# The bang-bang funnel controller: time delays and case study

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***Abstract*— We investigate the recently introduced bang-bang funnel controller with respect to its robustness to time delays. We present slightly modified feasibility conditions and prove that the bang-bang funnel controller applied to a relative-degree-two nonlinear system can tolerate sufficiently small time delays. A second contribution of this paper is an extensive case study, based on a model of a real experimental setup, where implementation issues such as the necessary sampling time and the conservativeness of the feasibility assumptions are explicitly considered.**

## I. INTRODUCTION

We consider SISO systems described by a nonlinear differential equation

$$\begin{aligned} \dot{x} &= F(x) + G(x)u, \quad x(0) = x^0 \in \mathbb{R}^n \\ y &= H(x) \end{aligned} \tag{1}$$

with known relative degree $r = 2$ and positive "high frequency" gain (see Section II-A for details). We also discuss the simpler case $r = 1$ briefly after the presentation of the main result for relative-degree-two case. In the precursor [4] of this paper we have introduced the bang-bang funnel controller for this system class which ensured – provided certain feasibility assumptions were satisfied – *approximate tracking* of a reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$. In fact, the bang-bang funnel controller ensures that the error

$$e := y - y_{\text{ref}}, \tag{2}$$

meets prespecified strict (time-varying) error bounds which are given by the funnel

$$\mathcal{F}_0 := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \;\middle|\; \varphi_0^-(t) \leq e \leq \varphi_0^+(t) \right\}, \tag{3}$$

where $\varphi_0^-, \varphi_0^+ : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are the prespecified (time-varying) error bounds, see also the upper part of Figure 1. To achieve this control objective a second funnel $\mathcal{F}_1$, analogously defined as in (3), for the derivative $\dot{e}$ of the error is introduced and a schematic illustration of a typical closed loop response is shown in Figure 1.

The new contribution of the current work is the allowance of time delays in the closed loop as shown in Figure 2. This is a more realistic scenario then the one studied in [4]. The usage of a switching logic (implemented on a digital machine) makes it necessary to sample the error signal which introduces a time delay $\tau_e > 0$ in the measured error signal. Furthermore, the calculation time introduces another time delay $\tau_q > 0$ in the switching signal. Additionally, in a digitally connected closed loop there might be time delays due to the network transmission times yielding bigger

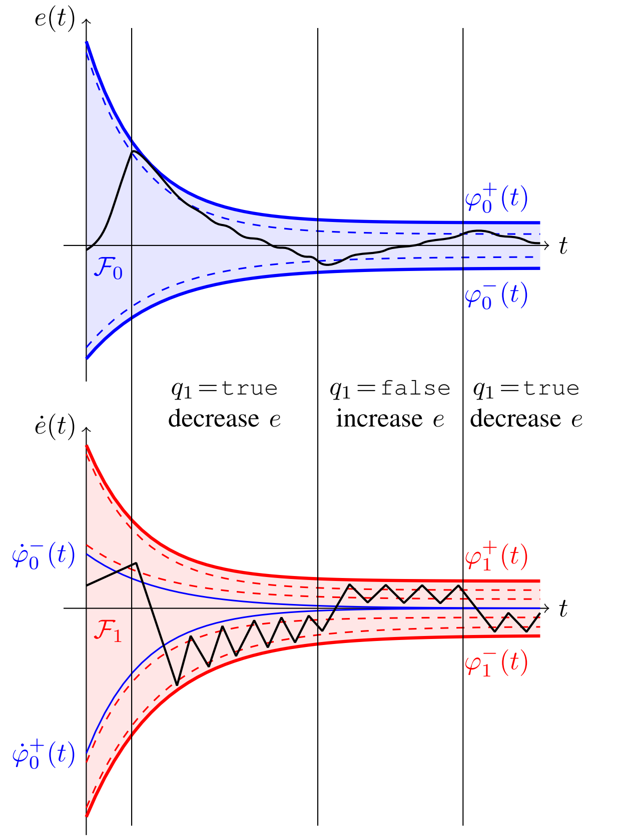


**Fig. 1:** A schematic illustration how the error and its derivative evolve in closed loop with the bang-bang funnel controller in the presence of time delays. The dashed lines denote the safety distances.

values for $\tau_e$ and $\tau_q$. We will present adjusted feasibility assumptions which take these time delays into account.

Furthermore, we will carry out an extensive case study of a relative degree two system stemming from a real

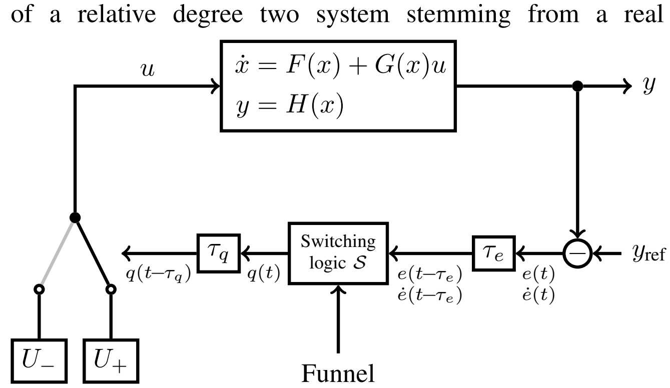


**Fig. 2:** Overall system structure with time delays in the error signal and the input.

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experimental setup of two stiff rotary machines as described in [2]; we also discuss design issues not discussed in [4].

For a literature overview and a further motivation of the considered topic we refer the reader [4].

## II. Controller design

In this section we will recall the definition of the switching logic which then leads to the bang-bang funnel controller. For this definition we only need to know the relative degree and the sign of the "high frequency gain". This structural assumption is formulated first before the actual definition of the switching logic is given.

### A. Structural assumptions on the system class and the reference signal

Throughout this work we assume that system (1) has known relative degree $r = 2$ with positive gain, i.e. the following structural assumption holds.

