unknown switching signals, but rearrange it to reduce the computational effort. In the sequel, denote the correct mode pair by (i^*, j^*). The procedure for unknown switching signals is now described in three steps.

1. Pre-Switch interval: On the pre-switch interval $[0, t_S]$, we use for each mode i a classical observers for its observable part (6): If the estimated output error $r := \|y - \hat{y}_i\|$ becomes sufficiently small, i.e. if the mode captures the input-output behavior sufficiently well, we consider this mode to be reasonable for the pre-switch interval and add it to the candidate set. In this case, we also save the partial state estimation $\hat{z}_i^{\text{pre}} := \hat{z}_i(t_S)$. The procedure for one mode i is described in Algorithm 1. It has to be repeated for each mode.

Note that one might get several reasonable mode candidates for the pre-switch interval. This is admissible (and not avoidable) for strongly (x, σ_1)-observable systems. Only with the information from the post-switch interval we will be able to find the correct pre-switch mode within the set of the now computed candidates. It is not sufficient to store the best candidate.

Definition 2: For given x_0, σ, u and an interval $\mathcal{I}$, the mode $p \in \mathcal{P}$ is called *reasonable* if there exists an initial value $\tilde{x}_0$ with

$$y_{(x_0,\sigma,u)} = y_{(\tilde{x}_0,p,u)} \quad \text{on } \mathcal{I},$$

i.e. if mode p can describe the dynamic of $y_{(x_0,\sigma,u)}$ on the interval $\mathcal{I}$.

Algorithm 1: Partial observer for pre-switch interval

Data: i, Z_i, t_S, y
Result: accept, $\hat{z}_i^{\text{pre}}$
accept $\leftarrow$ false, $\hat{z}_i^{\text{pre}} \leftarrow \emptyset$;
Compute observable part (6);
Construct Luenberger observer for (6) on $[0, t_S]$: State estimation $\hat{z}_i$, output estimation $\hat{y}_i$;
Set $r_i \leftarrow \|y - \hat{y}_i\|$;
if $r_i(t) < \varepsilon_r \ \forall t \in (t_S - \varepsilon_T, t_S)$ **then**
  accept $\leftarrow$ true;
  $\hat{z}_i^{\text{pre}} \leftarrow \hat{z}_i(t_S)$;

2. Post-Switch interval: The algorithm for the post-switch interval is very similar to that on the pre-switch interval. Note that one cannot make use of the pre-switch state estimations here as 1) they might be incomplete (as the modes do not have to be observable), 2) they can differ greatly for different modes. With the same computations as in the pre-switch observer, but on the interval $[t_S, T]$, we arrive at a set of of reasonable mode candidates $\mathcal{P}^+$.
We furthermore need partial state estimations at time t_S. One could propagate the partial state estimation $\tilde{z}_j(T)$ for mode j back to time t_S. For this, we need u on $[t_S, T]$. A more reliable procedure, which also requires y on $[t_S, T]$, is using a Luenberger-observer on the interval $[t_S, T]$ backwards in time. (We can use the estimation $\tilde{z}_j(T)$ as an initial value for the observer.) This yields $\hat{z}_j^{\text{post}} := \hat{z}_j(t_S)$. Such a "back- and forth-observer" has been used in [7]. The procedure for one mode j is described in Algorithm 2. It has to be repeated for each mode

3. Combination of partial results: The previous steps give us two sets $\mathcal{P}^-$, $\mathcal{P}^+$ of reasonable modes for the pre- and post-switch interval as well as the corresponding partial state estimations. We now have to use these partial state estimations to reduce the set $\mathcal{P}^- \times \mathcal{P}^+$ to the correct mode pair (i^*, j^*). Intuitively, we have to check if the partial state estimations for modes i and j "fit together",

