Averaging for Switched DAEs: Convergence, Partial Averaging and Stability

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Abstract

Averaging is a useful technique to simplify the analysis of switched systems. In this paper we present averaging results for the class of systems described by switched differential algebraic equations (DAEs). Conditions on the consistency projectors are given which guarantee convergence towards a non-switched averaged system. A consequence of this result is the possibility to stabilize switched DAEs via fast switching. We also study partial averaging in case the consistency projectors do not satisfy the conditions for convergence; the averaged system is then still a switched system, but is simpler than the original. The practical interest of the theoretical averaging results is demonstrated through the analysis of the dynamics of a switched electrical circuit.

Keywords: switched systems; descriptor systems; averaging; exponential stability; linear/nonlinear models.

1. Introduction

Hybrid systems encompass continuous and discrete behavior, see e.g. Schaft and Schumacher (2000). A switched system is a hybrid system consisting of a family of dynamical subsystems and a policy that at each time instant selects the active subsystem among a set of possible modes (Liberzon, 2003). The selection policy is usually described by means of a switching function, which here is assumed to be a function of time (in contrast to state dependent switching).

In this paper we study switched systems whose modes are given by linear differential algebraic equations (DAEs). Linear DAEs are a natural way of modeling electrical circuits, simple mechanical systems or, in general, (linear) systems with additional (linear) algebraic constraints (Kunkel and Mehrmann, 2006). If this kind of systems change their model at some time one obtains a switched system; for example one can add (ideal) switches to an electrical circuit or allow for sudden structural changes in mechanical systems. The potentially complex interaction between the modes dynamics and the switching signal complicates the analysis of switched models. A possible approach to circumvent some of these difficulties, when switchings occur at high frequencies, is to average the hybrid dynamics over a time interval and to base the analysis and control design on the simpler averaged system.

Averaging theory for switched systems has a big interest in the control literature considering different approaches and points of view related to the switched system characteristics: non-periodic switching functions (Porfiri et al., 2008; Almér and Jönsson, 2009), pulse modulations (Teel et al., 2004; Pedicini et al., 2011), dithering (Iannelli et al., 2008), effects of exogenous inputs (Iannelli et al., 2008), hybrid systems framework (Wang and Nešić, 2010; Wang et al., 2012). On the practical point of view, the averaging approach is a widely used technique in the power electronics community since 1970s (Sanders et al., 1991; Pedicini et al., 2012b) and has been also applied to other switched systems of practical interest, see (Pedicini et al., 2012a) and the references therein. This paper has three major contributions: 1) We establish an averaging result for linear switched DAEs, 2) we present a partial averaging result in case a smooth averaged model does not exist and 3) we show how the averaging result can be utilized to achieve stabilization via fast switching.

Averaging results for switched DAEs are presented in the conference papers (Iannelli et al., 2013a,b), but under strong limitations on the number of modes and on some properties of their matrices. An alternative averaged model is conjectured in Mostacciuolo and Vasca (2016), but without providing a formal proof of convergence. The averaging result presented in this paper is able to considerably relax the strong assumptions of the previous works. The regularity of the DAEs allow us to establish an equivalence of a DAE with a proper ordinary differential equation (ODE) and then to prove an averaging result which is also new for switched ODEs with jumps.

The partial averaging result is an extension of the averaging result when some parts are still switching. It is built upon our conference paper (Mostacciuolo et al., 2015b) which considers only two modes; here we present the result for arbitrarily many modes.

The stability property is a key topic for switched systems (Sun and Ge, 2011).The stabilization procedure for switched DAE that we propose, is via fast switching. Our use of averaging technique with this aim is new, but there is a strong connection to the results in Mironchenko et al. (2015); in particular, Mironchenko et al. (2015, Rem. 21) already discusses this connection and concludes that the averaging technique may

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be more powerful because commutativity of the flows is not needed, see also the recent detailed comparison of this two approaches in Trenn (2016).