**(F$_1$)** There exists a coordinate transformation (a diffeomorphism) $x \mapsto (y, \dot{y}, z^\top)^\top$ which transforms (1) to the Byrnes-Isidori normal form [3]:

$$\ddot{y} = f(y,\dot{y},z) + g(y,\dot{y},z)u, \ \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix} = y^0 \in \mathbb{R}^2 \quad \text{(4a)}$$

$$\dot{z} = h(y,\dot{y},z), \quad z(0) = z^0 \in Z_0 \subseteq \mathbb{R}^{n-2} \quad \text{(4b)}$$

where $f, g, h$ are locally Lipschitz continuous, $g$ is positive and the $z$-system does not have a finite escape time for any bounded "input" vector $(y, \dot{y})$, i.e.

$$\left.\begin{array}{l} \forall y, \dot{y} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \ \forall z_0 \in Z_0 \subseteq \mathbb{R}^{n-1} \\ \exists \text{ solution } z : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n-1} \text{ for (4b)} \end{array}\right\} \quad (5)$$

Since we will consider non-continuous inputs $u$ we have to allow for solutions in the sense of Carathéodory, i.e. $\dot{y}$ and $z$ are absolutely continuous and (4) holds almost everywhere. The original system (1) inherits this solution concept. For the implementation of the bang-bang funnel controller the knowledge of the Byrnes-Isidori normal form (and the corresponding coordinate transformation) is not needed, however in order to check the feasibility assumptions the knowledge of (at least certain bounds on) $f$, $g$ and $h$ is needed.

Furthermore, we assume that the controller is able to obtain the derivative $\dot{e}$ of the error signal $e := y - y_{\text{ref}}$, in particular we have to make the following assumption on the reference signal:

**(F$_2$)** $y_{\text{ref}} \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $\dot{y}_{\text{ref}}$ is absolutely continuous with right-continuous derivative.

### B. Definition of the switching logic

As indicated in Figure 2 the bang-bang control law is simply given by

$$u(t) = \begin{cases} U_-, & \text{if } q(t) = \texttt{true}, \\ U_+, & \text{if } q(t) = \texttt{false}. \end{cases} \quad (6)$$

where $q : [0, \infty) \to \{\texttt{true}, \texttt{false}\}$ is the output of the switching logic $\mathcal{S}$ which maps the error signal to the switching signal.

The switching logic is given by

$$\begin{aligned} q_1(t) &= \mathfrak{S}(e(t), \varphi_0^+(t) - \varepsilon_0^+, \varphi_0^-(t) + \varepsilon_0^-, q_1(t-)), \\ q_1(0-) &= q_1^0 \in \{\texttt{true}, \texttt{false}\}, \\ q(t) &= \begin{cases} \mathfrak{S}_t(t), & \text{if } q_1(t) = \texttt{true}, \\ \mathfrak{S}_f(t), & \text{if } q_1(t) = \texttt{false}, \end{cases} \\ q(0-) &= q^0 \in \{\texttt{true}, \texttt{false}\}, \end{aligned}$$

where $\mathfrak{S} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \{\texttt{true}, \texttt{false}\} \to \{\texttt{true}, \texttt{false}\}$ is a switching predicate given by

$$\mathfrak{S}(e, \overline{e}, \underline{e}, q_{\text{old}}) := [e \geq \overline{e} \ \vee \ (e > \underline{e} \ \wedge \ q_{\text{old}})]. \quad (7)$$

and

$$\begin{aligned} \mathfrak{S}_t(t) &= \mathfrak{S}(\dot{e}(t), \min\{\dot{\varphi}_0^+(t), 0\} - \varepsilon_1^+, \varphi_1^-(t) + \varepsilon_1^-, q(t-)) \\ \mathfrak{S}_f(t) &= \mathfrak{S}(\dot{e}(t), \varphi_1^+(t) - \varepsilon_1^+, \max\{\dot{\varphi}_0^-(t), 0\} - \varepsilon_1^-, q(t-)). \end{aligned}$$

Here the positive quantities $\varepsilon_0^+, \varepsilon_0^-, \varepsilon_1^+, \varepsilon_1^-$ denote the so called safety distance from the corresponding funnel boundaries; for an intuition of this switching rule see Figure 1. Note that this is the same switching rule as in [4] with one important difference: Also at the derivative funnel boundaries the safety distances $\varepsilon_1^+$ and $\varepsilon_1^-$ are introduced. This is the key change to allow for time delays and will play a role in the new feasibility assumptions.

## III. Time delays in the feedback loop

### A. The remaining feasibility assumptions

We first collect further feasibility assumptions which are needed to state our main theoretical result. We have to assume that the initial size of the funnel boundaries are large enough to contain the initial error with a "safe" distance, i.e.

**(F$_3$)** $e^{(i)}(0) \in [\varphi_i^-(0) + \varepsilon_i^-, \varphi_i^+(0) - \varepsilon_i^+]$ for $i = 0, 1$.

The funnel boundaries have to be at least as smooth as the corresponding error signal evolving within it, hence we make the following smoothness as well as boundedness assumptions on the funnel boundaries.

**(F$_4$)** $\varphi_i^+, \varphi_i^- \in \mathcal{W}^{2-i,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, i.e. $(\varphi_i^\pm)^{(1-i)}$ are absolutely continuous with right-continuous derivative for $i = 0, 1$ and $\|(\varphi_i^\pm)^{(j)}\|_\infty < \infty$ for $i = 0, 1$, $j = 0, \ldots, 2 - i$.

Since the control objective is to keep the error signal within the corresponding funnel the error must be able to decrease or increase at least as fast as the funnel boundary, hence we have to choose the funnel $\mathcal{F}_1$ large enough such that it contains the derivative of the funnel boundaries of $\mathcal{F}_0$. Additionally the safety regions are not allowed to overlap. Altogether, we obtain the following feasibility assumption (cf. Figure 1).

**(F$_5$)** $\forall t \geq 0 :\ \varphi_0^+(t) - \varepsilon_0^+ > \varphi_0^-(t) + \varepsilon_0^-$ and

$$\begin{aligned} \varphi_1^+(t) - \varepsilon_1^+ &> \max\{\dot{\varphi}_0^-(t), 0\} + \varepsilon_1^-, \\ \varphi_1^-(t) + \varepsilon_1^- &< \min\{\dot{\varphi}_0^+(t), 0\} - \varepsilon_1^+. \end{aligned}$$

The next feasibility assumption is a slight modification of the feasibility assumption [4, (17)] taking the time delay into account.