Algorithm 2: Partial observer for post-switch interval

Data: i, Z_i, t_S, y
Result: accept, $\hat{z}_i^{\text{post}}$
accept $\leftarrow$ false, $\hat{z}_i^{\text{post}} \leftarrow \emptyset$;
Compute observable part (6);
Construct Luenberger observer for (6) on $[t_S, t]$: State estimation $\tilde{z}_i$, output estimation $\tilde{y}_i$;
Set $r_{i,1} \leftarrow \|y - \tilde{y}_i\|$;
if $r_{i,1}(t) < \varepsilon_r \ \forall t \in (t_S - \varepsilon_T, t_S)$ **then**
  Construct Luenberger observer for (6) backwards in time on $[t_S, T]$: State estimation $\hat{z}_i$, output estimation $\hat{y}_i$. Initialize with $\hat{z}_i(T) := \tilde{z}_i(T)$;
  Set $r_{i,2} \leftarrow \|y - \hat{y}_i\|$;
  if $r_{i,2}(t) < \varepsilon_r \ \forall t \in (t_S, t_S + \varepsilon_T)$ **then**
    accept $\leftarrow$ true;
    $\hat{z}_i^{\text{post}} \leftarrow \hat{z}_i(t_S)$;

i.e. if they give rise to an overall state estimation whose corresponding output approximates the measured output on the whole observation interval $[0, T]$.

Assume that both modes i and j are observable. Then $\hat{z}_i^{\text{pre}}$ and $\hat{z}_j^{\text{post}}$ are both estimations of the full state $x(t_S)$. For the pair (i, j) to be correct we expect $\hat{z}_i^{\text{pre}} \approx \hat{z}_j^{\text{post}}$.

As the modes are in general not observable, we have to combine the partial state estimations to an overall estimation

$$\hat{x}_{i,j} = U_{i,j}^\top \begin{bmatrix} \hat{z}_i^{\text{pre}} \\ \hat{z}_j^{\text{post}} \end{bmatrix}.$$

There usually is some freedom in choosing $U_{i,j}$ (if it is not square, i.e. if the spaces $\operatorname{im} Z_i$ and $\operatorname{im} Z_j$ overlap). We assume henceforth that it is chosen as the Moore-Penrose-pseudoinverse of $Z_{i,j} := \begin{bmatrix} Z_i & Z_j \end{bmatrix}$, i.e.

$$U_{i,j} := Z_{i,j}^\top \left(Z_{i,j} Z_{i,j}^\top \right)^{-1}.$$

Other choices are possible. One can, for example, weight the influence of $\hat{z}_i^{\text{pre}}$ and $\hat{z}_j^{\text{post}}$ on the intersection of $\operatorname{im} Z_i$ and $\operatorname{im} Z_j$.

To assert that the mode pair $(i, j) \in \mathcal{P}^- \times \mathcal{P}^+$ is correct, we present two different methods methods:

1) Check that the overall state estimation $\hat{x}_{i,j}$ fits to the partial state estimations $\hat{z}_i^{\text{pre}}$ and $\hat{z}_j^{\text{post}}$, i.e. assert that

$$\left\| \begin{bmatrix} Z_i^\top \hat{x}_{i,j} - \hat{z}_i^{\text{pre}} \\ Z_j^\top \hat{x}_{i,j} - \hat{z}_j^{\text{post}} \end{bmatrix} \right\| \tag{8}$$

is sufficiently small.

2) Check that the output produced by the overall state estimation $\hat{x}_{i,j}$ fits to the measured output. As the modes $i \in \mathcal{P}^-$, $j \in \mathcal{P}^+$ are reasonable, they capture the input-output behavior on the pre- and post-switch interval sufficiently well. Hence we do not compare the output produced by $\hat{x}_{i,j}$ mode i and input u on $[0, t_S]$ with the correct output $y_{(x_0, \sigma, u)}$, but with the output of the partial estimation, i.e. the output produced by $\hat{z}_i^{\text{pre}}$, mode i and input u. Similarly, we proceed on the post-switch interval. As we compare solutions with the same switching signals, we can restrict our attention to the homogeneous case. Simulating the outputs can be avoided by making use of the Kalman-matrices and checking that

$$\left\| \begin{bmatrix} \mathcal{O}_i \\ \mathcal{O}_j \end{bmatrix} \hat{x}_{i,j} - \begin{bmatrix} \mathcal{O}_i^{\text{o.p.}} \hat{z}_i^{\text{pre}} \\ \mathcal{O}_j^{\text{o.p.}} \hat{z}_j^{\text{post}} \end{bmatrix} \right\| \tag{9}$$

is sufficiently small.

