Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability[☆]

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Abstract

We study switched nonlinear differential algebraic equations (DAEs) with respect to existence and nature of solutions as well as stability. We utilize piecewise-smooth distributions introduced in earlier work for switched linear DAE to establish a solution framework for switched nonlinear DAEs. In particular, we allow induced jumps in the solutions. To study stability, we first generalize Lyapunov's direct method to non-switched DAEs and afterwards obtain Lyapunov criteria for asymptotic stability of switched DAEs. Developing appropriate generalizations of the concepts of a common Lyapunov function and multiple Lyapunov functions for DAEs, we derive sufficient conditions for asymptotic stability under arbitrary switching and under sufficiently slow average dwell-time switching, respectively.

Keywords: Nonlinear differential algebraic equations, piecewise-smooth distributions, Lyapunov functions, asymptotic stability

1. Introduction

We consider switched nonlinear differential algebraic equations (DAEs) of the form

$$E_{\sigma(t)}(x(t))\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{1}$$

where $\sigma : \mathbb{R} \to \{1, \ldots, N\}$, $N \in \mathbb{N}$, is the switching signal and $E_p : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $f_p : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions. In particular, we assume that each subsystem is a DAE in *quasi-linear form*

$$E(x)\dot{x} = f(x). \tag{2}$$

Equations of this kind occur for example when modeling (nonlinear) electrical circuits [1] or mechanical coupled systems [2]. Classical linear DAEs (i.e. without

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switching) of the form $E\dot{x} = Ax$, with matrices $E, A \in \mathbb{R}^{n\times n}$, naturally appear when modeling electrical circuits because Kirchhoff's circuit laws add algebraic equations to the differential equations stemming from capacitors and inductances. For more details and further motivation for studying (non-switched) DAEs the reader is referred to [3]. Adding, for example, (ideal) switches to an electrical circuit or allowing for sudden structural changes in mechanical systems yield a switched DAE as in (1). When studying the zero dynamics of an ordinary differential equation (ODE) one arrives at a DAE because of the additional algebraic constraint $0 = y = h(x)$, where $h : \mathbb{R}^n \to \mathbb{R}^m$ is the output function. In particular, using a switched controller to stabilize the zero dynamics (as was done in [4]) yields a switched DAE (1) even if one starts with an ODE.

The aim of this paper is a stability analysis of (1) with the help of Lyapunov functions. For this we first need to establish a Lyapunov theory for non-switched DAEs in quasi-linear form (2) and secondly we need to define a suitable solution framework for the switched DAE (1).

The use of Lyapunov functions is a powerful tool to study stability of nonlinear differential equations. However, it is not immediately clear how Lyapunov functions can be defined for *implicit* differential equation such as (2). The main problem is that, given a function $x \mapsto V(x)$, its derivative along solutions $\dot{V}(x) = \nabla V(x)\dot{x}$ can not be expressed directly in terms of the right-hand side $f(x)$, because $\dot{x}$ is not explicitly given. We resolve this problem and generalize the well known Lyapunov's Direct Method to implicit differential equations of the form (2). In the linear case $E\dot{x} = Ax$ there have been generalizations of Lyapunov's Direct Method, e.g. in [5, 6], but no general definition of a Lyapunov function is given, hence our result is also new for the linear case. Note that a fully general implicit differential algebraic equation $F(x, \dot{x}) = 0$ can always be rewritten as (2) by introducing the new state variable $z = \dot{x}$. Therefore, the consideration of the special form (2) is not a hard restriction of generality. However, we later impose additional assumptions on (2) to ensure existence and uniqueness of solutions.

One major problem of studying switched DAEs of the form (1) is the presence of jumps in the solutions induced by the presence of so-called *consistency spaces*. We are using the piecewise-smooth distributional framework from [7, 8] to define solutions of the switched DAE (1). In this framework $\dot{x}$ is well defined even when x contains jumps, in which case $\dot{x}$ contains *Dirac impulses*. It should be noted that a general distributional solution framework (i.e. not considering the smaller space of piecewise-smooth distributions) will not work, because 1) the nonlinear function evaluations $E(x)$ and $f(x)$ are not defined for distributions and 2) the product $E(x)\dot{x}$ is not defined even when $E(x)$ is a piecewise-smooth function.

All results presented here apply of course also to a linear switched DAE

$$E_\sigma \dot{x} = A_\sigma x, \tag{3}$$

where $E_p, A_p \in \mathbb{R}^{n\times n}$ for $p \in \{1, \dots, N\}$. In this case some of the results simplify significantly and we will formulate corollaries to highlight the linear case. We have studied stability of the linear switched DAE (3) already in [9]. However, our nonlinear results presented here applied to the linear switched DAE (3) still generalize the results in [9]. In particular, the notion of a Lyapunov function as well as the dwell time results are significantly generalized.

Although the two research fields "DAEs" and "switched systems" are now relatively mature, see e.g. the textbooks [3] and [10], the combination of both has not been studied much even in the linear case. The existing literature available on switched DAEs [11, 12, 13, 14, 15] does not consider stability problems. Furthermore, the fundamental problem that one needs distributional solutions for a switched linear DAE (3) and at the same time the equation (3) cannot be evaluated for distributional x is not resolved there. This problem is resolved (at least for *linear* switched DAEs) if, as an underlying solution space, the recently introduced space of *piecewise-smooth distributions* [7, 8] is considered. For this space the products $E_\sigma \dot{x}$ and $A_\sigma x$ are always well defined (provided the switching signal does not switch arbitrarily fast). A summary of the corresponding definitions can be found in the Appendix.

It might be possible to reformulate the switched DAE (1) as a hybrid system in the framework of [16] by writing (1) as $\dot{x} \in E_\sigma(x)^{-1}\big(f_\sigma(x)\big)$; however, by doing so, we lose the special structure of (1). In particular, the jumps of the states are implicitly given by (1) and no additional jump map needs to be considered. This is a major difference between switched DAEs and switched ODEs with reset maps. Another related research subject are so-called complementarity systems as e.g. in [17] which is more general then the linear switched DAE (3) because it is used to model elctrical circuits with diodes. However, the complementarity condition contains inequalities which makes the analysis more involved.

The structure of the paper is as follows. In Section 2 we study the non-switched DAE (2) and generalize Lyapunov's Direct Method to the DAE case in Theorem 2.7. This result is based on a presumably new definition of a Lyapunov function for the DAE (2) as formulated in Definition 2.6. In Section 3 the distributional solution framework for switched DAEs of the form (1) is introduced. We formulate Assumption **A4** which under certain regularity assumptions on the subsystems guarantees existence and uniqueness of solutions of the switched DAE (1), see Theorem 3.5. In Section 3.2 we consider the linear case and observe with Corollary 3.10 that the linear equivalent of Assumption **A4** ensures the existence of impulse-free solutions of the linear switched DAE (3). Furthermore, we give an explicit formula for the consistency projectors in Definition 3.8. Finally, in Section 4 we generalize the well-known results that the existence of a "common Lyapunov function" implies asymptotic stability under arbitrary switching; the novel element is that this Lyapunov function must take into account the consistency projectors as formulated in Theorem 4.1. We also prove an average dwell time result in the spirit of [18] for switched nonlinear DAEs (1) in Theorem 4.4.

The following notation is used throughout the paper. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the natural numbers, integers, real and complex numbers, respectively. For a matrix $M \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$, the kernel (null space) of M is $\ker M$, the image (range, column space) of M is $\operatorname{im} M$, and the transpose of M is $M^\top \in$

$\mathbb{R}^{m\times n}$. For a matrix $M \in \mathbb{R}^{n\times n}$ and a set $\mathcal{S} \subset \mathbb{R}^n$, the image of $\mathcal{S}$ under M is $M\mathcal{S} := \{\ Mx \in \mathbb{R}^n \mid x \in \mathcal{S}\ \}$ and the pre-image of $\mathcal{S}$ under M is $M^{-1}\mathcal{S} := \{\ x \in \mathbb{R}^n \mid \exists y \in \mathcal{S} : Mx = y\ \}$. The identity matrix is denoted by I. For a piecewise-continuous function $f : \mathbb{R} \to \mathbb{R}$ the left-sided evaluation $\lim_{\varepsilon \searrow 0} f(t-\varepsilon)$ at $t \in \mathbb{R}$ is denoted by $f(t-)$. The space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ is denoted by $\mathcal{C}^1$, the space of piecewise-smooth functions is denoted by $\mathcal{C}^\infty_{\text{pw}}$, the space of distributions is denoted by $\mathbb{D}$ and the space of piecewise-smooth distributions is denoted by $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, for detailed definition of these spaces see the Appendix. The set of switching signals considered here is

$$\Sigma := \left\{ \sigma : \mathbb{R} \to \{1, \dots, N\} \;\middle|\; \begin{array}{l} \sigma \text{ is right continuous with a} \\ \text{locally finite number of jumps} \end{array} \right\}$$

where $N \in \mathbb{N}$ is the number of subsystems.