**(F$_6$)**

$$\begin{aligned} \varepsilon_0^+ &> (\tau_a + \tau_e)\|\dot{\varphi}_0^+ - \varphi_1^+\|_\infty + \tfrac{(\|\dot{\varphi}_0^+\|_\infty + \|\varphi_1^+\|_\infty)^2}{2\delta^-}, \\ \varepsilon_0^- &> (\tau_a + \tau_e)\|\dot{\varphi}_0^- - \varphi_1^-\|_\infty + \tfrac{(\|\dot{\varphi}_0^-\|_\infty + \|\varphi_1^-\|_\infty)^2}{2\delta^+}, \end{aligned}$$

where $\delta^+, \delta^-$, as in [4], additionally fulfill

**(F$_7$)** $\delta^+ > \max\{\dot\varphi_1^-(t), \ddot\varphi_0^-(t), 0\}$ and $-\delta^- < \min\{\dot\varphi_1^+(t), \ddot\varphi_0^+(t), 0\}$ for all $t \geq 0$.

The following feasibility assumption is again identical to [4, (16)] and states that $U^\pm$ must be sufficiently large in terms of the system data.

**(F$_8$)** $U_- < \frac{-\delta_- + \ddot y_{\text{ref}}(t) + f(y_t, \dot y_t, z_t)}{g(y_t, \dot y_t, z_t)}$ and, $U_+ > \frac{\delta_+ + \ddot y_{\text{ref}}(t) + f(y_t, \dot y_t, z_t)}{g(y_t, \dot y_t, z_t)}$, hold for all $t \geq 0$ and all $(y_t, \dot y_t, z_t) \in [y_{\text{ref}}(t) + \varphi_0^-(t), y_{\text{ref}}(t) + \varphi_0^+(t)] \times [\dot y_{\text{ref}}(t) + \varphi_1^-(t), \dot y_{\text{ref}}(t) + \varphi_1^+(t)] \times Z_t$, where $Z_t :=$
$$\left\{ z(t) \;\middle|\; \begin{array}{l} z : [0,t] \to \mathbb{R}^{n-1} \text{ solves (4b) for some} \\ z^0 \in Z_0 \text{ and for some } y : [0,t] \to \mathbb{R} \text{ with} \\ \varphi_0^-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi_0^+(\tau) \text{ and} \\ \varphi_1^-(\tau) \leq \dot y(\tau) - \dot y_{\text{ref}}(\tau) \leq \varphi_1^+(\tau) \ \forall \tau \in [0,t] \end{array} \right\}.$$

Different to the case without time delays it is not possible anymore to change the input instantaneously when $\dot e$ hits its funnel boundary $\varphi_1^\pm$. Hence the new safety distance $\varepsilon_1^\pm$ must be large enough so that the derivative of the error does not leave its funnel even in the case that the input steers the output in the wrong direction. This is captured by the final feasibility assumption which is new.

**(F$_9$)** $\varepsilon_1^+ > (\tau_e + \tau_d) E^+_{[t, t+\tau_e+\tau_q]}$ and $\varepsilon_1^- > (\tau_e + \tau_d) E^-_{[t, t+\tau_e+\tau_q]}$, where
$$E^+_{[t,t+\tau_e+\tau_q]} := \sup_{\substack{s \in [t, t+\tau_e+\tau_q], z_s \in Z_s \\ y_s - y_{\text{ref}}(s) \in [\varphi_0^-(s), \varphi_0^+(s)] \\ \dot y_s - \dot y_{\text{ref}}(s) \in [\varphi_1^-(s), \varphi_1^+(s)]}} \left\{ \begin{array}{l} -\ddot y_{\text{ref}}(s) + f(y_s, \dot y_s, z_s) \\ \quad + g(y_s, \dot y_s, z_s) U^+ - \dot\varphi_1^+(s), \end{array} \right.$$
and $E^-_{[t,t+\tau_e+\tau_q]}$ is defined analogously.

## B. Main result

*Theorem 3.1 (Bang-bang funnel controller & time delays):* Consider the nonlinear system (1) and the bang-bang funnel controller as in Section II with time delays as shown in Figure 2. Let the feasibility assumptions **(F$_1$)** – **(F$_9$)** be satisfied. Then the bang-bang funnel controller works, i.e. there exists a global solution of the closed loop such that $q$ has locally finitely many switches and the error and its derivatives evolve within the corresponding funnels, i.e. $e^{(i)}(t) \in [\varphi_i^-(t), \varphi_i^+(t)]$ for all $t \geq 0$ and $i = 0, 1$.

*Proof:* The well-posedness result [4, Cor. 3.3] and its proof in [5] remain valid without any modification, hence existence and uniqueness of a maximal solution $(x, q_0, q) : [0, \omega) \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\}^2$ for $0 < \omega \leq \infty$ is guaranteed. It remains to show that the switching logic ensures that the error and its derivative remain within its corresponding funnels and $\omega = \infty$. The proof follows the same steps as the one of [5, Thm. 3.4] with some slight modifications due to the time delays.