The first variant is faster, the second one takes into account the effect errors in the overall state estimation have on the output. We chose the second variant as a better error analysis is possible in this case, see Section IV.

Now this final part is described in Algorithm 3.

Algorithm 3: The switch-observer for switched ODEs

Data: t_S, T, $\mathcal{P}$, y
Result: $\mathcal{M}$, $\hat{x}_{i,j}$ for $(i, j) \in \mathcal{M}$
$\mathcal{P}^- \leftarrow \emptyset$, $\mathcal{P}^+ \leftarrow \emptyset$, $\mathcal{M} \leftarrow \emptyset$;
for $i \in \mathcal{P}$ **do**
  Compute Z_i;
  [accept, $\hat{z}_i^{\text{pre}}$] $\leftarrow$`PartObsPre(`i, Z_i, t_S, y`);`
  if accept **then**
    $\mathcal{P}^- \leftarrow \mathcal{P}^- \cup \{i\}$;
  [accept, $\hat{z}_i^{\text{post}}$] $\leftarrow$`PartObsPost(`i, Z_i, t_S, y`);`
  if accept **then**
    $\mathcal{P}^+ \leftarrow \mathcal{P}^+ \cup \{i\}$;
for $i \in \mathcal{P}^-$, $j \in \mathcal{P}^+$, $i \neq j$ **do**
  Construct $U_{i,j}$ with (7);
  $\hat{x}_{i,j} \leftarrow U_{i,j}^\top \begin{bmatrix} \hat{z}_i^{\text{pre}} \\ \hat{z}_j^{\text{post}} \end{bmatrix}$;
  if (9) $< \varepsilon$ *holds true* **then**
    $\mathcal{M} \leftarrow \mathcal{M} \cup \{(i, j)\}$;

The N classical observers on the pre- and post-switch interval can be replaced by other classical observers such as the Kalman filter. Note that the proposed algorithm does not correspond to the idea of Kalman filter banks as we allocate information from the intervals $[0, t_S]$, $[t_S, T]$ for an overall estimate. On the intervals $[0, t_S]$, $[t_S, T]$ there might be several suitable modes.

Remark 2 (Relaxing the assumptions): Strong (x, σ_1)-observability guarantees that the switch observer (Algorithm 3) works for any initial value and any input. It guarantees that we can reconstruct both state and switching signal and that the result is unique. The algorithm might still work for weaker assumptions:

- Analytic (or smooth) inputs were necessary for the formulation of strong (x, σ_1)-observability. For the observer with known switching times, this is not required. Any input that can be handled by the partial observers (Algorithms 1 and 2) is feasible.
- Condition (3) is not essential for the observer: It is required to cover the case of a zero state at the switch.

Without (3) we are still able to deal with all nonzero states, i.e. with most cases and, in some applications, all relevant cases.
- One can adapt the algorithm to work in a generic case for some weaker condition than (4).

Remark 3 (Switching time detection): The proposed observer assumes that the switching time is known. This assumption allows for significant reduction in the algorithm's complexity. It is suitable for systems where a switch leads to a state jump or, for DAEs, even to an impulse, and is thus easy to notice. It is also useful in applications where a switch has been detected, but not identified (fault identification).

Remark 4 (Time delay): The switch-observer gives a state and switching signal estimation only after time $T > t_S$, i.e. the information about the switching instant is obtained with a delay $T - t_S$. Note that a delay is in fact necessary as we have to wait for the individual observers to adapt (as we cannot equip them with a correct initial state). One can use other classical observers in Algorithms 1 and 2 to reduce the required delay. One could also reduce the delay by trying a smaller time delay $\tilde{T} < T$ in the switch-observer. At time $\tilde{T}$, one continues with step 3 (combination of partial results) and, parallel to that, continues with step 2 (Post-switch interval). The latter is only used if the third step (for $\tilde{T}$) does not yield a unique result.

In [3], the concept of switch observability has been extended to homogeneous switched differential-algebraic equations (DAEs) and some ideas on an observer were presented. A next step is to extend the procedure to inhomogeneous switched DAEs.

IV. Error analysis

We give bounds on the state error and the output error for the correct mode pair (i^*, j^*).