2. Non-switched DAEs and Lyapunov functions

2.1. Classical solutions and consistency spaces

Consider for now the (non-switched) nonlinear DAE (2)

$$E(x)\dot{x} = f(x).$$

A (classical, local) *solution* of (2) is any differentiable function $x : J \to \mathbb{R}^n$ which fulfills (2) on some interval $J \subseteq \mathbb{R}$. Due to the time-invariant nature of (2) we can always assume that $J = [0, T)$ for some $T \in (0, \infty]$.

Definition 2.1 (Consistency space). The *consistency space* of (2) is given by

$$\mathfrak{C}_{E,f} := \{\ x_0 \in \mathbb{R}^n \mid \exists \text{ solution } x : [0,T) \to \mathbb{R}^n \text{ with } x(0) = x_0\ \}.$$

Each $x_0 \in \mathfrak{C}_{(E,A)}$ is called *consistent initial value.*

Time-invariance of (2) implies that all solutions x of (2) evolve within $\mathfrak{C}_{E,f}$, i.e. $x(t) \in \mathfrak{C}_{E,f}$ for all $t \in [0, T)$. In general, it is not easy to characterize the solution behavior of (2); for details see e.g. [19, 20, 21]. Here we just assume that the solution behavior is not drastically different from the regular linear case:

Assumption 2.2. The nonlinear DAE (2) satisfies:

- **A1** $f(0) = 0$, in particular $0 \in \mathfrak{C}_{(E,A)}$,
- **A2** $\mathfrak{C}_{E,f}$ is a closed manifold (possibly with boundary) in $\mathbb{R}^n$.
- **A3** For each $x_0 \in \mathfrak{C}_{E,f}$ there exists a unique solution $x : [0, \infty) \to \mathbb{R}^n$ with $x(0) = x^0$ and $x \in (\mathcal{C}^1 \cap \mathcal{C}^\infty_{\text{pw}})^n$.

Remark 2.3 (On A3). First note that due to the time-invariance of (2) the initial time in **A3** can be chosen arbitrarily. Secondly, we exclude systems which exhibit finite escape time. Finally, the assumption that the differentiable solution is also piecewise-smooth is just a technical assumption which will be needed later for studying switched DAEs. It should be possible to avoid this assumption, but then the distributional framework used to study the switched case must be adjusted accordingly. For example, the product of a differentiable function with the Dirac impulse is formally not defined in the forthcoming distributional framework, but it is straightforward (although cumbersome in all generality) to extend the definition to deal also with this case.

For the linear case

$$E\dot{x} = Ax \tag{4}$$

with $E, A \in \mathbb{R}^{n\times n}$ Assumptions **A1** and **A2** are fulfilled trivially (by linearity the consistency space is a linear subspace, see also the forthcoming Theorem 2.4), and **A3** is fulfilled if and only if the matrix pair (E, A) is *regular*, i.e. the polynomial $\det(Es-A) \in \mathbb{R}[s]$ is not the zero polynomial (for details see e.g. the textbook [3]). Furthermore, regularity of the matrix pair (E, A) is equivalent to the existence of invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that a coordinate transformation of the codomain and domain by S and T yields the *quasi-Weierstrass form* [22]

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{5}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $n_1 \in \mathbb{N}$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 = n - n_1$, is *nilpotent*, i.e. $N^{n_2} = 0$. The smallest number $\nu \in \mathbb{N}$ such that $N^\nu = 0$ is called the *index* of the corresponding linear DAE $E\dot{x} = Ax$. It is not difficult to see that the consistency space $\mathfrak{C}_{E,A}$ is spanned by the first n_1 columns of T. Note that the quasi-Weierstrass form implies that any classical solution is not only differentiable but actually analytic. A convenient way to calculate the matrices S and T is the usage of the so-called *Wong sequences* of subspaces.

Theorem 2.4 ([22]). *Consider a regular matrix pair (E, A) with index ν and define the associated Wong sequences by*

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), \quad i = 0, 1, \ldots, & \mathcal{V}^* &:= \bigcap_i \mathcal{V}_i \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i), \quad i = 0, 1, \ldots, & \mathcal{W}^* &:= \bigcup_i \mathcal{W}_i. \end{aligned}$$

The Wong sequences are nested and get stationary after exactly ν steps. For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^* := \mathcal{V}_\nu$ and $\operatorname{im} W = \mathcal{W}^* := \mathcal{W}_\nu$ the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (5) holds. In particular

$$\mathfrak{C}_{E,A} = \mathcal{V}^*.$$

The intuition behind the first Wong sequence $\mathcal{V}_0, \mathcal{V}_1, \ldots$ is the following: Putting no constraint on $\dot{x}$ still yields a constraint on x, because $E\dot{x} = Ax$ implies that $x \in A^{-1}(\operatorname{im} E) = A^{-1}(E\mathcal{V}_0) = \mathcal{V}_1$. Constraining x also constrains $\dot{x}$. Hence $\dot{x} \in \mathcal{V}_1$ and using the same argument as above implies that $x \in A^{-1}(E\mathcal{V}_1) = \mathcal{V}_2$ and so forth. To some extent this idea can also be applied to obtain the consistency space for non-linear DAEs [19, 21]. For ODEs, a similar approach of constructing nested subspace sequences was also used to study zero dynamics [23, Sec. 6]. In fact, the similarity is striking when rewriting the linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$ with zero output as a linear DAE

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Remark 2.5 (Linear index-one case). From the quasi-Weierstrass form (5) it follows that the (classical) solutions of (4) do not depend on N, or, in other words, the solutions remain the same when N is set to be the zero matrix. By definition this is equivalent to assuming that the matrix pair (E, A) is *index-one*. The importance of N only shows up when studying switched DAEs, where a non-zero N might produce impulses in the solutions (we will study impulse free solutions in more detail in Section 3.2). An easy way to exclude impulsive behaviors is an index-one assumption for all subsystems, i.e. assuming that in each quasi-Weierstrass form (5) the nilpotent matrix is the zero matrix. However this assumption excludes a large class of interesting switched DAEs. For example, if all subsystems have the same consistency space, then all solutions of the corresponding switched systems will have neither jumps nor impulses, independently of the index of the subsystems. In Section 3 we propose Assumption **A4**, whose linear equivalent (13) ensures impulse-free solutions and is implied by the above two stricter conditions (index-one or same consistency spaces).

2.2. Stability and Lyapunov functions

We call the DAE (2) *asymptotically stable* when all solutions converge to zero as $t \to \infty$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each consistent initial value $x^0 \in \mathfrak{C}_{E,f}$ with $|x^0| < \delta$ the corresponding solution $x : [0, \infty) \to \mathfrak{C}_{E,f}$ fulfills $|x(t)| < \varepsilon$ for all $t \geq 0$. The only difference with the classical definition of asymptotic stability is the restriction to consistent initial values. Later, in the switched case, we have to reconsider this restriction, because due to the switching it is not guaranteed that the initial value at a switching instant is consistent.

Definition 2.6 (Lyapunov function). Consider the DAE (2) satisfying **A1**-**A3**. A continuously differentiable non-negative function $V : \mathfrak{C}_{E,f} \to \mathbb{R}_{\geq 0}$ with the following properties:

L1 V is positive definite, i.e. $V(x) = 0 \;\Leftrightarrow\; x = 0$, and for all $x \in \mathfrak{C}_{E,f}$ each sublevel set $V^{-1}[0, V(x)] \subseteq \mathfrak{C}_{E,f}$ is bounded (hence compact by **A2**),

L2 $\exists F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous such that $\nabla V(x)z = F(x, E(x)z)$ for all $x \in \mathfrak{C}_{E,f}$, $z \in T_x\mathfrak{C}_{E,f}$, where $T_x\mathfrak{C}_{E,f} \subseteq \mathbb{R}^n$ is the tangent space of $\mathfrak{C}_{E,f}$ at x,

L3 $\dot{V}(x) := F\big(x, f(x)\big) < 0$ for all $x \in \mathfrak{C}_{E,f}\backslash\{0\}$

is called *Lyapunov function* for (2).