The paper is organized as following: in Section 2 we recall some mathematical notions, present some concepts regarding switched ODEs with jumps and some results from the theory of switched DAEs. In Section 3 we present the averaging result for switched DAEs; the stability analysis is carried out in Section 4 resulting in a method for stabilization via fast switching. In Section 5 the partial averaging result is presented. The conclusions of the work are summarized in Section 6.

2. Notation and preliminaries

In the following subsections some preliminary definitions are recalled. Furthermore, in order to present the averaging technique, some results regarding switched ODEs and some concepts of the theory of switched DAEs are illustrated. In the sequel the following notation is adopted: $\mathbb{R}^n$ is the set of n-th dimensional real vectors, $\mathbb{R}_+$ is the set of nonnegative real numbers, $\mathbb{N}$ is the set of nonnegatives integers, the product of any q matrices $\{M_i\}_{i=1}^{\mathsf{q}}$ is defined as (note the order)

$$\prod_{i=1}^{\mathsf{q}} M_i = M_{\mathsf{q}} M_{\mathsf{q}-1}, \dots, M_2 M_1,$$

$\|\cdot\|$ is the Euclidean norm and $\|\cdot\|_\infty$ is the infinity norm. Recall that a function $f : \mathbb{R} \to \mathbb{R}^n$ is called Lipschitz, if there exists a positive constant $L > 0$ such that $\forall p_1, p_2 \in \mathbb{R}$ the inequality

$$\|f(p_1) - f(p_2)\| \leq L\,|p_1 - p_2|$$

holds.

2.1. Big-O notation

Definition 1 (Big-O notation). *Given any functions $f : \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$, we say that f is an $\mathrm{O}(g(p))$ function as $p \to 0$ ($f(p) = \mathrm{O}(g(p))$ for short), if there exist positive constants α and $\bar{p}$ such that*

$$\|f(p)\| \leq \alpha g(p), \quad \forall p \in (0, \bar{p}).$$

In the case that f is a matrix-valued Definition 1 above can be directly extended by using an induced matrix norm.

In the following, we are mainly concerned with the case $g(p) = p$ and $f(p) = \mathrm{O}(p)$. Clearly any linear combination of functions which are $\mathrm{O}(p)$ is an $\mathrm{O}(p)$ function itself. Moreover if f is Lipschitz and $f(0) = 0$ then it is also $\mathrm{O}(p)$ but the converse does not necessarily hold because Definition 1 does not require $f(p)$ to be continuous. If $f(p) = \mathrm{O}(p)$ then $f(p) \to 0$ as $p \to 0$. Given a compact set $\Im \subset (0, \infty)$ and functions $x_p : \mathbb{R}_+ \to \mathbb{R}^n$ parameterized by $p > 0$, we implicitly indicate by

$$x_p(t) = \mathrm{O}(p), \quad \forall t \in \Im$$

that these values are $\mathrm{O}(p)$ *uniformly* in t, i.e. the big-O-constant α is independent of t.

By considering the Taylor approximation we can write, for any matrix $M \in \mathbb{R}^{n\times n}$ and any $s \in [0, p]$

$$e^{Ms} = I + Ms + \mathrm{O}(p^2) = I + \mathrm{O}(p), \tag{1}$$

where I is the identity matrix.

2.2. Projectors

Recall that a matrix $\Pi \in \mathbb{R}^{n\times n}$ (or its associated linear map) is a projector by definition if, and only if, it is idempotent, i.e. $\Pi^2 = \Pi$. There is a one-to-one correspondence between projectors in $\mathbb{R}^n$ and direct sums $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W}$, via

$$\operatorname{im} \Pi = \mathcal{V}, \quad \ker \Pi = \mathcal{W};$$

the projector is then said to map onto $\mathcal{V}$ along $\mathcal{W}$.