After that, we discuss the size of (9) for incorrect, but reasonable mode pairs (p, q). We show that it is bounded from below. This implies that for suitable tolerances the switch-observer works correctly, i.e. the output $\mathcal{M}$ contains only the correct mode pair (i^*, j^*).

The Luenberger observers on $[0, t_S]$ and on $[t_S, T]$ are exponentially convergent for the correct mode i^* and j^*, respectively. The same holds true for reasonable modes, i.e. modes that can model the given input-output behavior on the relevant interval. Note that also other observers for the individual modes can be used in Algorithms 1 and 2.

A. Error in the state estimation, correct mode pair

For the correct mode pair (i^*, j^*) we consider the error propagation of the partial state estimations $\hat{z}_{i^*}^{\text{pre}}$, $\hat{z}_{j^*}^{\text{post}}$ to the overall estimation $\hat{x}_{i^*,j^*}$. Let $x_{(x_0,\sigma,u)}$ be the correct solution and the errors made by the Luenberger observers be given by

$$\begin{aligned} \varepsilon^{\text{pre}} &:= \left\| \hat{z}_{i^*}^{\text{pre}} - Z_{i^*}^\top x_{(x_0,\sigma,u)}(t_S) \right\|, \\ \varepsilon^{\text{post}} &:= \left\| \hat{z}_{j^*}^{\text{post}} - Z_{j^*}^\top x_{(x_0,\sigma,u)}(t_S) \right\|. \end{aligned}$$

Then we have

$$\left\| x_{(x_0,\sigma,u)}(t_S) - \hat{x}_{i^*,j^*} \right\| \le \left\| U_{i^*,j^*}^\top \right\| \sqrt{(\varepsilon^{\text{pre}})^2 + (\varepsilon^{\text{post}})^2}.$$

Hence the norm of $U_{i^*,j^*}^\top$ is crucial. We consider several cases:

- Full redundancy in the information: Assume that both modes i^*, j^* are observable. This implies that Z_{i^*}, Z_{j^*} are orthonormal and U_{i^*,j^*} is given by $U_{i^*,j^*} = \frac{1}{2} Z_{i^*,j^*}^\top$, hence $\left\| U_{i^*,j^*}^\top \right\|_2 = \sqrt{2}$.
- No redundancy in the information, orthogonal information: Assume that the subspaces spanned by Z_{i^*} and Z_{j^*} are orthogonal, i.e. Z_{i^*,j^*} is orthonormal. Then it hold $U_{i^*,j^*} = Z_{i^*,j^*}^\top$ and $\left\| U_{i^*,j^*}^\top \right\|_2 = 1$.
- Almost identical information: The subspaces spanned by the matrices
$$Z_{i^*} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Z_{j^*} = \frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 \\ a \end{bmatrix}$$
are almost parallel for $|a|$ small. U_{i^*,j^*} is given by $Z_{i^*,j^*}^{-\top}$, i.e.
$$U_{i^*,j^*} = \begin{bmatrix} 1 & -a^{-1} \\ 0 & \sqrt{1+a^{-2}} \end{bmatrix}$$
and its norm can be estimated by
$$\|U_{i^*,j^*}\|_2^2 \ge \langle U_{i^*,j^*} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, U_{i^*,j^*} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle = 1 + 2a^{-2},$$
i.e. it becomes arbitrarily large for $a \to 0$.

B. Error in the output estimation, correct mode pair

We want to give a bound on (9) in Algorithm 3 for the correct mode pair (i^*, j^*). Using

$$\begin{aligned} \mathcal{O}_{i^*} \hat{x}_{i^*,j^*} - \mathcal{O}_{i^*}^{\text{o.p.}} \hat{z}_{i^*}^{\text{pre}} &= \mathcal{O}_{i^*} \left(U_{i^*,j^*}^\top \begin{bmatrix} \hat{z}_{i^*}^{\text{pre}} \\ \hat{z}_j^{\text{post}} \end{bmatrix} - Z_{i^*} \hat{z}_{i^*}^{\text{pre}} \right) \\ &= \mathcal{O}_{i^*} \left(U_{i^*,j^*}^\top \begin{bmatrix} z_{i^*}^{\text{pre}} - \hat{z}_{i^*}^{\text{pre}} \\ z_{j^*}^{\text{post}} - \hat{z}_{j^*}^{\text{post}} \end{bmatrix} - Z_{i^*} (z_{i^*}^{\text{pre}} - \hat{z}_{i^*}^{\text{pre}}) \right), \end{aligned}$$