Note that in the linear case (4) the tangent space $T_x\mathfrak{C}_{E,A}$ is identical to the consistency space $\mathfrak{C}_{E,A}$ for all $x \in \mathfrak{C}_{E,A}$, hence **L2** simplifies in this case. Furthermore, for any non-trivial solution x of (2) with a Lyapunov function V it holds that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \nabla V\big(x(t)\big)\dot{x}(t) \overset{\textbf{L2}}{=} F\big(x(t), E(x(t))\dot{x}(t)\big)$$
$$= F\big(x(t), f(x(t)\big) = \dot{V}\big(x(t)\big) \overset{\textbf{L3}}{<} 0, \quad (6)$$

hence V is *decreasing along solutions.*

Theorem 2.7 (Lyapunov's direct method). *Consider the DAE* (2) *satisfying **A1**-**A3**. If there exists a Lyapunov function for* (2) *then* (2) *is (globally) asymptotically stable.*

Proof. *Stability*
For $\varepsilon > 0$ consider the set $B_\varepsilon := \left\{ x \in \mathfrak{C}_{(E,A)} \;\middle|\; |x| = \varepsilon \right\}$ which is empty or compact by Assumption **A2**. If $B_\varepsilon = \emptyset$ then each solution starting within the set enclosed by B_ε cannot leave this set, hence stability follows in this case. Otherwise, let $b := \min_{x\in B_\varepsilon} V(x)$ where positive definiteness of V implies $b > 0$. Continuity of V and $V(0) = 0$ guarantees the existence of $\delta > 0$ such that $V(x) < b$ for all $|x| < \delta$, in particular $\delta < \varepsilon$. From (6) it follows that $t \mapsto V\big(x(t)\big)$ is decreasing for any solution x of (2), hence any solution x with $|x(0)| < \delta$ fulfills $V(x(t)) < b$ for all $t \geq 0$. Seeking a contradiction, assume there exists $t > 0$ such that $|x(t)| \geq \varepsilon$, then, by continuity of x together with $|x(0)| < \delta < \varepsilon$, there exists $t_1 \in (0, t)$ such that $|x(t_1)| = \varepsilon$ which leads to $b \leq V(x(t_1)) < b$.
Convergence to zero
Step 1: $V\big(x(t)\big) \to 0$ as $t \to \infty$.
Let $x : [0, \infty) \to \mathfrak{C}_{E,f}$ be any non-trivial solution, then the non-negative function $t \mapsto v(t) := V\big(x(t)\big) \geq 0$ is strictly decreasing by (6). Therefore, $\underline{v} = \lim_{t\to\infty} v(t)$ is well defined. Seeking a contradiction, assume $\underline{v} > 0$. Then $v(t) \in [\underline{v}, v(0)]$ for all $t \geq 0$. By **L1** and continuity of V, $K := V^{-1}[\underline{v}, v(0)]$ is a compact set, hence $M := \dot{V}(K) \subseteq \mathbb{R}$ is also compact (since $\dot{V}$ is continuous) and $0 \notin M$. This implies that $m := -\max M > 0$ and, in particular, $v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t)\big) = \dot{V}\big(x(t)\big) < -m < 0$ for all $t \geq 0$. Hence $v(t) \leq v(0) - mt$ for all $t \geq 0$, which contradicts $v(t) \geq 0$ for all $t \geq 0$, hence $\underline{v} = 0$ must hold.
Step 2: $x(t) \to 0$ as $t \to \infty$.
Seeking a contradiction, assume $x(t) \not\to 0$, then there exists a sequence $(t_n)_{n\in\mathbb{N}}$

with $t_n \to \infty$ as $n \to \infty$ and $\varepsilon > 0$ such that $|x(t_n)| > \varepsilon$. By **L1** and (6), each solution x evolves within the compact set $V^{-1}[0, V(x(0))]$, hence there exists a convergent subsequence of $x(t_n)$ with limit $x^* \neq 0$. By continuity and positive definiteness of V we arrive at the contradiction $0 = \lim_{t\to\infty} V(x(t)) = V(x^*) > 0$.

Remark 2.8 (Local asymptotic stability). The sublevel-sets-compactness assumption in **L1** (or the commonly used stronger assumption of radial unboundedness) of V is not needed to show local asymptotic stability. This follows from the observation that continuity and positive definiteness of V already implies that $V^{-1}[0, V(x)] \cap \{|x| < 1\}$ is compact for sufficiently small x.

Remark 2.9 (The linear case). In the linear, regular case it is well-known [5][1] that $E\dot{x} = Ax$ is asymptotically stable if, and only if, there exists a solution $(P, Q) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ of the generalized Lyapunov equation

$$A^\top PE + E^\top PA = -Q, \tag{7}$$

where $P = P^\top$ is positive definite and $Q = Q^\top$ is positive definite on $\mathfrak{C}_{E,A}$. In fact, it is easy to see that then $V(x) = (Ex)^\top PEx$ is a Lyapunov function in the sense of Definition 2.6 with

$$\nabla V(x)z = (Ex)^\top PEz + (Ez)^\top PEx =: F(x, Ez)$$

and

$$\dot{V}(x) = x^\top (E^\top PA + A^\top PE)x = -x^\top Qx < 0 \text{ on } \mathfrak{C}_{(E,A)}.$$

Compare also the result in [24] which yields the same Lyapunov function under weaker assumptions, at least when the matrix pair is already in Weierstrass normal form [25].

If the linear system $E\dot{x} = Ax$ is index-one, i.e. $N = 0$ in the quasi-Weierstrass form (5), it is shown in [6, 25] that asymptotic stability is also equivalent to the existence of a solution $P \in \mathbb{R}^{n\times n}$ of

$$P^\top A + A^\top P = -Q, \quad P^\top E = E^\top P \geq 0,$$

for any positive definite $Q \in \mathbb{R}^{n\times n}$. The corresponding "unsymmetric" Lyapunov function[2] is given by $V(x) = (Ex)^\top Px$ with

$$\nabla V(x)z = \underbrace{(Ex)^\top Pz}_{=x^\top P^\top Ez} + (Ez)^\top Px =: F(x, Ez)$$

and

$$\dot{V}(x) = x^\top (P^\top A + A^\top P)x = -x^\top Qx < 0.$$

[1] Actually, in [5] only the *complex-valued* case is studied; however, by considering the real part of the generalized Lyapunov equation (7) we also obtain real-valued matrix pairs (P, Q) with the desired properties.

[2] We thank Emilia Fridman for making us aware of this Lyapunov function construction.

We conclude this section with an example which illustrates the application of Theorem 2.7.

Example 2.10. Consider the nonlinear DAE

$$\begin{bmatrix} \sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_1 \sin x_3 - x_2 \cos x_3 \\ x_1 \cos x_3 - x_2 \sin x_3 \\ x_3 - x_1^2 - x_2^2 \end{pmatrix}, \tag{8}$$

which fulfills our Assumptions **A1**, **A2** and **A3**. The consistency space is given by the equation $x_3 = x_1^2 + x_2^2$ and $x_1 \cos x_3 = x_2 \sin x_3$; the projection to the x_1-x_2-plane is illustrated in Figure 1. Note that the consistency space can be parametrized by

$$\mathfrak{C}_{E,f} = \left\{ \ (\theta \sin \theta^2, \theta \cos \theta^2, \theta^2)^\top \ \middle| \ \theta \in \mathbb{R} \ \right\}.$$

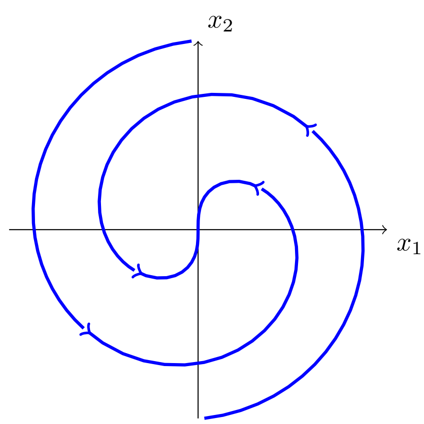

Figure 1: Consistency space of Example 2.10 in the x_1-x_2-plane (the dynamics within the consistency space are shown by the arrows).

The corresponding tangent space is given by

$$T_x\mathfrak{C}_{E,f} = \operatorname{span}\left\{(x_1 + 2x_2x_3, x_2 - 2x_1x_3, 2x_3)^\top\right\} \text{ for } x \neq 0 \tag{9}$$

and $T_0\mathfrak{C}_{E,f} = \operatorname{span}\left\{(0,1,0)^\top\right\}$. We propose the following Lyapunov function candidate:

$$V(x) = x_3.$$

For all $x \in \mathfrak{C}_{E,f}$ it follows that $x_3 = x_1^2 + x_2^2$, hence V fulfills **L1**. For $x \in \mathfrak{C}_{E,f}$ and $v \in E(x)^{-1}(T_x\mathfrak{C}_{E,f})$ let

$$F(x,v) := \frac{2x_3 v_1}{x_1 \sin x_3 + x_2 \cos x_3},$$

then for all $x \in \mathfrak{C}_{E,f}$, $z \in T_x\mathfrak{C}_{E,f}$ and by using (9) as well as $x_1 \cos x_3 = x_2 \sin x_3$ we get

$$F(x, E(x)z) = z_3 = \nabla V(x)z,$$

hence **L2** is fulfilled. Finally,

$$\dot{V}(x) = F\left(\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right), \left(\begin{smallmatrix} -x_1 \sin x_3 - x_2 \cos x_3 \\ x_1 \cos x_3 - x_2 \sin x_3 \\ x_3 - x_1^2 - x_2^2 \end{smallmatrix}\right)\right) = \frac{2x_3(-x_2 \sin x_3 - x_2 \cos x_3)}{x_1 \sin x_3 + x_2 \cos x_3} = -2x_3,$$

hence **L3** is fulfilled and V is a Lyapunov function for (8) and Theorem 2.7 shows that (8) is globally asymptotically stable.