Lemma 2. *Let $\Pi \in \mathbb{R}^{n\times n}$ be a projector and $M \in \mathbb{R}^{n\times n}$ then*

$$\begin{aligned} \operatorname{im} M \subseteq \operatorname{im} \Pi \quad &\Leftrightarrow \quad \Pi M = M, \\ \ker M \supseteq \ker \Pi \quad &\Leftrightarrow \quad M\Pi = M. \end{aligned}$$

Proof. Necessity in both cases is trivial. Since Π is the identity on $\operatorname{im} \Pi$ sufficiency for the first case is also clear. Considering the transpose and orthogonal complements, sufficiency of the second case follows with analogous arguments. □

For a family of projectors $\{\Pi_i\}_{i=1}^{\mathsf{q}}$ we will introduce now the following Projector Assumption (PA) which will play a crucial role for our averaging results

$$\operatorname{im} \Pi_\cap \subseteq \operatorname{im} \Pi_i, \tag{PA.1}$$

$$\ker \Pi_\cap \supseteq \ker \Pi_i, \tag{PA.2}$$

$\forall i \in \Sigma := \{1, \dots, \mathsf{q}\}$ with $\Pi_\cap$ given by

$$\Pi_\cap := \prod_{i=1}^{\mathsf{q}} \Pi_i. \tag{2}$$

Corollary 3. *If a family of projectors $\{\Pi_i\}_{i=1}^{\mathsf{q}}$ with $\Pi_\cap$ given by* (2)*, satisfies the Projector Assumption (PA) then $\Pi_\cap^2 = \Pi_\cap$, i.e. $\Pi_\cap$ itself is a projector.*

Remark 4. *Consider a family of projectors $\{\Pi_i\}_{i=1}^{\mathsf{q}}$ which commute, i.e.*

$$\Pi_i \Pi_j = \Pi_j \Pi_i, \quad \forall i, j \in \Sigma, \tag{3}$$

then $\Pi_\cap$ given as in (2) *satisfies $\Pi_i \Pi_\cap = \Pi_\cap = \Pi_\cap \Pi_i$ for all $i \in \Sigma$, hence Lemma 2 implies* (PA.1) *and* (PA.2)*, but it is* not true *in general that (PA) implies commutativity of the projectors, see e.g. the forthcoming Example 16.*

Lemma 5. *(Iannelli et al., 2013a, Lem. 2 & Lem. 3) Let $\ell(p) : \mathbb{R}_+ \to \mathbb{N}$ be such that $p\ell(p) = \mathrm{O}(1)$ and let $\Pi \in \mathbb{R}^{n\times n}$ be a projector. Then*

$$(\Pi + \mathrm{O}(p))^{\ell(p)} = \mathrm{O}(1). \tag{4}$$

Furthermore, for any matrices $M, \tilde{M} \in \mathbb{R}^{n\times n}$ with

$$\Pi M \Pi = M = \Pi \tilde{M} \Pi$$

it holds that

$$\Pi\Big((\Pi + \tilde{M}p + O(p^2))^{\ell(p)} - (\Pi + Mp + O(p^2))^{\ell(p)}\Big)\Pi = \ell(p)O(p^2). \quad (5)$$

In the following an interpretation for $\ell(p)$ in Lemma 5 will be the number of consecutive periods of length p inside a fixed time interval $[0, \Delta]$. Indeed for this case $\ell(p)$ tends to infinity when p goes to zero but $\Delta - p < p\ell(p) \leq \Delta = \mathrm{O}(1)$.