we obtain

$$\begin{aligned} \left\| \mathcal{O}_{i^*} \hat{x}_{i^*,j^*} - \mathcal{O}_{i^*}^{\text{o.p.}} \hat{z}_{i^*}^{\text{pre}} \right\| &\le \|\mathcal{O}_{i^*}\| \, \varepsilon^{\text{pre}} \\ &+ \|\mathcal{O}_{i^*}\| \left\| U_{i^*,j^*}^\top \right\| \sqrt{(\varepsilon^{\text{pre}})^2 + (\varepsilon^{\text{post}})^2}, \end{aligned}$$

or, separating the effects of the individual errors,

$$\begin{aligned} \left\| \mathcal{O}_{i^*} \hat{x}_{i^*,j^*} - \mathcal{O}_{i^*}^{\text{o.p.}} \hat{z}_{i^*}^{\text{pre}} \right\| &\le \left\| \mathcal{O}_{i^*} \left(Z_{i^*,j^*} Z_{i^*,j^*}^\top \right)^{-1} Z_{j^*} \right\| \varepsilon^{\text{post}} \\ &+ \left\| \mathcal{O}_{i^*} \left(\left(Z_{i^*,j^*} Z_{i^*,j^*}^\top \right)^{-1} - I \right) Z_{i^*} \right\| \varepsilon^{\text{pre}}. \end{aligned}$$

A similar bound can be computed for $\left\| \mathcal{O}_{j^*} \hat{x}_{i^*,j^*} - \mathcal{O}_{j^*}^{\text{o.p.}} \hat{z}_{j^*}^{\text{post}} \right\|$.

C. Error in output estimation, wrong mode pair

We now want to show that for strongly (x, σ_1)-observable systems the error in the output estimation is bounded from below for wrong mode pairs. This implies that for a sufficiently small error bound in (9) we can prevent labeling false mode pairs as correct. Then sufficiently strong Luenberger-observers ensure that the correct mode is detected.

In the first part of this subsection, we assume that Algorithms 1 and 2 worked correctly, i.e. that $\mathcal{P}^-$ and $\mathcal{P}^+$ in

Algorithm 3 contain only reasonable modes for the pre- and post-switch interval.

Now let $p \in \mathcal{P}^-$, $q \in \mathcal{P}^+$. Let $x_{i^*,j^*} = x_{(x_0,\sigma,u)}(t_S)$ be the state at the switching time and $U = u^{[n]}(t_S)$. Then there exist unique z_p^{pre}, z_q^{post} with

$$\begin{aligned} \mathcal{O}_{i^*} x_{i^*,j^*} + \Gamma_{i^*} U &= \mathcal{O}_p^{\text{o.p}} z_p^{\text{pre}} + \Gamma_p U, \\ \mathcal{O}_{j^*} x_{i^*,j^*} + \Gamma_{j^*} U &= \mathcal{O}_q^{\text{o.p}} z_q^{\text{post}} + \Gamma_q U. \end{aligned} \tag{10}$$

However, due to (4) there does not exist a $x_{p,q}$ with

$$\begin{bmatrix} \mathcal{O}_{i^*} \\ \mathcal{O}_{j^*} \end{bmatrix} x_{i^*,j^*} + \begin{bmatrix} \Gamma_{i^*} \\ \Gamma_{j^*} \end{bmatrix} U = \begin{bmatrix} \mathcal{O}_p \\ \mathcal{O}_q \end{bmatrix} x_{p,q} + \begin{bmatrix} \Gamma_p U \\ \Gamma_q \end{bmatrix} U.$$

This means for given x_0, (i^*, j^*) and U we have

$$\delta := \text{dist}\left(\text{im} \begin{bmatrix} \mathcal{O}_p \\ \mathcal{O}_q \end{bmatrix}, \begin{bmatrix} \mathcal{O}_{i^*} x_{i^*,j^*} + (\Gamma_{i^*} - \Gamma_p) U \\ \mathcal{O}_{j^*} x_{i^*,j^*} + (\Gamma_{j^*} - \Gamma_q) U \end{bmatrix} \right) > 0.$$

