Supplementary material for: The bang-bang funnel controller for uncertain nonlinear systems with arbitrary relative degree

Daniel Liberzon and Stephan Trenn

All notation, equation numbering etc. are taken from the paper "The bang-bang funnel controller for uncertain nonlinear systems with arbitrary relative degree".

I. Explicit definition of the switching logic for $r = 1$, $r = 2$ and $r = 3$

For $r = 1$ the definition of $\mathcal{S}$ results (by "merging" the definitions for $\mathcal{S}_0$ and $\mathcal{S}_{r-1}$ in the obvious way) in the following DLS:

$$q(t) = \mathfrak{S}\left(e(t), \varphi_0^+(t) - \varepsilon_0^+, \varphi_0^-(t) + \varepsilon_0^-, q(t-)\right), \qquad q(0-) = q^0 \in \{\texttt{true}, \texttt{false}\},$$

for $r = 2$ we obtain

$$\begin{aligned}
q_1(t) &= \mathfrak{S}\left(e(t), \varphi_0^+(t) - \varepsilon_0^+, \varphi_0^-(t) + \varepsilon_0^-, q_1(t-)\right), \\
q_1(0-) &= q_1^0 \in \{\texttt{true}, \texttt{false}\}, \\
q(t) &= \begin{cases} \mathfrak{S}\left(\dot e(t), \min\{\dot\varphi_0^+(t), -\lambda_1^-\} - \varepsilon_1^+, \varphi_1^-(t) + \varepsilon_1^-, q(t-)\right), & \text{if } q_1(t) = \texttt{true}, \\ \mathfrak{S}\left(\dot e(t), \varphi_1^+(t) - \varepsilon_1^+, \max\{\dot\varphi_0^-(t), \lambda_1^+\} - \varepsilon_1^-, q(t-)\right), & \text{if } q_1(t) = \texttt{false}, \end{cases} \\
q(0-) &= q^0 \in \{\texttt{true}, \texttt{false}\},
\end{aligned}$$

and for $r = 3$

$$\begin{aligned}
q_1(t) &= \mathfrak{S}\left(e(t), \varphi_0^+(t) - \varepsilon_0^+, \varphi_0^-(t) + \varepsilon_0^-, q_1(t-)\right), \\
q_1(0-) &= q_1^0 \in \{\texttt{true}, \texttt{false}\}, \\
q_2(t) &= \begin{cases} \mathfrak{S}\left(\dot e(t), \min\{\dot\varphi_0^+(t), -\lambda_1^-\} - \varepsilon_1^+, \varphi_1^-(t) + \varepsilon_1^-, q_2(t-)\right), & \text{if } q_1(t) = \texttt{true}, \\ \mathfrak{S}\left(\dot e(t), \varphi_1^+(t) - \varepsilon_1^+, \max\{\dot\varphi_0^-(t), \lambda_1^+\} + \varepsilon_1^-, q_2(t-)\right), & \text{if } q_1(t) = \texttt{false}, \end{cases} \\
q_2(0-) &= q_2^0 \in \{\texttt{true}, \texttt{false}\}, \\
q(t) &= \begin{cases} \mathfrak{S}\left(\ddot e(t), \min\{\ddot\varphi_0^+(t), -\lambda_2^-\} - \varepsilon_2^+, \varphi_2^-(t) + \varepsilon_2^-, q(t-)\right), & \text{if } q_1(t) \wedge q_2(t), \\ \mathfrak{S}\left(\ddot e(t), \varphi_2^+(t) - \varepsilon_1^+, \max\{\dot\varphi_1^-(t), \lambda_2^+\} + \varepsilon_2^-, q(t-)\right), & \text{if } q_1(t) \wedge \neg q_2(t), \\ \mathfrak{S}\left(\ddot e(t), \min\{\dot\varphi_1^+(t), -\lambda_2^-\} - \varepsilon_1^+, \varphi_2^-(t) + \varepsilon_2^-, q(t-)\right), & \text{if } \neg q_1(t) \wedge q_2(t), \\ \mathfrak{S}\left(\ddot e(t), \varphi_2^+(t) - \varepsilon_1^+, \max\{\ddot\varphi_0^-(t), \lambda_2^+\} + \varepsilon_2^-, q(t-)\right), & \text{if } \neg q_1(t) \wedge \neg q_2(t), \end{cases} \\
q(0-) &= q^0 \in \{\texttt{true}, \texttt{false}\},
\end{aligned}$$