Remark 6. *The big-O-bounds in* (4) *and* (5) *can be given a bit more explicit: by using an iterative approach with tedious but standard algebraic manipulation, it can be shown that for any projector $\Pi \in \mathbb{R}^{n\times n}$, any $M \in \mathbb{R}^{n\times n}$ and any $O(p^2)$ matrix $F : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ we have*

$$\|(\Pi + Mp + F(p))^{\ell(p)}\| \leq \alpha_1 e^{\alpha_2\alpha_{3,M}\Delta}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}_+$ are such that $\|\cdot\| \leq \alpha_1 |||\cdot|||$ and $|||\cdot||| \leq \alpha_2\|\cdot\|$ with a norm $|||\cdot|||$ on $\mathbb{R}^n$ defined such that for the induced matrix norm it holds that $|||\Pi||| = 1$ (it is easily seen that such a norm always exists); and $\alpha_{3,M} = 2\max\{\|M\|, \alpha_F\}$ where $\alpha_F := \sup_{p\in(0,1)} \|F(p)\|/p^2$. Moreover for any $\tilde{M} \in \mathbb{R}^{n\times n}$ and any $O(p^2)$ matrix $\tilde{F} : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ it holds

$$\Pi\Big((\Pi + \tilde{M}p + \tilde{F}(p))^{\ell(p)} - (\Pi + Mp + F(p))^{\ell(p)}\Big)\Pi \leq \Big|\Big(\Pi\tilde{M}^2\Pi - \Pi M^2\Pi\Big)\Big| \Delta p + \eta_1 e^{\eta_2\Delta} p^2$$

where $\eta_1 = 32\alpha_1\alpha_4\|\Pi\|^2$ with $\alpha_4 = \max(\alpha_F, \alpha_{\tilde{F}}, \|M\|^2, \|\tilde{M}\|^2)$, $\eta_2 = 4\alpha_2\max(\alpha_{3,M}, \alpha_{3,\tilde{M}})$ and where $\alpha_{\tilde{F}}$ and $\alpha_{3,\tilde{M}}$ are given analogously as above.

2.3. Class of switching signals

Let $\sigma : \mathbb{R}_+ \to \Sigma$ be a piecewise constant right-continuous function, that selects at each time instant the index of the active mode from the finite index set $\Sigma := \{1, 2, \ldots, \mathsf{q}\}$. In the sequel σ is called the switching signal. Here we assume that σ is periodic with switching period $p > 0$. Without restriction we assume that σ is monotone on each interval $[kp, (k+1)p)$, $k \in \mathbb{N}$, i.e., we consider the switching signal

$$\sigma(t) = \begin{cases} 1, & t \in [t_k, s_{k,2}), \\ 2, & t \in [s_{k,2}, s_{k,3}), \\ \vdots & \\ \mathsf{q}, & t \in [s_{k,\mathsf{q}}, t_{k+1}), \end{cases} \quad (6)$$

where the switching time instants t_k, $s_{k,i}$, $k \in \mathbb{N}$, $i \in \Sigma$ are defined as follows

$$t_k := kp, \qquad s_{k,i} := t_k + \sum_{j=1}^{i-1} d_j p, \quad (7)$$

where $d_i \in (0, 1)$ is the duty cycle of the i-th mode; in particular, $\sum_{i=1}^{\mathsf{q}} d_i = 1$. Note that $s_{k,1} = t_k$. Furthermore, let $c_i > 0$ be the time interval between the beginning of any period and the end of the i-th mode, i.e.

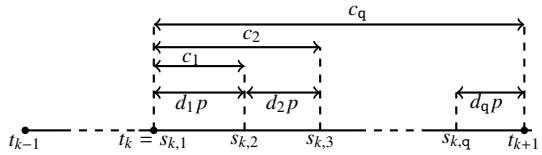

Figure 1: Illustration of the switching times notation.

$$c_i := \sum_{j=1}^{i} d_j p, \quad i \in \Sigma. \quad (8)$$

Note that $c_p = p$ and, by convention, $c_0 := 0$. The notation is illustrated in Figure 1.

2.4. Averaging for linear switched ODEs

Consider the switched ODE

$$\dot{w}(t) = A_{\sigma(t)}w(t) + B_{\sigma(t)}u(t), \quad \forall t \in \mathbb{R}_+, \quad w(0) = w_0,$$

with $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$, $i \in \Sigma$, switching signal given by (6) and continuous input $u : \mathbb{R}_+ \to \mathbb{R}^m$.