If $e(t)$ leaves the funnel $\mathcal{F}_0$ then let $\omega_1 > 0$ be the first time the error crosses the funnel boundary, otherwise let $\omega_1 = \omega$.
*Step 1: We show that $\dot e$ evolves within $\mathcal{F}_1$ on $[0, \omega_1)$.*
First observe, that the feasibility assumption **(F$_9$)** yields the following implications.
$$\begin{aligned} \dot e(t_0) = \varphi_1^+(t_0) - \varepsilon_1^+ \quad &\Rightarrow \quad \forall t \in [t_0, t_0 + \tau_e + \tau_q] : \\ &\qquad \dot e(t) \leq \varphi_1^+(t), \\ \dot e(t_0) = \varphi_1^-(t_0) + \varepsilon_1^- \quad &\Rightarrow \quad \forall t \in [t_0, t_0 + \tau_e + \tau_q] : \\ &\qquad \dot e(t) \geq \varphi_1^-(t). \end{aligned}$$
Furthermore, the switching logic guarantees, for all $t \in [0, \omega_1)$,
$$\begin{aligned} &\dot e(t) \geq \varphi_1^+(t) - \varepsilon_1^+ \\ &\qquad \Rightarrow \ q(t + \tau_e) = \texttt{true} \ \Rightarrow \ u(t + \tau_e + \tau_q) = U_-, \\ &\dot e(t) \leq \varphi_1^-(t) + \varepsilon_1^- \\ &\qquad \Rightarrow \ q(t + \tau_e) = \texttt{false} \ \Rightarrow \ u(t + \tau_e + \tau_q) = U_+ \end{aligned}$$
and the feasibility assumption **(F$_8$)** together with
$$\begin{aligned} \ddot e(t) &= f(y_{\text{ref}}(t) + e(t), \dot y_{\text{ref}}(t) + \dot e(t), z(t)) \\ &\quad + g(y_{\text{ref}}(t) + e(t), \dot y_{\text{ref}}(t) + \dot e(t), z(t)) u(t) - \ddot y_{\text{ref}}(t) \end{aligned} \tag{8}$$
yields, for all $t \in [0, \omega_1)$,
$$\begin{aligned} \dot e(t) \geq \varphi_1^+(t) - \varepsilon_1^+ \ &\Rightarrow \ \ddot e(t + \tau_e + \tau_d) < -\delta_- < \dot\varphi_1^+(t), \\ \dot e(t) \leq \varphi_1^-(t) + \varepsilon_1^- \ &\Rightarrow \ \ddot e(t + \tau_e + \tau_d) > \ \ \delta_+ > \dot\varphi_1^-(t). \end{aligned}$$
Altogether, invoking also **(F$_3$)**, it follows that $\mathcal{F}_1$ is positively invariant for $\dot e$ on the interval $[0, \omega_1)$.
*Step 2: We show that $\omega_1 = \omega$.*
Let $t_0 \in [0, \omega_1)$ be such that $e(t_0) = \varphi_+(t_0) - \varepsilon_0^+$. The switching logic ensures $q_1(t) = \texttt{true}$ for all $t \in [t_0 + \tau_e, t_1 + \tau_e)$ where $t_1 > t_0$ is the smallest time when $e(t_1) = \varphi_-(t_1) + \varepsilon_-$ or $t_1 = \omega_1$. Choose a maximal $s_0 \in [t_0, t_1]$ such that $q(t) = \texttt{true}$ for all $t \in [t_0 + \tau_e, s_0 + \tau_e)$, i.e. $u(t) = U_-$ for all $t \in [t_0 + \tau_e + \tau_q, s_0 + \tau_e + \tau_q)$. The feasibility assumption **(F$_8$)** and (8) now ensures that $\ddot e(t) < -\delta_-$ on $[t_0 + \tau_e + \tau_q, s_0 + \tau_e + \tau_q)$. Step 1 also ensures that $\dot e(t) \leq \varphi_1^+(t)$ on $[t_0, t_0 + \tau_e + \tau + q]$, hence **(F$_6$)** together with Corollary 6.2 from the Appendix ensures that $e(t) < \varphi_0^+(t)$ on $[t_0, s_0 + \tau_e + \tau_q) \cap [t_0, t_1)$.

As long as $q_1(t) = \texttt{true}$ the same arguments as in Step 1, invoking **(F$_6$)**, ensure that the set $\left\{ (t, \dot e) \;\middle|\; \varphi_1^-(t) \leq \dot e \leq \min\{\dot\varphi_0^+(t), 0\} \right\}$ is positively invariant for $\dot e$ on $[t_0, t_1)$ if $\dot e(t_0) \in [\varphi_1^-(t_0) + \varepsilon_1^-, \min\{\dot\varphi_0^+(t_0), 0\} - \varepsilon_1^+]$. By definition, $\dot e(s_0) = \varphi_1^-(s_0) + \varepsilon_1^-$ if $s_0 < t_1$, hence $\dot e(t) \leq \dot\varphi_0^+(t)$ on $[s_0, t_1)$. Altogether this ensures $e(t) < \varphi_0^+(t)$ for all $t \in [t_0, t_1)$.

For $t_0 \in [0, \omega_1)$ with $e(t_0) = \varphi_-(t_0) + \varepsilon_-$ an analogous argument shows $e(t) > \varphi_-(t)$ for all $t \in [t_0, t_1)$ where $t_1 > t_0$ is the smallest time when $e(t_1) = \varphi_+(t_1) - \varepsilon_+$ or $t_1 = \omega_1$. We can now inductively argue that the error cannot leave the funnel and $\omega_1 = \omega$ is shown.
*Step 3: We show $\omega = \infty$.*
Since $e$ and $\dot e$ evolve within the funnels, finite escape time for $y$ and $\dot y$ is not possible. By **(F$_1$)** this also precludes finite escape time for $z$. In particular $y, \dot y, z$ are bounded on $[0, \omega)$, therefore $\omega < \infty$ is only possible when the switching times accumulate for $t \to \omega$. However, if $\omega < \infty$ then **(F$_5$)** ensures a positive distance between the switching surfaces on the compact interval $[0, \omega]$. Hence, invoking analogous arguments as in the proof of [5, Thm. 2.4], an accumulation of switching times would imply unboundedness of $\dot e$ or $\ddot e$

which is not possible as $\dot{e}$ is contained within the bounded funnel $\mathcal{F}_1$ and $\ddot{e}$ is governed by the bounded right-hand-side of (8). ■