3. Solutions of switched DAEs

3.1. The general nonlinear case

Recall the switched nonlinear DAE (1)

$$E_\sigma(x)\dot{x} = f_\sigma(x),$$

where each subsystem $E_p(x)\dot{x} = f_p(x)$, $p = \{1, \ldots, N\}$, fulfills Assumptions **A1-A3** and $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ is an admissible switching signal, i.e.

$$\sigma \in \Sigma := \{\ \sigma : \mathbb{R} \to \{1, \ldots, N\}\ \mid \sigma \text{ is right continuous with a locally finite number of jumps }\}.$$

As an underlying solution framework for (1), we will use the space $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ of *piecewise smooth distributions* which was introduced in [7, 8] for studying linear switched DAE. For a short summary of the basic definition and the main properties of piecewise-smooth distributions see the Appendix.

Definition 3.1 (Solution of switched nonlinear DAE). A *solution* of (1) on some interval $J \subseteq \mathbb{R}$ is any *piecewise-smooth function* $x \in (\mathcal{C}^\infty_{\mathrm{pw}})^n$ such that (1) restricted to J holds as an equation of piecewise-smooth distributions, i.e.

$$(E_\sigma(x)(x_{\mathbb{D}})')_J = (f_\sigma(x)_{\mathbb{D}})_J,$$

The product $E(x)(x_{\mathbb{D}})'$ in Definition 3.1 is well defined, since by assumption $t \mapsto E(x(t))$ is piecewise smooth and $(x_{\mathbb{D}})'$ is a piecewise-smooth distribution. Note that this definition of a solution does not allow for Dirac impulses in the solution. There are two reasons for this: 1) It is not clear how a nonlinear function of a Dirac impulse should be defined in general and 2) for stability analysis the existence of Dirac impulses in the solution can be interpreted as an undesired unstable solution. However, in Section 3.2 we will also study solutions with impulses for *linear* switched DAEs.

In the following we will give sufficient conditions which ensure existence and uniqueness of solutions of the switched DAE (1).

Assumption 3.2. The switched DAE (1) and the corresponding consistency spaces $\mathfrak{C}_p := \mathfrak{C}_{E_p,f_p}$, $p \in \{1, \ldots, N\}$, satisfy:

A4 $\forall p,q \in \{1,\dots,N\}\ \forall x_0^- \in \mathfrak{C}_p\ \exists$ unique $x_0^+ \in \mathfrak{C}_q: \quad x_0^+ - x_0^- \in \ker E_q(x_0^+)$.

Note that Assumption **A4** is always fulfilled, with $x_0^+ = x_0^-$, if the subsystems are ODEs (i.e. $E_p(x)$ is an invertible matrix for each x and p). Clearly, Assumption **A4** makes it possible to define nonlinear consistency projectors Π_q, $q \in \{1,\dots,N\}$:

$$\Pi_q : \bigcup_p \mathfrak{C}_p \to \mathfrak{C}_q, \quad x_0^- \mapsto x_0^+,$$

where x_0^+ is the unique value given by Assumption **A4**. In particular, $\Pi_q(x) = x$ for all $x \in \mathfrak{C}_q$. In general it might not be easy to check **A4** and to give an explicit definition of the consistency projector. However, in the linear case matters simplify significantly, see Section 3.2.

Remark 3.3 (Assumption A4 for a single system). Note that Assumption **A4** applied to each single system, i.e. $p = q$, additionally restricts the possible nonlinear DAEs even without switching: In **A4** one can always choose $x_0^+ = x_0^-$ if $p = q$ and the asserted uniqueness of x_0^+ implies therefore

$$\forall x_0^+ \in \mathfrak{C}_p: \quad \ker E_p(x_0^+) \cap \left\{\ x_0^+ - x_0^-\ \middle|\ x_0^- \in \mathfrak{C}_p\ \right\} = \{0\}. \tag{10}$$

So in addition to **A1-A3** each subsystem must also fulfill (10). In the linear case it can be shown that **A3** already implies (10), but in the general case this is not true as the following example shows:

$$\begin{aligned} x_2\dot{x}_1 &= 0\\ \dot{x}_2 &= 1 \end{aligned}$$

Considering the initial value $x_2(0) = t_0$, the DAE reduces to $\dot{x}_1(t) = 0$ for $t \neq -t_0$. For any initial value x_0 there exists a unique classical solution, namely $x(t) = x_0$ for all $t \in \mathbb{R}$, hence **A3** holds. However, condition **A4** is not fulfilled because (10) does not hold. In fact, when allowing jumps in solutions (as in the case for switched DAEs) uniqueness of solutions is lost, because x can have an arbitrary jump at $t = t_0$ without violating the DAE (in a distributional sense).

Example 3.4 (Example 2.10 revisited). It is not difficult to show that Assumption **A4** or, equivalently, (10) is *not* fulfilled for Example 2.10, hence when allowing jumps in solutions the uniqueness of solutions cannot be guaranteed anymore. However, we can use the fact that from $x_1^2 + x_2^2 - x_3 = 0$ it follows that $2x_1\dot{x}_1 + 2x_2\dot{x}_2 - \dot{x}_3 = 0$, hence the altered nonlinear DAE

$$\begin{bmatrix} \sin x_3 & \cos x_3 & 0\\ 2x_1 & 2x_2 & -1\\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1\\ \dot{x}_2\\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_1\sin x_3 - x_2\cos x_3\\ x_1\cos x_3 - x_2\sin x_3\\ x_3 - x_1^2 - x_2^2 \end{pmatrix}, \tag{11}$$

has the same (classical) solution behavior as the DAE (8). One can show that now (11) fulfills Assumption **A4** for $p = q$. The reason is that by including the term $2x_1\dot{x}_1 + 2x_2\dot{x}_2 - \dot{x}_3$ we imposed more differentiability conditions on the state space variables and therefore prohibited certain jumps. In particular, all solutions of the DAE (11) must have a differentiable third component; this property did not directly follow from the DAE (8), because the derivative of x_3 did not appear.

Theorem 3.5 (Existence and uniqueness of solutions of switched DAE). *Consider the switched nonlinear DAE* (1) *satisfying* ***A4*** *and* ***A1****-****A3*** *for each subsystem. Then for every switching signal* $\sigma \in \Sigma$ *and every* $x_0 \in \mathfrak{C}_{\sigma(0-)}$ *there exists a unique solution* $x \in (\mathcal{C}^\infty_{\text{pw}})^n$ *of* (1) *on* $[0,\infty)$ *with* $x(0-) = x_0$*. Furthermore, for all* $t \in [0,\infty)$ *and all solutions* x *of* (1)*,*

$$x(t) = \Pi_{\sigma(t)}(x(t-)),$$

where Π_p*,* $p \in \{1,\dots,N\}$*, are the consistency projectors induced by* ***A4****. In particular, on each interval which does not contain a switching time,* x *is a classical solution of the corresponding subsystem.*

Proof. *Step 1:* Existence of a solution.
Let $t_0 = 0$ and $t_i > 0$, $i = 1,2,\dots$ be the ordered switching times of σ after t_0 and let $p_i := \sigma(t_i)$. Inductively and invoking Assumption **A3** choose $x^i \in (\mathcal{C}^1 \cap \mathcal{C}^\infty_{\text{pw}})^n$, $i \in \mathbb{N}$, such that x^i is the unique (classical) solution of $E_{p_i}(x^i)\dot{x}^i = f_{p_i}(x^i)$ on the interval $[t_i, t_{i+1})$ with $x^i(t_i) = \Pi_{p_i}(x^{i-1}(t_i-))$, where $x^{-1}(t_0-) := x_0$. We show that any $x \in (\mathcal{C}^\infty_{\text{pw}})^n$ with $x(0-) = x_0$ and $x_{[t_i,t_{i+1})} = {x^i}_{[t_i,t_{i+1})}$ for $i \in \mathbb{N}$ solves the switched DAE (1) on $[0,\infty)$. By definition x solves (1) on each open interval (t_i, t_{i+1}) and it remains to check that

$$(E_\sigma(x)(x_\mathbb{D})')[t_i] = (f_\sigma(x)_\mathbb{D})[t_i] = 0 \quad \text{for all } i \in \mathbb{N},$$

where $D[t]$ denotes the impulsive part of $D \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ at $t \in \mathbb{R}$ (see Appendix for details). Invoking the properties of piecewise-smooth distributions, it follows that

$$\begin{aligned}(E_\sigma(x)(x_\mathbb{D})')[t_i] &= E_{p_i}(x(t_i))\big(x(t_i) - x(t_i-)\big)\delta_{t_i} \\ &= E_{p_i}\big(\Pi_{p_i}\big(x(t_i-)\big)\big)\big(\Pi_{p_i}\big(x(t_i-)\big) - x(t_i-)\big)\delta_{t_i} = 0,\end{aligned}$$

where the last equation follows from Assumption **A4**. Hence x is a solution of (1) on $[0,\infty)$.
Step 2: Uniqueness of the solution.
With the notation as in Step 1 it suffices to show that the solution x as constructed above is unique on $[0, t_1)$, uniqueness on $[t_1,\infty)$ follows then inductively. Let $z \in (\mathcal{C}^\infty_{\text{pw}})^n$ be a solution of (1) on $[0,t_1)$ with $z(0-) = x_0$. With a similar argument as in Step 1 it follows that

$$E_{p_0}(z(0))(z(0) - x_0) = 0,$$

hence Assumption **A4** ensures $z(0) = \Pi_{p_0}(x_0) = x(0)$. Furthermore, Assumption **A4** also implies that $z(t) = z(t-)$ for all $t \in (0,t_1)$, hence z is continuous on $(0,t_1)$ which together with Assumption **A3** implies that $z_{(0,t_1)} = x_{(0,t_1)}$. Hence uniqueness of the solution is shown.