The corresponding averaged model is given by

$$\dot{w}_{\mathrm{av}}(t) = \sum_{i=1}^{\mathsf{q}} d_i(A_i w_{\mathrm{av}}(t) + B_i u(t)), \quad w_{\mathrm{av}}(0) = w_0, \quad (9)$$

see Pedicini et al. (2012b).

The approximation property between the averaged and the switched systems is $\mathrm{O}(p)$ assuming the same initial condition w_0 and that the exogenous input u is bounded, differentiable and with bounded derivative (Pedicini et al., 2011). No further assumptions on the matrices A_i and B_i are needed for this approximation result.

2.5. Switched ODEs with jumps

In this subsection we prove some results for the case that additional jumps are present in the switched ODE. The following averaging result are new and noteworthy by themselves, but mainly they will play important role for deriving our main results on averaging of switched DAEs.

Here we consider switched linear ODE with jumps of the form

$$\begin{aligned} \dot{w}(t) &= A_{\sigma(t)}w(t) + B_{\sigma(t)}u(t), \quad t \neq s_{k,i}, \\ w(s_{k,i}^+) &= \Pi_{\sigma(s_{k,i})}w(s_{k,i}^-) + Q_{\sigma(s_{k,i})}v(s_{k,i}), \end{aligned} \quad k \in \mathbb{N}, i \in \Sigma, \quad (10)$$

with initial condition $w(0^-) = w_0 \in \mathbb{R}^n$; σ is given by (6), $u : \mathbb{R}_+ \to \mathbb{R}^{m_u}$, $m_u \in \mathbb{N}$, is the flow input, $v : \mathbb{R}_+ \to \mathbb{R}^{m_v}$, $m_v \in \mathbb{N}$, is the jump input, $A_i \in \mathbb{R}^{n\times n}$, B_i, $Q_i \in \mathbb{R}^{n\times m}$ and $\Pi_i \in \mathbb{R}^{n\times n}$ are projectors determining the jumps. The first equation in (10) describes the dynamics in the different modes, while the second equation represents the jump rule at the switching time instants.

The following Lemma expresses the solution of (10) evaluated at the multiplies of the switching period in a compact form depending on the initial conditions and on the input.

Lemma 7. *Consider the switched ODE* (10) *with periodic switching signal* (6) *with period* $p > 0$. *Then there exist matrices* $H(p) \in \mathbb{R}^{n\times n}$, $N(p) \in \mathbb{R}^{n\times \mathrm{q}m_v}$ *and an operator* $\mathcal{I}(p)$ *such that every solution of* (10) *satisfies*

$$w(t_k^-) = H(p)w(t_{k-1}^-) + N(p)v_{k-1} + \mathcal{I}(p)\{u_{k-1}\} \quad \forall k \in \mathbb{N}, \quad (11)$$

where $v_{k-1} := [v_{s_{k-1,1}} v_{s_{k-1,2}} \dots v_{s_{k-1,\mathrm{q}}}]^\top$ *and* u_{k-1} *indicates the input function on the time interval* (t_{k-1}, t_k) *translated into the time interval* $(0, p)$, *i.e.,* $u_{k-1} : (0, p) \to \mathbb{R}^m$, $\xi \mapsto u(\xi + t_{k-1})$. *In particular,*

$$w(t_k^-) = H(p)^k w_0 + \sum_{i=0}^{k-1} H(p)^{k-1-i}(N(p)v_i + \mathcal{I}(p)\{u_i\}). \quad (12)$$

The explicit formulas for $H(p)$, $N(p)$ and $\mathcal{I}(p)$ and the proof are given in the Appendix.

We highlight in the following that for a zero initial value, "small" inputs and some additional assumptions on the jump maps, the solutions of (10) remain small in the $\mathrm{O}(p)$ sense.