### C. Short discussion of the relative-degree-one case

Assume there exist a coordinate transformation such that (1) is equivalent to

$$\dot{y} = f(y,z) + g(y,z)u$$
$$\dot{z} = h(y,z)$$

where $f, g, h$ are locally Lipschitz with $g$ being positive and the $z$-system has no finite escape time for bounded $y$. Assume the same feasibility assumption as in [4, Thm. 2.4] where [4, (8)] is replaced by

$$U_- < \frac{\dot{\varphi}_0^+(t)+\dot{y}_{\text{ref}}(t)+f(y_t,z_t)}{g(y_t,z_t)},$$
$$U_+ > \frac{\dot{\varphi}_0^-(t)+\dot{y}_{\text{ref}}(t)+f(y_t,z_t)}{g(y_t,z_t)},$$

for all $t \geq 0$, $y_t \in [y_{\text{ref}}(t)+\varphi_0^-(t), y_{\text{ref}}(t)+\varphi_0^+(t)]$ and $z_t \in Z_t$ where $Z_t$ is analogously defined as in $(\mathbf{F}_8)$. Additionally assume that $\varepsilon_0^+ > (\tau_e+\tau_d)E^+_{[t,t+\tau_e+\tau_q]}$ and $\varepsilon_0^- > (\tau_e+\tau_d)E^-_{[t,t+\tau_e+\tau_q]}$, where here

$$E^+_{[t,t+\tau_e+\tau_q]} := \sup_{\substack{s\in[t,t+\tau_e+\tau_q], z_s\in Z_s \\ y_s-y_{\text{ref}}(s)\in[\varphi_0^-(s),\varphi_0^+(s)]}} -\dot{y}_{\text{ref}}(s)+f(y_s,z_s)+g(y_s,z_s)U^+-\dot{\varphi}_1^+(s),$$

and $E^-_{[t,t+\tau_e+\tau_q]}$ is defined analogously. Then analogous arguments as in Step 1 of the proof of Theorem 3.1 show that the bang-bang funnel controller for the relative-degree-one case works in the presence of time delays.

## IV. Simulations

In this section we will apply the bang-bang funnel controller to a model of a laboratory setup of two stiffly coupled machines on which the continuos funnel controller was successfully applied [2]. The model has the following form

$$\begin{aligned}
\ddot{y} &= -\tfrac{1}{\Theta}\Big(\sigma_0 z_2+\sigma_D(\dot{y})\left(\dot{y}-\tfrac{|\dot{y}|z_2}{\beta(\dot{y})}\right)+\sigma_V(\dot{y})+u_L(z_1)\Big) \\
&\quad + \tfrac{1}{\Theta}u, \quad y(0)=\dot{y}(0)=0, \\
\dot{z}_1 &= 1, \quad z_1(0)=0, \\
\dot{z}_2 &= \dot{y}-\frac{|\dot{y}|}{\beta(\dot{y})}z_2, \quad z_2(0)=0,
\end{aligned} \tag{9}$$

where $y$ denotes the angle of the (rotating) machine, $u$ denotes the torque applied to the machine, $z_2$ denotes the average bristle deflection from the Lund-Grenoble friction model (cf. [2]), $\sigma_D(\dot{y}) = \sigma_1 e^{-(|\dot{y}|/\Omega_D)^{\delta_D}}$ models the damping (of the deflection rate $\dot{z}_2$), $\sigma_V(\dot{y}) = \sigma_2|\dot{y}|^{\delta_V}\operatorname{sgn}(\dot{y})$ models the viscous friction and $\beta(\dot{y}) = \frac{1}{\sigma_0}(u_C+(u_S-u_C)e^{-(|\dot{y}|/\Omega_S)^{\delta_S}}$ is the Stribeck function. The function $u_L$ describes the load on the mechanical systems, which is unknown but assumed bounded with known bound. The variable $z_1$ is the time written as a state-variable (in order to match our framework (4)). Note that $z_1$ will grow unbounded with time, however its influence on the systems dynamics is bounded (because of boundedness of $u_L(\cdot)$). For more details on this model see [2]. The system parameters $\Theta, u_C, u_S, \sigma_0, \sigma_1, \sigma_2, \Omega_D, \Omega_S, \delta_D, \delta_V, \delta_S$ are in general not known (and also not necessary to know to apply the bang-bang funnel controller), however for the simulations we assign (following [2], [1]): $\Theta = 0.3342$, $u_C = 1$, $u_S = 1.5$, $\sigma_0 = 10^4$, $\sigma_1 = \sqrt{\sigma_0}$, $\sigma_2 = 0.4$, $\Omega_D = 0.1$, $\Omega_S = 10^{-3}$, $\delta_D = 2$, $\delta_V = 1$, $\delta_S = 2$. Note that, compared to [1], we choose smaller values for $\sigma_0$ and $\sigma_1$ to reduce the numerical stiffness.

To make the simulations comparable to the experiments carried out with the continuous funnel controller we choose the load torque $u_L$, the reference output $y_{\text{ref}}$ and the funnel boundaries identical to the ones from [2], cf. Figure 3, in fact $\|\varphi_0^\pm\|_\infty = 6.2832$, $\|\dot{\varphi}_0^\pm\|_\infty = 7.353$, $\|\varphi_1^\pm\|_\infty = 8.853$, $\|\dot{\varphi}_0^\pm - \varphi_1^\pm\| = 16.206$, $\|\dot{\varphi}_1^\pm\|_\infty = 8.9791$ and $\inf_{t\geq 0}\big(\varphi_0^+(t)-\varphi_0^-(t)\big) = 0.52360$, $\inf_{t\geq 0}\big(\varphi_1^+(t)-\dot{\varphi}_0^-(t)\big) = \inf_{t\geq 0}\big(\dot{\varphi}_0^+(t)-\varphi_1^-(t)\big) = 1.5$, .