Remark 3.6 (Index-one systems). A sufficient condition for **A4** is the possibility to transform each nonlinear DAE subsystem into

$$\begin{aligned} \dot{x}_1 &= g(x_1, x_2) \\ 0 &= h(x_1, x_2) \end{aligned}$$

where h is such that x_2 can be solved in terms of x_1. This is often called the index-one case. However, Assumption **A4** is weaker because it could hold even when not all subsystems are index-one, see also Remark 2.5.

Remark 3.7 (No additional jump map). Although switches induce jumps in solutions, it is not necessary to define additional jump maps; these are implicitly given by the subsystems themselves which fulfill Assumption **A4**. This is a special feature of switched differential algebraic equations and is in contrast to switched ODEs with jumps.

3.2. The linear case

Consider the linear switched DAE (3)

$$E_\sigma \dot{x} = A_\sigma x,$$

where $E_p, A_p \in \mathbb{R}^{n\times n}$, $p \in \{1, \ldots, N\}$, and $\sigma \in \Sigma$ is the switching signal. As already mentioned above, the Assumptions **A1-A3** for each subsystem reduce to the regularity condition $\det(E_p s - A_p) \not\equiv 0$ for each subsystem. Under this assumption (in particular without assuming **A4**) it already follows from [7, 8] that existence and uniqueness of solutions of (3) is guaranteed. However, these solutions are then elements of the space of *piecewise-smooth distributions* and will therefore, in general, contain *Dirac impulses* and their derivatives. Since the presence of impulses in solutions can be seen as an undesired unstable behavior, we would like to give an easily checkable condition which ensures that for arbitrary switching all solutions of (3) are impulse-free. It will turn out that this condition is equivalent to Assumption **A4** but is easier to check in the linear case. For the formulation of this condition, we define the linear consistency projector of a regular matrix pair (E, A).

Definition 3.8 (Linear consistency projector). Consider a regular matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ and, invoking Theorem 2.4, choose invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that (SET, SAT) is in quasi-Weierstrass form (5) with $n_1 \times n_1$ and $n_2 \times n_2$ the corresponding diagonal block sizes. The *linear consistency projector* is then given by

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

where I is an $n_1 \times n_1$ identity matrix.

Let $\mathcal{V}^*$ and $\mathcal{W}^*$ be the limits of the Wong sequences as in Theorem 2.4. Then it is easy to see that the definition of $\Pi_{(E,A)}$ is independent of the choice of T and that it is a linear projection onto $\mathcal{V}^* = \mathfrak{C}_{(E,A)}$ along $\mathcal{W}^*$, i.e.

$$\Pi_{(E,A)}^2 = \Pi_{(E,A)}, \quad \operatorname{im}\Pi_{(E,A)} = \mathfrak{C}_{(E,A)} = \mathcal{V}^*, \quad \text{and } \ker\Pi_{(E,A)} = \mathcal{W}^*. \tag{12}$$

With the help of the linear consistency projectors it is now possible to give an easily checkable characterization of Assumption **A4**.

Theorem 3.9 (Linear version of Assumption A4). *Consider the switched linear DAE* (3) *and let* $\Pi_p := \Pi_{(E_p,A_p)}$, $p \in \{1,\dots,N\}$ *given as in Definition 3.8. Then Assumption* ***A4*** *is equivalent to*

$$\forall p,q \in \{1,\dots,N\}: \quad E_q(\Pi_q - I)\Pi_p = 0 \tag{13}$$

and the linear mapping $x_0^- \mapsto x_0^+ := \Pi_q x_0^-$ *coincides with the consistency projector associated with Assumption* ***A4***.

Proof. Let $p,q \in \{1,\dots,N\}$ and $x_0^- \in \mathfrak{C}_p := \mathfrak{C}_{(E_p,A_p)}$ be arbitrary and fixed in the rest of the proof.
Step 1: We show (13) $\Rightarrow$ **A4**.
Let $x_0^+ := \Pi_q x_0^- \in \mathfrak{C}_q := \mathfrak{C}_{(E_q,A_q)}$, then, since $\Pi_p x_0^- = x_0^-$,

$$E_q(x_0^+ - x_0^-) = E_q(\Pi_q\Pi_p x_0^- - \Pi_p x_0^-) = E_q(\Pi_q - I)\Pi_p x_0^- \overset{(13)}{=} 0,$$

hence the existence assertion of Assumption **A4** is shown. To show uniqueness of x_0^+, let $z \in \mathfrak{C}_q$ be such that

$$z - x_0^- \in \ker E_q \subseteq \mathcal{W}_q^* = \ker \Pi_q,$$

where $\mathcal{W}_q^*$ is the limit of the corresponding Wong sequence for (E_q, A_q) as in Theorem 2.4. Together with $\Pi_q z = z$ this implies $z = \Pi_q x_0^- = x_0^+$.
Step 2: We show **A4** $\Rightarrow$ (13).
Choose $x_0^+ \in \mathfrak{C}_q$ such that $x_0^+ - x_0^- \in \ker E_q \subseteq \mathcal{W}_q^* = \ker\Pi_q$, hence $x_0^+ = \Pi_q x_0^+ = \Pi_q x_0^-$. Therefore, by $\Pi_p x_0^- = x_0^-$,

$$0 = E_q(x_0^+ - x_0^-) = E_q(\Pi_q\Pi_p x_0^- - \Pi_p x_0^-) = E_q(\Pi_q - I)\Pi_p x_0^-.$$

Since $x_0^- \in \mathfrak{C}_p = \mathcal{V}_p^*$ is arbitrary it follows from $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$ together with $\mathcal{W}^* = \ker\Pi_p$ that $E_q(\Pi_q - I)\Pi_p = 0$, hence (13).

Combining Theorems 3.5 and 3.9 yields that for every switched linear DAE 3 with regular matrix pairs (E_p, A_p), $p = 1,\dots,N$ satisfying (13) there *exists* a solution $x \in (\mathcal{C}_{\text{pw}}^\infty)$. By definition, this solution also solves 3.9 in the distributional framework of [7, 8]. Since the switched DAE (3) with regular pairs (E_p, A_p), $p = 1,\dots,N$ has a *unique* distributional solution (for a fixed initial value $x(0-)$) we obtain the following result.

Corollary 3.10 (Impulse-free solutions for linear switched DAE). *Consider the switched DAE* (3) *with arbitrary switching signal* $\sigma \in \Sigma$ *and regular matrix pairs* (E_p, A_p) *with corresponding consistency projectors* $\Pi_p \in \mathbb{R}^{n\times n}$ *given by Definition 3.8. If* (13) *holds, then every distributional solution* $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ *of* (3) *is impulse-free.*

4. Asymptotic stability of (nonlinear) switched DAEs

The definition for asymptotic stability of (1) is basically identical to the definition in the non-switched case; the only difference is that the solutions might have jumps, so we have to decide where to evaluate the initial value. In view of Theorem 3.5, we consider the initial value $x(0-)$. Note that in the linear case Assumption **A4** excludes impulses in the solution, which is reasonable for the definition of stability, because an impulse can be interpreted as an infinite peak which remains infinite even when the corresponding solution is scaled so that $|x(0-)|$ gets arbitrarily small. Finally note that we do not assume that the ε-δ-definition of stability is uniform in the switching signal $\sigma \in \Sigma$.

Theorem 4.1 (Asymptotic stability under arbitrary switching). *Consider the switched DAE* (1) *satisfying Assumption* ***A4*** *and Assumptions* ***A1-A3*** *for each subsystem with corresponding consistency space* $\mathfrak{C}_p := \mathfrak{C}_{E_p,f_p}$ *and consistency projectors* Π_p, $p \in \{1,\ldots,N\}$ *induced by* ***A4***. *Assume for each subsystem that there exists a Lyapunov function* $V_p : \mathfrak{C}_p \to \mathbb{R}_{\geq 0}$ *in the sense of Definition 2.6. If*

$$\forall p,q \in \{1,\ldots,N\}\ \forall x \in \mathfrak{C}_p: \quad V_q(\Pi_q(x)) \leq V_p(x), \tag{14}$$

then the switched DAE (1) *is asymptotically stable for any switching signal* $\sigma \in \Sigma$.