The switching logic can be illustrated by state diagrams, for $r = 1$ and $r = 2$ see [12], for $r = 3$ see Figure 1.

II. Relative degree four simulation

In this section we carry out simulations for a relative degree four example, where we take time delays due to the time sampling into account. To circumvent the problem of competing control objectives as highlighted in Remark 4.3 and also to simplify the feasibility assumptions we consider constant funnel boundaries; in particular, the transient behavior is not in the focus of this simulation. As an academic example we consider the following nonlinear system

$$\begin{aligned}
y^{(4)} &= z\,\dddot y^{2} + e^z u, \qquad & y^{(i)}(0) &= y_{\text{ref}}^{(i)}(0),\ i = 0,1,2,3, \\
\dot z &= z(a - z)(z + b) - cy, \qquad & z(0) &= 0,
\end{aligned} \tag{1}$$

where $a, b, c \in \mathbb{R}$ are parameters of which only the following bounds are known: $0 < a \leq 0.1$, $0 < b \leq 0.1$, $|c| \leq 0.01$. Note that the system with zero input and for $c > 0$ will exhibit finite escape time if $\dddot y(0) \neq 0$. As reference signal we choose

This work was partially supported by NSF grants CNS-0614993 and ECCS-0821153
D. Liberzon is with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA; S. Trenn is with the Department of Mathematics, University of Kaiserslautern, 67663 Kaiserslautern, Germany
email: trenn@mathematik.uni-kl.de