In order to simplify the analysis we assume in the following symmetric safety distances, i.e. $\varepsilon_i^+ = \varepsilon_i^- =: \varepsilon_i^\pm$, $i = 0, 1$. According to the feasibility assumptions $(\mathbf{F}_5)$ the safety distances must fulfill

$$0 < 2\varepsilon_0^\pm < 0.52360, \ 0 \leq 2\varepsilon_1^\pm < 1.5.$$

For our simulation we choose

$$\varepsilon_0^\pm = 0.23562, \ \varepsilon_1^\pm = 0.5.$$

Using the latter in $(\mathbf{F}_6)$ we obtain

$$\delta^\pm > \frac{(\|\dot{\varphi}_0^\pm\|_\infty + \|\varphi_1^\pm\|_\infty)^2}{2\varepsilon_0^\pm} \approx 557$$

and, when choosing $\delta^\pm = 600$,

$$(\tau_e+\tau_q) < \frac{\varepsilon_0^\pm - \frac{(\|\dot{\varphi}_0^\pm\|_\infty+\|\varphi_1^\pm\|_\infty)^2}{2\lambda_2^\pm}}{\|\dot{\varphi}_0^\pm - \varphi_1^\pm\|_\infty} \approx 0.001.$$

Hence the sampling time of the switching logic (or the numerical integration step size) must be $10^{-3}$ or smaller. The feasibility assumption $(\mathbf{F}_9)$ might further reduce the needed step size for the numerical integration, however we have to invoke $(\mathbf{F}_8)$ first to obtain feasible values for $U^+$ and $U^-$. In [2] it was shown that all solutions of (9) fulfill

$$\begin{aligned}
&\left\|\sigma_0 z_2+\sigma_D(\dot{y})\left(\dot{y}-\tfrac{|\dot{y}|}{\beta(\dot{y})}z_2\right)+\sigma_V(\dot{y})+u_L(z_1)\right\|_\infty \\
&\qquad \leq u_s+\sigma_1\|\dot{y}\|_\infty(1+\tfrac{u_S}{u_C})+\sigma_2\|\dot{y}\|_\infty^{\delta_V}+\|u_L\|_\infty.
\end{aligned}$$

Invoking $\|\dot{y}\|_\infty \leq \|\varphi_1^\pm+\dot{y}_{\text{ref}}\|_\infty \leq \|\varphi_1^\pm\|_\infty$, $\|u_L\|_\infty \leq 4$, $\|\ddot{y}_{\text{ref}}\|_\infty = 6.05$ and with

$$\begin{aligned}
M_z &:= u_s+\sigma_1\|\varphi_1^\pm\|_\infty(1+\tfrac{u_S}{u_C})+\sigma_2\|\varphi_1^\pm\|_\infty^{\delta_V}+\|u_L\|_\infty \\
&\approx 2222.3
\end{aligned}$$

we obtain a lower bound for $U^+$ and an upper bound for $U^-$ via $(\mathbf{F}_8)$:

$$U^+ \geq \frac{\delta^+ + \|\ddot{y}_{\text{ref}}\|_\infty + \frac{M_z}{\Theta}}{\frac{1}{\Theta}} \approx 2424.8$$
$$U^- \leq \frac{-\delta^- - \|\ddot{y}_{\text{ref}}\|_\infty - \frac{M_z}{\Theta}}{\frac{1}{\Theta}} \approx -2424.8$$

Choosing $U^+ = -U^- = 2425$ we obtain via $(\mathbf{F}_9)$

$$\tau_e+\tau_q < \frac{\varepsilon_1^\pm}{\|\ddot{y}_{\text{ref}}\|_\infty + \frac{M_z}{\Theta} + \frac{1}{\Theta}U^+ + \|\dot{\varphi}_1^\pm\|_\infty} \approx 3.6\cdot 10^{-5},$$

which yields a much smaller upper bound for the integration step size. For the simulation we use the simple explicit Euler method with a step size $h = 10^{-5}$ and the switching logic is initialized with $q(0-) = \texttt{true}$, $q_1(0-) = \texttt{true}$. The simulation result for the output $y(\cdot)$ and its capability to

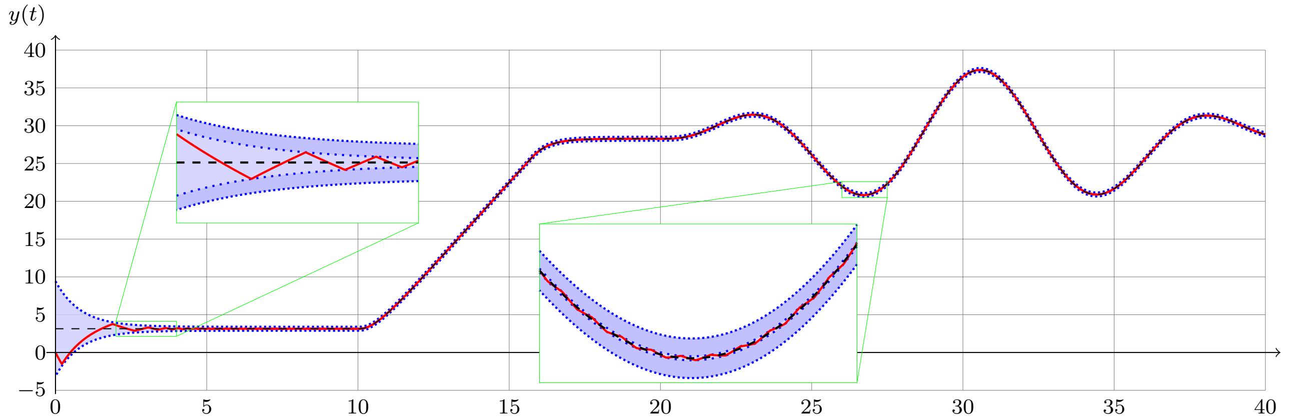


**Fig. 3:** Simulation of the Bang-Bang funnel controller for a relative degree two example with constant safety distance and $U^+ = -U^- = 2425$, $y(t)$: —, $y_{\text{ref}}(t)$: - - -, $\varphi_0^{\pm}(t)$: ······, $\varphi_0^{\pm}(t) \mp \varepsilon_0^{\pm}$ (only in zoomed areas): ····

follow the reference signal $y_{\text{ref}}(\cdot)$ is shown in Figure 3.