Proof. *Step 1:* Definition of a common Lyapunov function candidate.
If $x \in \mathfrak{C}_p \cap \mathfrak{C}_q$ for some $p,q \in \{1,\ldots,N\}$ then $x = \Pi_p(x) = \Pi_q(x)$ hence (14) implies $V_p(x) = V_q(x)$. Therefore

$$V : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \begin{cases} V_p(x), & x \in \mathfrak{C}_p, \\ 0, & \text{otherwise,} \end{cases}$$

is well defined.
Step 2: $V\big(x(t)\big) \to 0$ as $t \to \infty$.
Fix $\sigma \in \Sigma$, let $I \subseteq \mathbb{R}$ be an interval without switching times and consider a solution $x : \mathbb{R} \to \mathbb{R}^n$ of (1). By Theorem 3.5, this solution is a classical (local) solution of $E_p(x)\dot{x} = f_p(x)$ on I where $p := \sigma(\tau)$ for $\tau \in I$. From $x(\tau) \in \mathfrak{C}_p$ for all $\tau \in I$ it follows that $V(x(\tau)) = V_p(x(\tau))$ and, by Definition 2.6 together with (6),

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_p\big(x(\tau)\big) = \dot{V}_p\big(x(\tau)\big) < 0 \quad \forall \tau \in I.$$

Let $t \in \mathbb{R}$ be a switching time of σ, then $x(t) = \Pi_{\sigma(t)}\big(x(t-)\big)$ and $x(t-) \in \mathfrak{C}_{\sigma(t-)}$, hence, by (14),

$$\begin{aligned} V\big(x(t)\big) &= V_{\sigma(t)}\big(x(t)\big) = V_{\sigma(t)}\big(\Pi_{\sigma(t)}(x(t-))\big) \\ &\leq V_{\sigma(t-)}\big(x(t-)\big) = V\big(x(t-)\big) \end{aligned}$$

Hence $t \mapsto v(t) = V\big(x(t)\big)$ is monotonically decreasing and therefore $\underline{v} := \lim_{t\to\infty} v(t) \geq 0$ is well defined. Seeking a contradiction, assume $\underline{v} > 0$. Analogously to the proof of Theorem 2.7 let $K_p := V_p^{-1}[\underline{v}, v(0)]$, $M_p := \dot{V}(K_p)$ and

$m_p := -\max M > 0$. Let $m = \min_p m_p > 0$ then $\frac{\mathrm{d}}{\mathrm{d}t} v(t) < -m < 0$ for all non-switching (hence almost all) times $t \geq 0$, which contradicts $v(t) \geq 0$ and the assertion of Step 2 is shown.
Step 3: $x(t) \to 0$ as $t \to \infty$.
Seeking a contradiction, assume $x(t) \not\to 0$. Then there exists $\varepsilon > 0$ and a sequence $(s_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $s_i \to \infty$ as $i \to \infty$ such that $\|x(s_i)\| > \varepsilon$ for all $i \in \mathbb{N}$. There is at least one $p \in \{1, \ldots, N\}$ such that the set $\{ \ i \in \mathbb{N} \mid \sigma(s_i) = p \ \}$ has infinitely many elements, therefore, without loss of generality, assume that $\sigma(s_i) = p$ for some p and all $i \in \mathbb{N}$. Since each $x(s_i)$ is contained within the compact set $V_p^{-1}[0, V(x(0))]$, the same argument as in the proof of Theorem 2.7 shows existence of $x^* \neq 0$ such that we arrive at the contradiction $0 = \lim_{t\to\infty} V(x(t)) = \lim_{i\to\infty} V_p(x(s_i)) = V_p(x^*) \neq 0$.
Step 4: Stability of the switched DAE.
We first show that

$$\forall \varepsilon > 0 \ \exists b_\varepsilon > 0 \ \forall p \in \{1, \ldots, N\} \ \forall x \in \mathfrak{C}_p : \quad V_p(x) < b_\varepsilon \ \Rightarrow \ |x| < \varepsilon. \tag{15}$$

Assume the contrary, then there exists $\varepsilon > 0$ and sequences $(p_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that $V_{p_n}(x_n) < 1/n$ and $|x_n| \geq \varepsilon$. There exist at least one $p \in \{1, \ldots, N\}$ which occurs infinitely often in the sequence (p_n), so we can, without loss of generality, assume that $p_n = p$ for all $n \in \mathbb{N}$ and some $p \in \{1, \ldots, N\}$. Then, by **L1**, all x_n are contained in the compact set $V_p^{-1}[0, V_p(x_{n_{\max}})]$ where $n_{\max} := \operatorname{argmax}_n V_p(x_n) < \infty$. This implies that there exists $x^* \in \mathfrak{C}_p$ which is a limit of a subsequence of (x_n) and with $|x^*| \geq \varepsilon$. Hence we arrive at the contradiction $0 = \lim_{n\to\infty} V_p(x_n) = V_p(x^*) \neq 0$ and the claim (15) is shown.

For a given $\varepsilon > 0$ choose $b_\varepsilon > 0$ according to (15). Let $p_0 := \sigma(0-)$, then by continuity of V_{p_0} there exists $\delta > 0$ such that $|x| < \delta$ implies $V_{p_0}(x) < b_\varepsilon$ for all $x \in \mathfrak{C}_{p_0}$. In Step 2 it was shown that $t \mapsto V_{\sigma(t-)}(x(t-))$ is monotonically decreasing, hence $V_{\sigma(t-)}(x(t-)) < b_\varepsilon$ for all $t \geq 0$. Hence (15) yields $|x(t-)| < \varepsilon$ for all $t \geq 0$.

Remark 4.2 (Stability and compactness of sublevel sets). In contrast to the stability proof of the non-switched case in Theorem 2.7 we needed the compactness-of-sublevel-sets assumption of each Lyapunov function to prove stability of the switched DAE (1). In particular, the solutions of the switched DAE can exhibit jumps and therefore the continuity argument used in the proof of Theorem 2.7 cannot be applied here.

Condition (14) implies that any two Lyapunov functions V_p and V_q coincide on the intersection $\mathfrak{C}_p \cap \mathfrak{C}_q$, hence Theorem 4.1 is a generalization of the switched ODE case where the existence of a common Lyapunov function is sufficient to ensure stability under arbitrary switching [10, Thm. 2.1]. However, the existence of a common Lyapunov function is not enough in the DAE case [9]. Under arbitrary switching, solutions will in general exhibit jumps; these jumps are described by the consistency projectors, and these projectors must “fit together” with the Lyapunov functions in the sense of (14) to ensure stability of the switched DAE under arbitrary switching. If one assumes that the switching signal is chosen in such a way that no jumps occur, then the conditions on the consistency projectors are not needed. To be precise, we consider the following set of switching signals

$$\Sigma_{x^0}^{\mathrm{jf}} := \left\{ \sigma \in \Sigma \;\middle|\; \begin{array}{l} \exists \text{ solution } x \text{ of } (1) \\ \text{with } x(0) = x^0 \text{ and} \\ x \text{ is jump free} \end{array} \right\},$$

where $x_0 \in \mathbb{R}^n$. Note that these switching signals can be realized by state-dependent switching, where a switch from subsystem p to subsystem q is only possible when the state is contained in $\mathfrak{C}_p \cap \mathfrak{C}_q$. However, for linear systems it is possible to describe the switching signals $\Sigma_{x^0}^{\mathrm{jf}}$ without explicit reference to the state, see the examples in [9]. Since the allowed switching signal depends on the initial value, it is not possible to speak of asymptotic stability as defined above, because there the switching signal is fixed first and afterwards the initial values are considered. Nevertheless, we can formulate the following result.

Corollary 4.3 (Solutions without jumps). *Consider the switched DAE* (1) *satisfying* ***A1**-**A4*** *and assume each DAE $E_p(x)\dot{x} = f_p(x)$, $p = 1, \ldots, N$, has a Lyapunov function V_p. If*

$$\forall p, q = 1, \ldots, N \;\forall x \in \mathfrak{C}_p \cap \mathfrak{C}_q : \quad V_p(x) = V_q(x) \tag{16}$$

then all solutions x of (1) *with $x(0) = x^0 \in \mathbb{R}^n$ and $\sigma \in \Sigma_{x^0}^{jf}$ converge to zero as $t \to \infty$.*

If $\mathfrak{C}_p \cap \mathfrak{C}_q = \{0\}$ for all $p, q \in \{1, \ldots, N\}$ then the only jump-free solution for non-constant switching signals is the trivial solution, hence although Corollary 4.3 is applicable it is not very useful in this case.

It is well-known for switched ODEs that by restricting the class of switching signals one can obtain asymptotic stability also in cases where no common Lyapunov function exists. Denote by $N_\sigma(t, T)$ the number of switchings of σ in the interval $[t, T)$ and define the class of *average dwell time* switching signals with average dwell time $\tau_a > 0$ [18]

$$\Sigma_{\tau_a} := \left\{ \sigma \in \Sigma \;\middle|\; \exists N_0 > 0 \;\forall t \in \mathbb{R} \;\forall \Delta t > 0 : \; N_\sigma(t, t + \Delta t) < N_0 + \tfrac{\Delta t}{\tau_a} \right\}.$$

The number $N_0 > 0$ is called *chatter bound* of the switching signal $\sigma \in \Sigma_{\tau_a}$. Note that the subset of average dwell time switching signals with chatter bound $N_0 = 1$ is precisely the class of switching signals with *dwell time* τ_a.