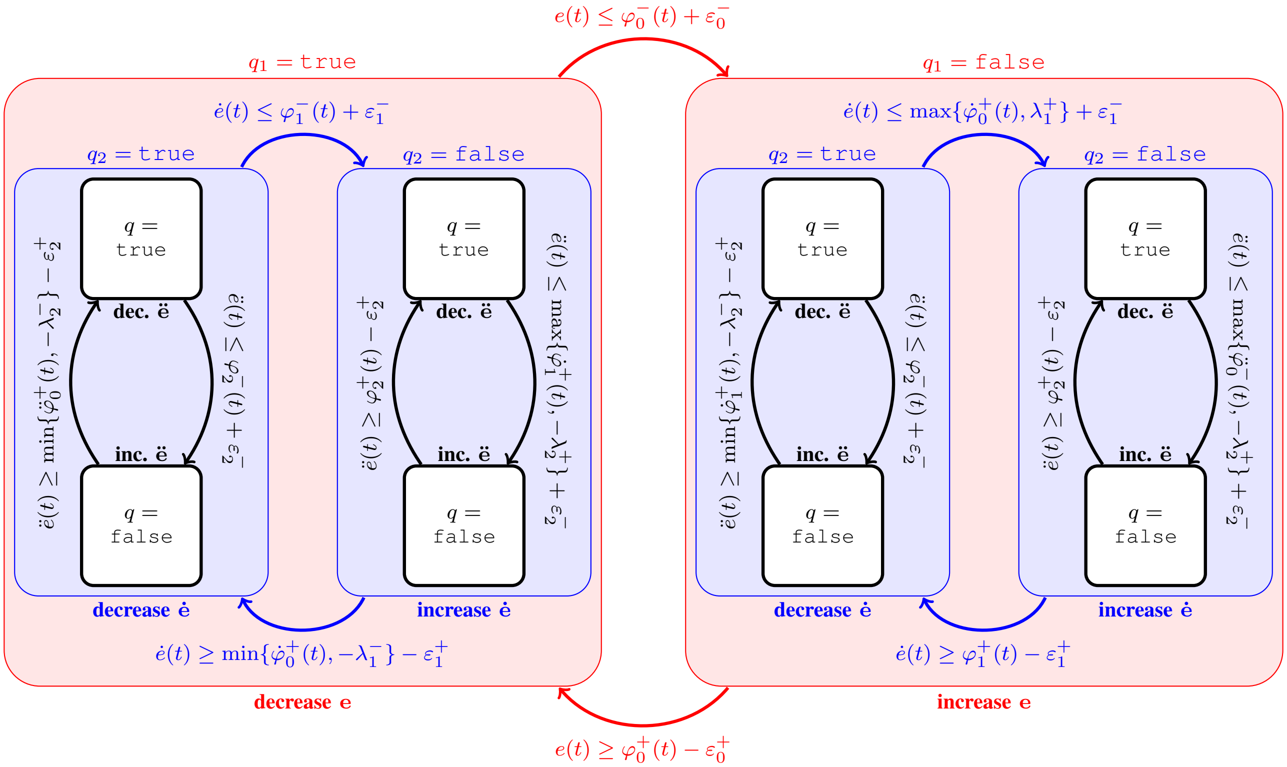

Fig. 1: The switching logic for the relative degree three case.

$y_{\text{ref}}(t) = 5\sin(t)$ which satisfies **(F$_2$)**. We choose the funnels, the bang-bang funnel controller parameters and the settling times as follows:

$$\begin{aligned}
&\varphi_0^+ = -\varphi_0^- \equiv 1, && \varepsilon_0^+ = \varepsilon_0^- = 0.9, && && \Delta_0^+ = \Delta_0^- = \infty,\\
&\varphi_1^+ = -\varphi_1^- \equiv 0.5, && \varepsilon_1^+ = \varepsilon_1^- = 0.1, && \lambda_1^+ = \lambda_1^- = 0, && \Delta_1^+ = \Delta_1^- = \Delta_0^\pm/2 = \infty,\\
&\varphi_2^+ = -\varphi_2^- \equiv 0.5, && \varepsilon_2^+ = \varepsilon_2^- = 0.1, && \lambda_2^+ = \lambda_2^- = 0.2, && \Delta_2^+ = \Delta_2^- = 0.4,\\
&\varphi_3^+ = -\varphi_3^- \equiv 4.5, && \varepsilon_3^+ = \varepsilon_3^- = 0.1, && \lambda_3^+ = \lambda_3^- = 4, && \Delta_3^+ = \Delta_3^- = 0.1,\\
& && && \lambda_4^+ = \lambda_4^- = 102, && \Delta_4^+ = \Delta_4^- = 0.0001.
\end{aligned}$$

It is not difficult to verify that the feasibility conditions **(F$_3$)-(F$_8$)** are fulfilled. Note that these parameters do not depend on the actual system. The only control parameters which depend on the system are U^+ and U^-. In order to choose feasible values for U^+ and U^- we have to find bounds for the terms in **(F$_9$)**. First observe that, for all $t \ge 0$,

$$\Phi_t^{y_{\text{ref}}} \subseteq \left\{ \, (y_0, y_1, y_2, y_3) \in \mathbb{R}^4 \;\middle|\; |y_0| \le 6, |y_1| \le 5.5, |y_2| \le 5.5, |y_3| \le 9.5 \, \right\}.$$

With $Z_0 = [-0.5, 0.5]$ it can now easily be verified that

$$Z_t^{y_{\text{ref}}} \subseteq [-0.5, 0.5] \quad \forall t \ge 0.$$

Hence, for all $t \ge 0$, $(y_t^0, y_t^1, y_t^2, y_t^3) \in \Phi_t^{y_{\text{ref}}}$ and $z_t \in Z_t^{y_{\text{ref}}}$,

$$|z_t (y_t^3)^2| \le 45.125 \quad \text{and} \quad e^{z_t} \ge e^{-0.5} \ge 0.6.$$