On a first glance the results look satisfactory and comparable to the experimental results from the application of the continuous funnel controller in [2]. However, the used input values, $u(t) = \pm 2425\, Nm$, are much bigger than the physically possible, $-22\, Nm \le u(t) \le 22\, Nm$, and the ones actually needed in the experimental setup of [2], $u(t) \in [-1.5\, Nm, 7.5\, Nm]$, which sufficed to keep the tracking error within its funnel. Furthermore, the switching frequency is very high (about $10^4\, Hz$) and is too high for the real actuators used in the experimental setup (which allow a frequency of at most $10^3\, Hz$). Since the fast switching is a consequence

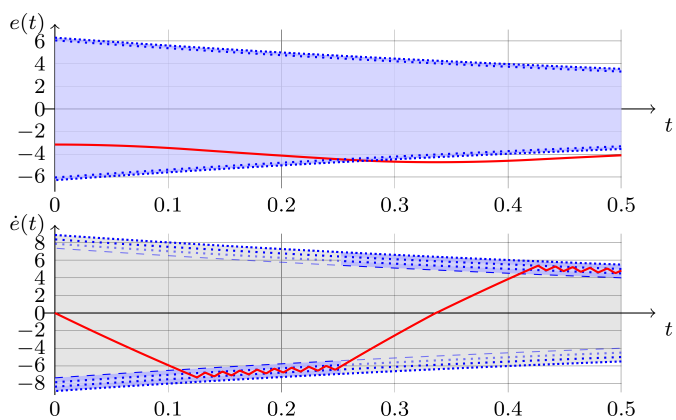


**Fig. 4:** Simulation with $U^+ = -U^- = 22$ showing how the error leaves the funnel (top) and how the derivative of the error evolves (below), $e(t)$ (top), $\dot{e}(t)$ (below): —, $\varphi_0^{\pm}(t)$ (top), $\varphi_1^{\pm}(t)$ (below): ······, $\dot{\varphi}_0^{\pm}(t)$: – –, safety distance: ····.

of the high values of $U^+$ and $U^-$ one might wonder whether the feasibility assumption are just too conservative (as was already pointed out in [2] in a comparable context), however a simulation with $U^+ = -U^- = 22$ (which are the maximal values of the considered physical system) shows that the bang-bang funnel controller is not able to keep the error within the funnel, see Figure 4. The underlying problem is that the bang-bang funnel controller "realizes too late" that the error approaches the funnel boundary (observe how in Figure 4 the error actually moves in the wrong direction on the interval $[0, 0.25]$) and is then not able to "turn the corner" fast enough. This problem is due to the too small safety distance $\varepsilon_0^{\pm}$ as already observed in [4, Rem. 3.5.4], where a time-varying safety distance is suggested. We therefore repeat the simulations with a time varying safety distance $\varepsilon_0^{\pm}$ given by

$$\varepsilon_0^{\pm}(t) = 0.9\varphi_0^{+}(t).$$

Note that this yields the same value for the safety distance for large $t$ as before. As can be seen in Figure 5 the simulation improves significantly: We can choose $U^+ = -U^- = 8$ and the switching frequency is less then $100\, Hz$ and therefore feasible for a real physical setup.

To further reduce the switching frequency, it might be beneficial to introduce an additional "neutral" input value $u(t) = 0$ to the switching logic, however this idea and carrying out the real experiments is still ongoing research.

## V. Conclusions

We were able to show that the bang-bang funnel controller can tolerate sufficiently small time delays and presented feasibility assumptions which made it possible to check whether the time delays are tolerable. Furthermore, we have presented a case study, which confirmed our theoretical result. However, the used input values exceeded by a magnitude of about one hundred the physical bounds of the experimental setup the model was based on. Restricted to the physical bounds the bang-bang funnel controller could not guarantee the desired error bounds. However, with only a slight modification of the switching logic we were able to recover error tracking with the prespecified error bounds with much smaller input values.

## VI. Appendix: A technical lemma

*Lemma 6.1 (Overshoot bound):* Assume $\eta : [t_0, t_2] \to \mathbb{R}$ is twice differentiable and let a continuous $\psi^d : [t_0, t_1) \to \mathbb{R}$ be such that $\dot{\eta}(t) \le \psi^d(t)$ for all $t \in [t_0, t_1) \subseteq [t_0, t_2]$. Furthermore, assume there exists $\lambda > 0$ such that $\ddot{\eta}(t) \le -\lambda$ for all $t \in [t_1, t_2)$. Then, for every absolutely continuous $\psi : [t_0, t_2] \to \mathbb{R}$ with essentially bounded derivative and