Theorem 4.4 (Asymptotic stability under average dwell time switching). *Consider the switched DAE* (1) *with corresponding consistency space $\mathfrak{C}_p$ and consistency projectors Π_p, $p \in \{1, \ldots, N\}$. Assume that all subsystems permit a Lyapunov function V_p, $p \in \{1, \ldots, N\}$, which additionally fulfill*

ADT1 $\exists \lambda > 0 \;\forall p \in \{1, \ldots, N\} \;\forall x \in \mathfrak{C}_p : \quad \dot{V}_p(x) \leq -\lambda V_p(x)$ *and*

ADT2 $\exists \mu > 0\ \forall p, q \in \{1, \ldots, N\}\ \forall x \in \mathfrak{C}_p: \quad V_q(\Pi_q(x)) \leq \mu V_p(x).$

If

$$\tau_a > \frac{\ln \mu}{\lambda} \tag{17}$$

then the switched DAE (1) *with switching signal* $\sigma \in \Sigma_{\tau_a}$ *is asymptotically stable.*

Proof. For any switching signal $\sigma \in \Sigma_{\tau_a}$ with average dwell time τ_a satisfying (17) let $0 := t_0 < t_1 < t_2 < \ldots$ be its positive switching times and let $x \in (\mathcal{C}^\infty_{\text{pw}})^n$ be a solution of (1). Between two consecutive switching times $t_i, t_{i+1} \in \mathbb{R}$, $i \in \mathbb{N}$, we have, by **ADT1**, $V(t-) \leq e^{-\lambda(t-t_i)}V(t_i)$ for all $t_i < t \leq t_{i+1}$. Furthermore, let $p_i := \sigma(t_{i+1}-)$ and $q_i := \sigma(t_i)$, then it holds that $x(t_i) = \Pi_{q_i}(x(t_i-))$ and $x(t_i-) \in \mathfrak{C}_{p_i}$, $i \in \mathbb{N}$. Hence, by ADT2, $V_{q_i}(x(t_i)) \leq \mu V_{p_i}(x(t_i-)$ for all $i \in \mathbb{N}$. Combining both inequalities inductively we get, for all $t > 0$,

$$\begin{aligned} V_{\sigma(t-)}(t-) &\leq e^{-\lambda(t-t_\nu)}\mu \cdots \underbrace{e^{-\lambda(t_2-t_1)}\mu\ \underbrace{e^{-\lambda(t_1-t_0)}\mu\ V_{p_0}(x(t_0-))}_{V_{\sigma(t_1-)}(x(t_1-))}}_{V_{\sigma(t_2-)}(x(t_2-))} \\ &= e^{-\lambda t}\mu^{N_\sigma(0,t)} \leq \mu^{N_0} e^{(-\lambda + \frac{\ln \mu}{\tau_a})t} V_{\sigma(0-)}(x(0-)), \end{aligned}$$

where $\nu \in \mathbb{N}$ is such that t_ν is the last switch before t.

By (17), the non-negative function $t \mapsto V_{\sigma(t-)}(x(t-))$ is therefore bounded by an exponentially decreasing function and hence converges to zero. Arguments analogous to those in Step 3 and Step 4 of the proof of Theorem 4.1 now conclude the proof.

In the linear case the Lyapunov function can be chosen according to Remark 2.9; in this case it is possible to express the inequality (17) for the average dwell time directly in terms of eigenvalues of corresponding matrices.

Lemma 4.5 (ADT1 and ADT2 always fulfilled for linear case). *Consider the linear switched DAE* (3) *with the regular matrix pairs* (E_p, A_p), $p \in \{1, \ldots, N\}$ *with corresponding consistency spaces* $\mathfrak{C}_p$, *and let* (P_p, Q_p) *be the solutions of the corresponding generalized Lyapunov equations*

$$A_p^\top P_p E_p + E_p^\top P_p A_p = -Q_p, \quad p = 1, \ldots, N$$

with $Q_p = Q_p^\top > 0$ *on* $\mathfrak{C}_p$ *and* $P_p = P_p^\top > 0$. *Choose a matrix* O_p *with orthonormal columns such that* $\operatorname{im} O_p = \operatorname{im} \Pi_p = \mathfrak{C}_p$, *where* Π_p *is the linear consistency projector corresponding to* (E_p, A_p) *as in Definition 3.8. Then, for* $p, q \in \{1, \ldots, N\}$,

$$\forall x \in \mathfrak{C}_p: \ V_q(\Pi_q x) \leq \mu_{p,q} V_p(x), \quad \textit{where } \mu_{p,q} := \frac{\lambda_{\max}(O_p^\top \Pi_q^\top E_q^\top P_q E_q \Pi_q O_p)}{\lambda_{\min}(O_p^\top E_p^\top P_p E_p O_p)} > 0$$

and

$$\forall x \in \mathfrak{C}_p: \ \dot V_p(x) \le -\lambda_p V_p(x), \quad \textit{where } \lambda_p := \frac{\lambda_{\min}(O_p^\top Q_p O_p)}{\lambda_{\max}(O_p^\top E_p^\top P_p E_p O_p)} > 0,$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimal and maximal eigenvalue of a symmetric matrix, respectively.

Proof. Let $d_p := \dim \mathfrak{C}_p$, i.e. $O_p \in \mathbb{R}^{n\times d_p}$, then $x \in \mathfrak{C}_p$ if, and only if, there exists a unique $z \in \mathbb{R}^{d_p}$ with $x = O_p z$, $O_p^\top x = z$ and $|x| = |z|$. Hence, by choosing z corresponding to $x \in \mathfrak{C}_p$ as above,

$$\begin{aligned}
V_p(x) &= z^\top O_p^\top E_p^\top P_p E_p O_p z =: z^\top P_p^z z \ge \lambda_{\min}(P_p^z)|z|^2 = \lambda_{\min}(P_p^z)|x|^2\\
V_p(x) &\le \lambda_{\max}(P_p^z)|x|^2\\
V_q(\Pi_q x) &= z^\top O_p^\top \Pi_q^\top E_q^\top P_q E_q \Pi_q O_p z =: z^\top M_{p,q}^z z \le \lambda_{\max}(M_{p,q}^z)|x|^2\\
\dot V_p(x) &= -z^\top O_p^\top Q_p O_p z =: -z^\top Q_p^z z \le -\lambda_{\min}(Q_p^z)|x|^2
\end{aligned}$$

By assumption, the matrices $Q_p^z = Q_p^{z\top} \in \mathbb{C}^{d_p\times d_p}$ and $P_p^z = P_p^{z\top} \in \mathbb{C}^{d_p\times d_p}$ are positive definite, hence $\lambda_{\min}(Q_p^z) > 0$ and $\lambda_{\max}(P_p^z) \ge \lambda_{\min}(P_p^z) > 0$. Therefore,

$$\mu_{p,q} := \frac{\lambda_{\max}(M_{p,q}^z)}{\lambda_{\min}(P_p^z)} \ge 0, \lambda_p := \frac{\lambda_{\min}(Q_p^z)}{\lambda_{\max}(P_p^z)} > 0$$

are well defined. Note that $\lambda_{\max}(M_{p,q}^z) = 0$ is possible, however $\lambda_{\max}(M_{p,p}^z) = \lambda_{\max}(P_p^z) \ge \lambda_{\min}(P_p^z)$, hence $\mu_{p,p} \ge 1$ and $\max_{p,q} \ln \mu_{p,q} \ge 0$.

Corollary 4.6 (Average dwell time for the linear case). *For the switched linear DAE* (3) *with asymptotically stable subsystems, let $\mu_{p,q}$ and λ_p, $p, q \in \{1,\dots,N\}$ be given as in Lemma 4.5. Then the linear switched DAE* (3) *is asymptotically stable if $\sigma \in \Sigma_{\tau_a}$ with*

$$\tau_a > \frac{\max_{p,q} \ln \mu_{p,q}}{\min_p \lambda_p}$$

Note that the obtained results cannot in general be expressed in terms of the eigenvalues of the matrices Q_p and P_p (or $E_p^\top P_p E_p$); the consistency projectors and basis transformation must be incorporated as formulated in Lemma 4.5. We show the application of Lemma 4.5 with a simple linear example.