Altogether this guarantees that

$$U^+ = -U^- := 254 \ge \frac{102 + 5 + 45.125}{0.6} \approx 253.54$$

is feasible (in the sense of **(F$_9$)**) for the bang-bang funnel controller. Finally for carrying out the simulation we have to check the maximal step size in view of the time delay introduced by the sampled time axis. The feasibility assumption **(F$_{10}$)** yields the following upper bound for the simulation step size h

$$h \le \min\left\{ \Delta_4^\pm, \frac{\varepsilon_3^\pm}{\|y_{\text{ref}}^{(4)}\|_\infty + \|z(y^{(3)})^2\|_\infty + \|e^z\|_\infty U^+ \|\dot\varphi_3^\pm\|_\infty} \right\} = \min\left\{ 10^{-4}, \frac{0.1}{5 + 45.125 + e^{0.5} 300 + 0} \right\} = 10^{-4}.$$

The simulation where carried out with the step size $h = 10^{-4}$ and the parameters of (1) are

$$a = 0.09, \quad b = 0.05, \quad c = 0.008.$$

The overall tracking accuracy is shown in Figure 2, which clearly shows that the error follows the reference signal within the specified error bounds (given by $\varphi_0^\pm$).

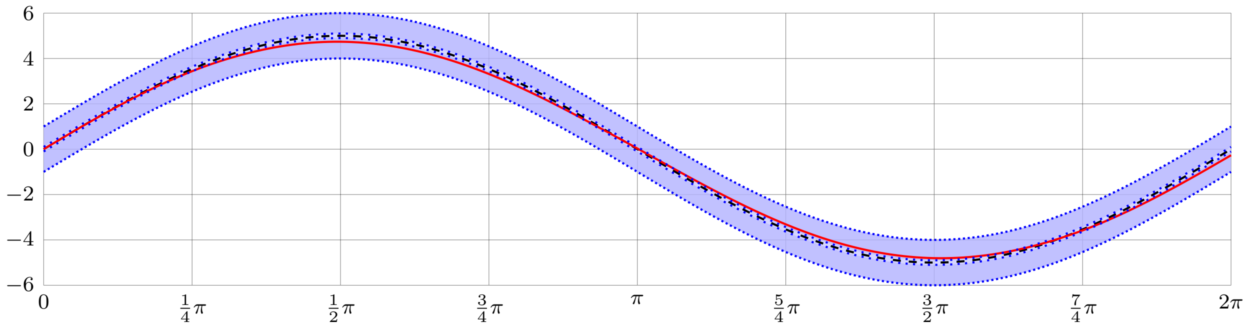

Fig. 2: The bang-bang funnel controller applied to the nonlinear relative degree four system (1): The output y —— follows the reference signal y_{ref} - - within the prespecfied bounds $\varphi_0^\pm$ ········, the safety distance $\varepsilon_0^\pm$ is shown as ····.

The behavior of the bang-bang funnel controller in detail is shown in Figure 3 where the error $e(t)$ and its derivatives $\dot{e}(t)$, $\ddot{e}(t)$, $\dddot{e}(t)$ for $t \in [0, 2\pi]$ are plotted. In addition the internal switching variables $q_1(t)$, $q_2(t)$ and $q_3(t)$ are shown as well as the resulting (external) switching signal $q(t)$ which determines directly $u(t)$ via

$$u(t) = \begin{cases} U^-, & \text{if } q(t) = \texttt{true}, \\ U^+, & \text{if } q(t) = \texttt{false}, \end{cases}$$

The switching frequency of the input $u(\cdot)$ is locally up to $10^3 Hz$ and might seem high. However, it should be noted that a relative degree four model in reality could arise from modeling a mechanical system (relative degree two) in combination with a model of the electro-mechanical actuator (relative degree two). Since the electrical input is often realized with a digital controller, a frequency of $10^3 Hz$ should be no problem.