This δ now enables us to give a lower bound on the error. Due to (10) we have

$$\delta = \text{dist}\left(\text{im} \begin{bmatrix} \mathcal{O}_p \\ \mathcal{O}_q \end{bmatrix}, \begin{bmatrix} \mathcal{O}_p^{\text{o.p.}} z_p^{\text{pre}} \\ \mathcal{O}_q^{\text{o.p.}} z_q^{\text{post}} \end{bmatrix} \right).$$

For exact partial state estimation, this would be a lower bound on (9). For non-ideal partial state estimations $\hat{z}_p^{\text{pre}}$, $\hat{z}_q^{\text{post}}$ we obtain

$$\begin{aligned} &\left\| \begin{bmatrix} \mathcal{O}_p \hat{x}_{p,q} - \mathcal{O}_p^{\text{o.p.}} \hat{z}_p^{\text{pre}} \\ \mathcal{O}_q \hat{x}_{p,q} - \mathcal{O}_q^{\text{o.p.}} \hat{z}_q^{\text{post}} \end{bmatrix} \right\| \\ &\geq \left\| \begin{bmatrix} \mathcal{O}_p \hat{x}_{p,q} - \mathcal{O}_p^{\text{o.p.}} z_p^{\text{pre}} \\ \mathcal{O}_q \hat{x}_{p,q} - \mathcal{O}_q^{\text{o.p.}} z_q^{\text{post}} \end{bmatrix} \right\| - \left\| \begin{bmatrix} \mathcal{O}_p^{\text{o.p.}} \left(z_p^{\text{pre}} - \hat{z}_p^{\text{pre}} \right) \\ \mathcal{O}_q^{\text{o.p.}} \left(z_q^{\text{post}} - \hat{z}_q^{\text{post}} \right) \end{bmatrix} \right\| \\ &\geq \delta - \underbrace{\left\| \mathcal{O}_p^{\text{o.p.}} \right\| \left\| z_p^{\text{pre}} - \hat{z}_p^{\text{pre}} \right\|}_{:=\delta_2} - \underbrace{\left\| \mathcal{O}_q^{\text{o.p.}} \right\| \left\| z_q^{\text{post}} - \hat{z}_q^{\text{post}} \right\|}_{:=\delta_3} \end{aligned}$$

As $p \in \mathcal{P}^-$ and $q \in \mathcal{P}^+$ are assumed to be reasonable, their "errors" $\left\| z_p^{\text{pre}} - \hat{z}_p^{\text{pre}} \right\|$ as well as $\left\| z_q^{\text{post}} - \hat{z}_q^{\text{post}} \right\|$ can be made arbitrarily small by choosing a suitable observer gain.

We can also give a (very conservative) lower bound on (8). Using $\mathcal{O}_p = \mathcal{O}_p^{\text{o.p}} Z_p^\top$ we get

$$\left\| \begin{bmatrix} \mathcal{O}_p^{\text{o.p.}} \left(Z_p^\top \hat{x}_{p,q} - z_p^{\text{pre}} \right) \\ \mathcal{O}_q^{\text{o.p.}} \left(Z_q^\top \hat{x}_{p,q} - z_q^{\text{post}} \right) \end{bmatrix} \right\| \geq \delta.$$

This implies

$$\left\| \begin{bmatrix} \mathcal{O}_p^{\text{o.p.}} \left(Z_p^\top \hat{x}_{p,q} - \hat{z}_p^{\text{pre}} \right) \\ \mathcal{O}_q^{\text{o.p.}} \left(Z_q^\top \hat{x}_{p,q} - \hat{z}_q^{\text{post}} \right) \end{bmatrix} \right\| \geq \delta - \delta_2 - \delta_3,$$

which in turn gives

$$\left\| \begin{bmatrix} Z_p^\top \hat{x}_{p,q} - \hat{z}_p^{\text{pre}} \\ Z_q^\top \hat{x}_{p,q} - \hat{z}_q^{\text{post}} \end{bmatrix} \right\| \geq \frac{\delta - \delta_2 - \delta_3}{\left\| \begin{bmatrix} \mathcal{O}_p^{\text{o.p.}} & 0 \\ 0 & \mathcal{O}_q^{\text{o.p.}} \end{bmatrix} \right\|}.$$