Example 4.7 (Example 1 from [9] revisited). Let

$$(E_1, A_1) = \left(\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right]\right), (E_2, A_2) = \left(\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}-1&-1\\1&0\end{smallmatrix}\right]\right).$$

The corresponding consistency spaces and consistency projectors are given by

$$\mathfrak{C}_1 := \mathfrak{C}_{(E_1,A_1)} = \operatorname{im}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \quad \mathfrak{C}_2 := \mathfrak{C}_{(E_2,A_2)} = \operatorname{im}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$$

and

$$\Pi_1 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right], \quad \Pi_2 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right].$$

As basis matrices for the consistency space choose

$$O_1 = \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}, \quad O_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right].$$

Consider the Lyapunov functions $V_1(x) = \frac{1}{2}x_2^2$ and $V_2(x) = \frac{1}{2}(x_1 + x_2)^2$, corresponding to

$$P_1 = P_2 = \frac{1}{2}\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$$

and

$$Q_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \quad Q_2 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right].$$

Then

$$\begin{aligned} O_1^\top E_1^\top P_1 E_1 O_1 &= \tfrac{1}{4}, & O_2^\top E_2^\top P_2 E_2 O_2 &= \tfrac{1}{2}, \\ O_1^\top \Pi_2^\top E_2^\top P_2^\top E_2 \Pi_2 O_1 &= 1, & O_2^\top \Pi_1^\top E_1^\top P_1^\top E_1 \Pi_1 O_2 &= \tfrac{1}{2}, \\ O_1^\top Q_1 O_1 &= \tfrac{1}{2}, & O_2^\top Q_2 O_2 &= 1, \end{aligned}$$

hence $\mu := \max_{p,q} \mu_{p,q} = 2$ and $\lambda := \min_p \lambda_p = 2$. Therefore the corresponding switched DAE is asymptotically stable for all switching signals $\sigma \in \Sigma_\tau$ with $\tau_a > \frac{\ln 2}{2}$. This bound is actually sharp in this example [9].

5. Conclusion

We have studied switched nonlinear DAEs with respect to solution and stability theory. For the non-switched nonlinear DAE subsystems we generalized the classical Lyapunov's Direct Method, in particular, we defined a Lyapunov function for quasi-linear DAE in general terms. This definition seems to be new even for the linear case. Furthermore, we studied existence and uniqueness of solutions of a switched nonlinear DAE, provided the subsystems are regular in a certain sense. Finally, we were able to generalize existing stability results of switched ODEs to switched DAEs.

Appendix A. Distribution theory

Appendix A.1. Classical distribution theory

We start by summarizing the definitions and properties of classical distributions as formalized by Schwartz [26]. The space of *test functions* is $\mathcal{C}_0^\infty$, i.e. the space of smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support $\operatorname{supp} \varphi$, where the latter is the closure of the set $\{ t \in \mathbb{R} \mid \varphi(t) \neq 0 \}$. The space $\mathcal{C}_0^\infty$ can be equipped with a suitable topology such that convergence of a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of test functions to zero is characterized by

C1 $\exists C \subseteq \mathbb{R}$ compact $\forall n \in \mathbb{N}:\ \operatorname{supp} \varphi_n \subseteq C$ and

C2 $\forall i \in \mathbb{N}: \ \lim_{n\to\infty} \|\varphi_n^{(i)}\|_\infty = 0$,

where $\|\cdot\|_\infty$ denotes the supremum norm of a function. Hence a linear operator $D : \mathcal{C}_0^\infty \to \mathbb{R}$ is continuous if and only if $\lim_{n\to\infty} D(\varphi_n) = 0$ for all sequences (φ_n) of test functions fulfilling **C1** and **C2**. The *space of distributions* is the dual space of the space of test functions, i.e.

$$\mathbb{D} := \{ \ D : \mathcal{C}_0^\infty \to \mathbb{R} \ | \ D \text{ is linear and continuous } \}.$$

The main two properties of distributions are 1) that they can be interpreted as generalized functions and 2) that they are arbitrarily often differentiable. To be more precise, let $\mathcal{L}_{1,\text{loc}}$ be the space of locally integrable functions, then the mapping

$$\mathcal{L}_{1,\text{loc}} \to \mathbb{D}, \ f \mapsto f_{\mathbb{D}} := \left(\varphi \mapsto \int_{\mathbb{R}} f\varphi \right)$$

is well defined (i.e. $f_{\mathbb{D}}$ is indeed a distribution) and an injective homomorphism. Distributions which can be represented by a locally integrable function are called *regular distributions*.

The derivative of an arbitrary distribution $D \in \mathbb{D}$ is given by

$$D'(\varphi) := -D(\varphi'), \quad \varphi \in \mathcal{C}_0^\infty.$$

It is easy to see (from integration by parts), that this definition generalizes the classical derivative of differentiable functions:

$$\forall f : \mathbb{R} \to \mathbb{R} \text{ differentiable}: \quad (f_{\mathbb{D}})' = (f')_{\mathbb{D}}.$$

The simplest and most famous non-regular distribution is the *Dirac impulse* (or Dirac Delta function or unit impulse function) given by

$$\delta(\varphi) := \varphi(0),$$

or, in general for $t \in \mathbb{R}$, $\delta_t(\varphi) := \varphi(t)$ for $\varphi \in \mathcal{C}_0^\infty$.

The support $\operatorname{supp} D$ of a distribution $D \in \mathbb{D}$ is the complement of the largest open set $O \subseteq \mathbb{R}$ with the property $\operatorname{supp} \varphi \subseteq O \ \Rightarrow \ D(\varphi) = 0$, which generalizes the classical support definition of (continuous) functions. Note that the support of the Dirac impulse δ is $\{0\}$; in fact the following much stronger results holds:

$$\operatorname{supp} D \subseteq \{t\} \quad \Leftrightarrow \quad \exists N \in \mathbb{N} \ \exists \alpha_0, \alpha_1, \ldots, \alpha_N \in \mathbb{R}: \ D = \sum_{i=0}^{N} \alpha_i \delta_t^{(i)},$$

where $\delta_t^{(i)}$ the i-th (distributional) derivative of the Dirac impulse δ_t.

Distributions can be *multiplied with smooth functions*:

$$(\alpha D)(\varphi) := D(\alpha\varphi), \quad \alpha \in \mathcal{C}^\infty, D \in \mathbb{D}, \varphi \in \mathcal{C}_0^\infty,$$

and it is easy to see that the product rule $(\alpha D)' = \alpha' D + \alpha D'$ holds.

Appendix A.2. Piecewise-smooth distribution $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$

Let $\mathcal{C}^\infty_{\mathrm{pw}}$ be the space of piecewise-smooth function, where $\alpha : \mathbb{R} \to \mathbb{R}$ is called piecewise-smooth when there exists a locally finite ordered set $T = \{ t_i \in \mathbb{R} \mid i \in \mathbb{Z} \}$ and smooth functions $\alpha_i \in \mathcal{C}^\infty$, $i \in \mathbb{Z}$, such that $\alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[t_i,t_{i+1})}$. Here, f_I denotes the restriction (or truncation) of a function $f : \mathbb{R} \to \mathbb{R}$ to the interval $I \subseteq \mathbb{R}$, i.e. $f_I(\tau) = f(\tau)$ for $\tau \in I$ and $f_I(\tau) = 0$ otherwise. The space of *piecewise-smooth distributions* is then given by

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, T \subseteq \mathbb{R} \text{ locally finite,} \\ \forall \tau \in T : D_\tau \in \mathbb{D} \text{ with } \operatorname{supp} D_\tau \subseteq \{\tau\} \end{array} \right\}.$$

As mentioned above, the condition $\operatorname{supp} D_\tau \subseteq \{\tau\}$ for some $\tau \in \mathbb{R}$ is equivalent to $D_\tau \in \operatorname{span}\{\delta_\tau, \delta'_\tau, \delta''_\tau, \ldots\}$. Note that for a piecewise-smooth distribution $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau$ the set of jumps of f and the set T of locations of impulses are in general independent of each other. The properties of $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and corresponding definitions are summarized in the following:

1. closed under differentiation: $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \;\Rightarrow\; D' \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$,
2. left- and right-evaluation: $D(t+) := f(t)$, $D(t-) := f(t-)$, where $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau$, and $t \in \mathbb{R}$
3. impulsive part: $D[t] := D_t$ if $t \in T$, $D[t] = 0$ otherwise and $D[\cdot] := \sum_{\tau\in T} D_\tau$, where $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau$ and $t \in \mathbb{R}$,
4. restriction to interval: $D_I := (f_I)_{\mathbb{D}} + \sum_{\tau\in T\cap I} D_t$, where $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau$ and $I \subseteq \mathbb{R}$ is some interval,
5. multiplication with piecewise-smooth function: $\alpha D := \sum_{i\in\mathbb{Z}} \alpha_i D_{[t_i,t_{i+1})}$, where $\alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[t_i,t_{i+1})}$ as above; in particular, $\alpha\delta_t = \alpha(t)\delta_t$.

For more details see [7, 8]. In the proof of Theorem 3.5 we actually need the fact that for any $\alpha \in \mathcal{C}^\infty_{\mathrm{pw}}$, $D \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})$ and $t \in \mathbb{R}$

$$(\alpha D)[t] = \alpha D[t] \text{ and } (\alpha_D)' = \sum_{i\in\mathbb{Z}} (\alpha'_i)_{[t_i,t_{i+1})} + \sum_{i\in\mathbb{Z}} (\alpha_i(t_i) - \alpha_{t_{i-1}}(t_i))\delta_{t_i},$$

where $\alpha = \sum_{i\in\mathbb{N}} (\alpha_i)_{[t_i,t_i+1)}$ as above.

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