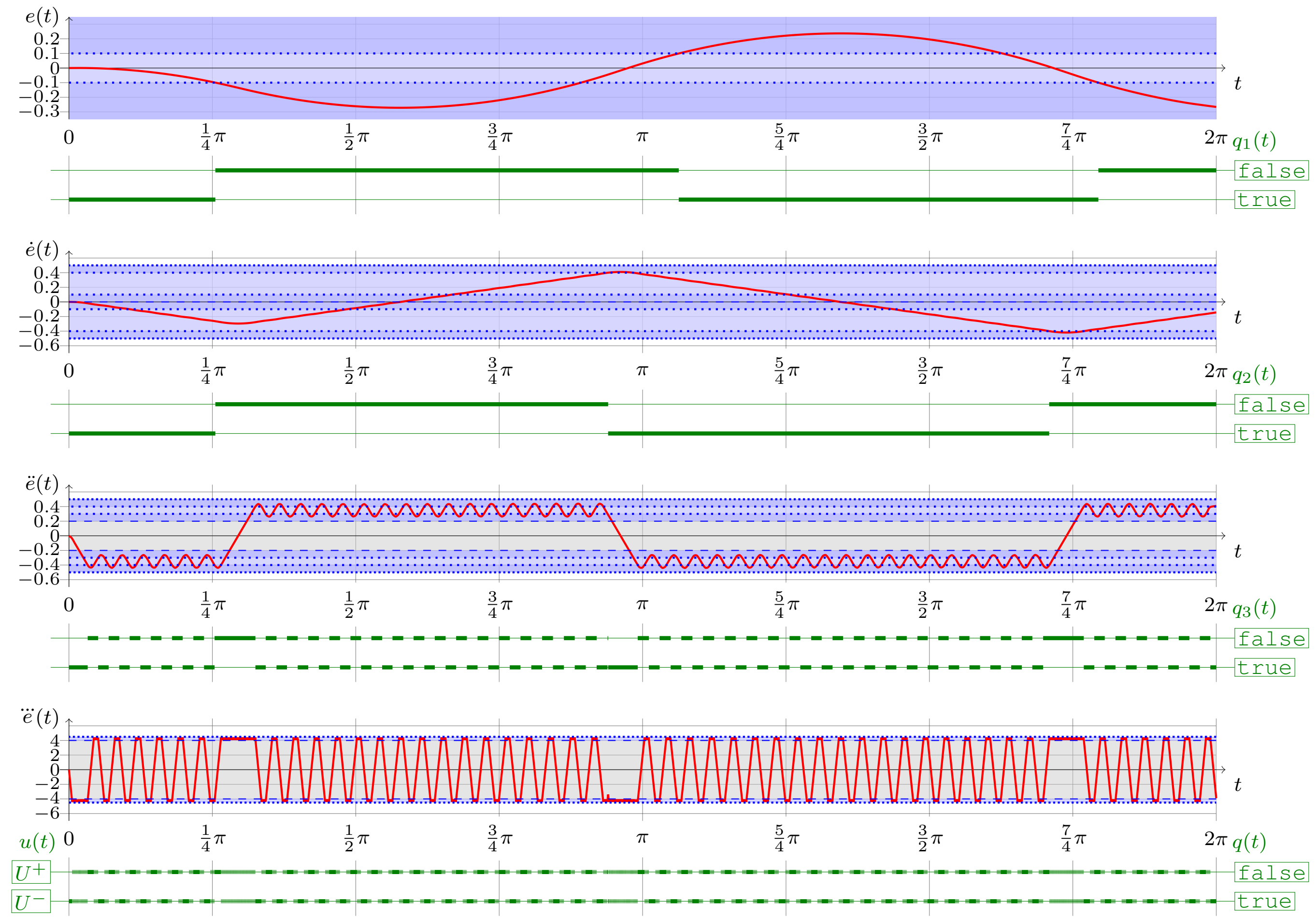

Fig. 3: The error and its derivatives — with corresponding switching variables —. The funnel boundaries are drawn as ····· (note that the funnel boundaries $\varphi_0^{\pm} \equiv 1$ are not in the picture), the safety distances are shown as ····,