V. Example

Example 1: We consider the system given by the modes

$$\begin{aligned} (A_1, B_1, C_1) &= \left(\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix} \right), \\ (A_2, B_2, C_2) &= \left(\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right), \\ (A_3, B_3, C_3) &= \left(\begin{bmatrix} -1 & 0 \\ -32 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 8.2 \end{bmatrix}, \begin{bmatrix} -42 & 5 \end{bmatrix} \right). \end{aligned}$$

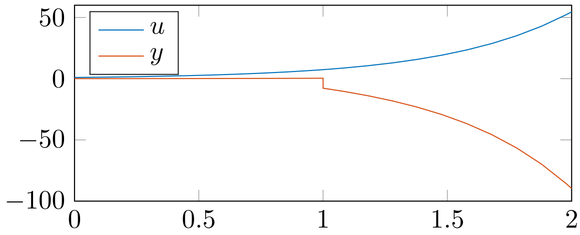

Fig. 1. Input and output for Example 1.

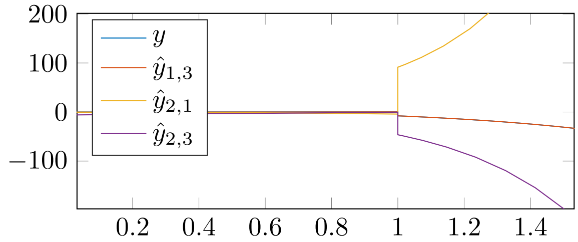

Fig. 2. Output and output estimations for Example 1.

This system is strongly (x, σ_1)-observable. The output for $u(t) = -e^{2t}$, $x_0 = \begin{bmatrix} \frac{1}{40} & 0 \end{bmatrix}^\top$ and

$$\sigma(t) = \begin{cases} 1, & t < 1, \\ 3, & t \geq 1. \end{cases}$$

is illustrated in Figure 1, together with the input u. On the interval $[0, 1]$, these dynamics can be achieved by modes 1 and 2. On the interval $[1, 2]$, modes 1 and 3 are candidates for the correct mode. Hence the observer (Algorithm 3) gives $\mathcal{P}^- = \{1, 2\}$, $\mathcal{P}^+ = \{1, 3\}$. The partial state estimations $\hat{z}_1^{\text{pre}}$, $\hat{z}_2^{\text{pre}}$, $\hat{z}_1^{\text{post}}$, $\hat{z}_3^{\text{post}}$ lead to the overall state estimations

$$\hat{x}_{1,3} \approx \begin{bmatrix} 0.1798 \\ -0.0418 \end{bmatrix}, \hat{x}_{2,1} \approx \begin{bmatrix} -2.1701 \\ 0.0492 \end{bmatrix}, \hat{x}_{2,3} \approx \begin{bmatrix} 1.0110 \\ -0.7807 \end{bmatrix}.$$

The correct value is given by $x(1) \approx \begin{bmatrix} 0.1853 & 0 \end{bmatrix}^\top$. We already see that the wrong mode pairs $(2, 1)$, $(2, 3)$ lead to completely wrong state estimations. This is advantageous as it indicates that the wrong modes will lead to completely wrong output estimations, making them easy to detect. These wrong candidate pairs can easily be rejected in the final part of Algorithm 3 as we have:

(i, j)	Condition (8)	Condition (9)
$(1, 3)$	0.084	0.049
$(2, 1)$	8.596	4.976
$(2, 3)$	2.177	5.517

In Figure 2, the output estimations $\hat{y}_{i,j}$ based on the mode pairs (i, j) are compared to the actual output y.

VI. Conclusions

The switch-observer can provide information on the switching signal and the state even for unobservable switching signals. It combines the information before and after a switching instant for obtaining an estimate on the switching signal and the state. It is particularly useful if the considered modes are not observable or have some common dynamic.

We investigated the algorithm as well as its error propagation and considered an example.

A generalization of this observer to DAEs is currently under investigation.

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