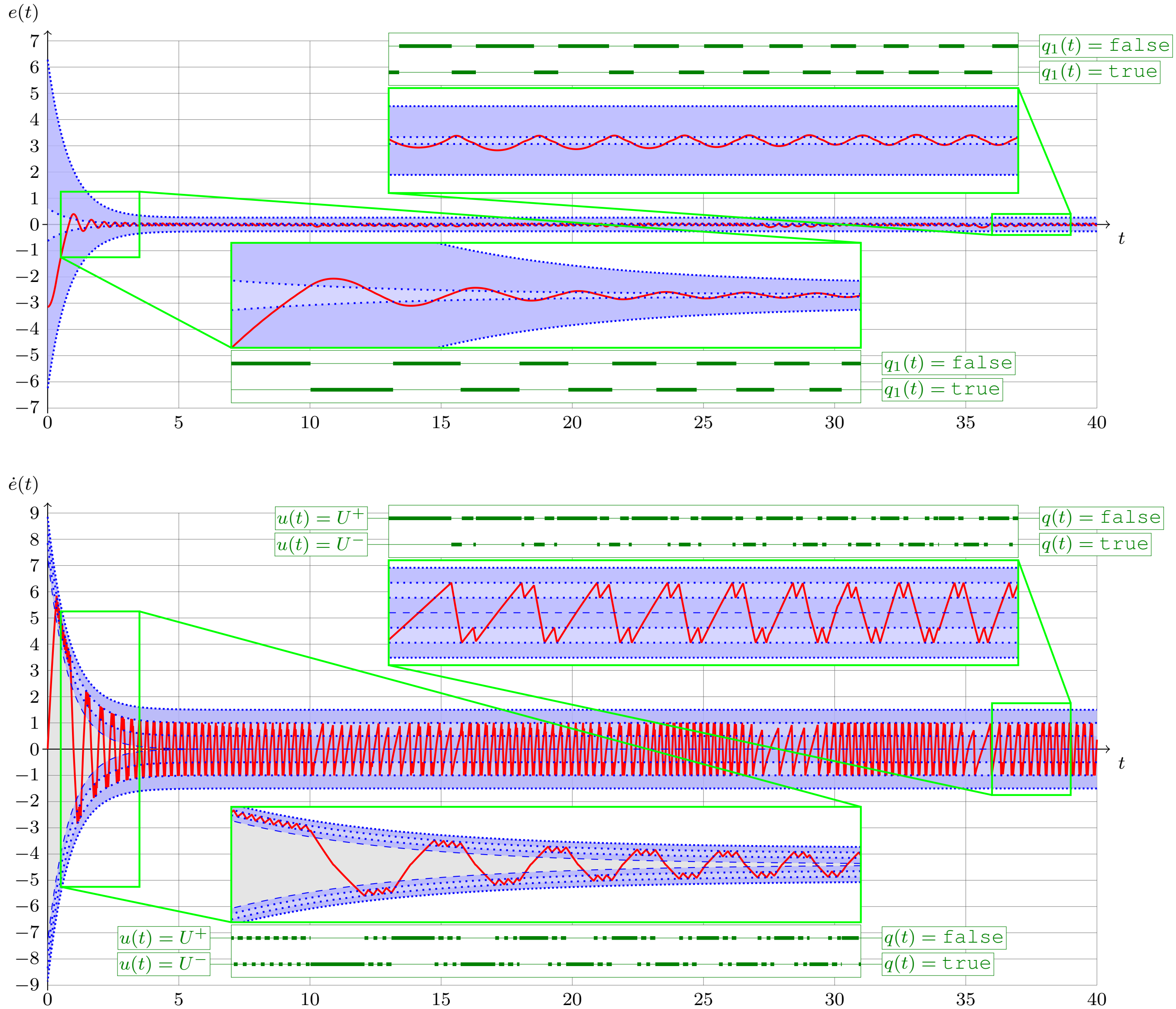


**Fig. 5:** Simulation results for a time-varying safety distance ⋯ and $U^+ = -U^- = 8$; for the time intervals $[0.5, 3.5]$ and $[36, 39]$ zoomed plots are provided with additional plots of the discrete variables $q_1$ and $q$ (equivalently $u(t)$) ▬. The remaining variables are encoded as in Figure 4.

$\varepsilon := \psi(t_0) - \eta(t_0) > 0$, it holds that $\eta(t) < \psi(t)$ for all $t \in [t_0, t_2]$ if

$$\varepsilon > (t_1 - t_0)\max\left\{0, -\underline{(\dot\psi - \psi^d)}_{[t_0,t_1]}\right\} + \frac{\left(\underline{\dot\psi}_{[t_1,t_2]} - \psi^d(t_1)\right)\max\left\{0, \underline{\dot\psi}_{[t_1,t_2]} - \psi^d(t_1)\right\}}{2\lambda}, \qquad (10)$$

where, for any interval $I \subseteq \mathbb{R}$ and any function $f : \mathbb{R} \to \mathbb{R}$,

$$\underline{f}_I := \operatorname*{ess\,inf}_{t \in I} f(t).$$

*Proof:* Since, by assumption $\dot\eta(t) \leq \psi^d(t)$ on $[t_0, t_1)$ it follows that, for all $t \in [t_0, t_1]$,

$$\begin{aligned}\psi(t) - \eta(t) &\geq \psi(t_0) - \eta(t_0) + (t - t_0)\underline{(\dot\psi - \psi^d)}_{[t_0,t_1]} \\ &\geq \varepsilon - (t_1 - t_0)\max\left\{0, -\underline{(\dot\psi - \psi^d)}_{[t_0,t_1]}\right\} > 0.\end{aligned}$$

From $\ddot\eta(t) \leq -\lambda$ on $[t_1, t_2)$ it follows that, for all $t \in [t_1, t_2]$,

$$\begin{aligned}\eta(t) &\leq \eta(t_1) + (t - t_1)\dot\eta(t_1) - \frac{\lambda}{2}(t - t_1)^2 \\ &\leq \eta(t_1) + (t - t_1)\psi^d(t_1) - \frac{\lambda}{2}(t - t_1)^2,\end{aligned}$$

hence

$$\begin{aligned}\psi(t) - \eta(t) &\geq \psi(t_1) + (t - t_1)\underline{\dot\psi}_{[t_1,t_2]} \\ &\quad - \left(\eta(t_1) + (t - t_1)\psi^d(t_1) - \frac{\lambda}{2}(t - t_1)^2\right) \\ &\geq \underbrace{\psi(t_0) - \eta(t_0)}_{=\varepsilon} + p(t),\end{aligned}$$

where $p(t) := (t_1 - t_0)\underline{(\dot\psi - \psi^d)}_{[t_0,t_1]} + (t - t_1)\left(\underline{\dot\psi}_{[t_1,t_2]} - \psi^d(t_1)\right) + \frac{\lambda}{2}(t - t_1)^2$. The parabola $t \mapsto p(t)$ obtains its minimum on $[t_1, \infty)$ at $t_{\min} = t_1 + \max\left\{0, -\left(\underline{\dot\psi}_{[t_1,t_2]} - \psi^d(t_1)\right)/\lambda\right\}$ with value

$$p(t_{\min}) = (t_1 - t_0)\underline{(\dot\psi - \psi^d)_{[t_0,t_1]}} + \frac{\left(\psi^d(t_1) - \underline{\dot\psi_{[t_1,t_2]}}\right)\max\left\{0, \psi^d(t_1) - \underline{\dot\psi_{[t_1,t_2]}}\right\}}{2\lambda}.$$

This concludes the proof. ∎

*Corollary 6.2 (Conservative overshoot bound):* Assume $\eta, \psi, \psi^d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ have the same properties as in Lemma 6.1 with respect to the intervals $[t_0, t_1] \subseteq [t_1, t_2] \subseteq \mathbb{R}_{\geq 0}$, then $\eta(t) < \psi(t)$ for all $t \in [t_0, t_2]$ if, with $\varepsilon := \psi(t_0) - \eta(t_0)$,

$$\varepsilon > (t_1 - t_0)\|\dot\psi - \psi^d\|_\infty + \frac{(\|\psi^d\|_\infty + \|\dot\psi\|_\infty)^2}{2\lambda}.$$

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