Lemma 8. *Consider the switched ODE* (10) *with initial condition* $w(0^-) = 0$ *and periodic switching signal* (6) *with period* $p > 0$. *Consider any given interval* $[0, \Delta]$ *where* $\Delta \in \mathbb{R}_+$, *and assume that the following conditions hold*

(i) $u(t) = \mathrm{O}(p)$, $\forall t \in [0, \Delta]$,
(ii) $v_{s_{k,i}} = \mathrm{O}(p)$, $\forall k \in \mathbb{N}$, $i \in \Sigma$,
(iii) $\Pi_\cap$ *given by* (2) *is a projector,*
(iv) $\Pi_i Q_{i-1} = 0$, $i \in \Sigma$ *with* $Q_0 := Q_\mathrm{q}$.

Then $w(t) = \mathrm{O}(p)$, $\forall t \in [0, \Delta]$.

The proof is carried out in the Appendix.

Remark 9. *Lemma 8 is similar to classical input to state stability results, in the sense that a small input (of order* $\mathrm{O}(p)$*) results in a small state (also* $\mathrm{O}(p)$*) on any fixed time interval. A stability result utilizing averaging for general hybrid systems has been investigated in Wang et al. (2012).*

2.6. Switched DAEs

A non homogeneous switched linear DAE is given by

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \in \mathbb{R}_+, \quad (13)$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state, $u : \mathbb{R}_+ \to \mathbb{R}^m$ is the input, $x(0^-) = x_0$ is the initial condition and the periodic switching signal σ is given by (6).

The dynamic of each mode i of the system is given by the following linear DAE

$$E_i\dot{x}(t) = A_i x(t) + B_i u(t) \quad (14)$$

where E_i, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ are constant matrices for each $i \in \Sigma$. All solutions of each mode evolve within a consistency space that is a linear subspace of $\mathbb{R}^n$. The value $x(s_{k,i}^-)$ just before a switching instant $s_{k,i}$ is not necessarily in the consistency space of the mode after the switch. Therefore it is necessary to allow solutions with jumps; this leads to problems in evaluating the derivative in (13). To resolve this problem we use the distributional solution framework as introduced in Trenn (2012). Furthermore, the solutions of switched DAE can also contain Dirac impulses (in addition to possible jumps), i.e., each mode can have impulsive modes of arbitrary degree but in this paper we only consider the impulse-free part of the solution (which may still contain jumps). Recently, some preliminary results concerning the convergence of the Dirac impulses were obtained in Trenn (2015).

If the matrix pairs (E_i, A_i) are regular, i.e. $m = n$ and the polynomial $\det(sE_i - A_i)$ is not the zero polynomial, then the following result is well known:

Proposition 10 (Quasi Weierstrass form). *A matrix pair* $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ *is regular if and only if there exist invertible transformation matrices* $S, T \in \mathbb{R}^{n\times n}$ *which put* (E, A) *into quasi Weierstrass form*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right) \quad (15)$$

where $N \in \mathbb{R}^{n_2\times n_2}$, *with* $0 \le n_2 \le n$ *is a nilpotent matrix,* $J \in \mathbb{R}^{n_1\times n_1}$ *with* $n_1 = n - n_2$ *is some matrix and* I *is the identity matrix of the appropriate size.*

Note that, the transformation matrices S and T can easily be obtained via the so called Wong sequences, see Berger et al. (2012).

Definition 11 (Flow matrix and projectors). *Consider a regular matrix pair* (E, A) *and its quasi Weierstrass form* (15). *The consistency projector* Π *and the flow matrix* A^{diff} *of* (E, A) *are given by*

$$\Pi = T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}T^{-1}, \quad A^{\mathrm{diff}} = T\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}T^{-1};$$

the differential and the impulsive projectors of (E, A) *are given by*

$$\Pi^{\mathrm{diff}} = T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}S, \quad \Pi^{\mathrm{imp}} = T\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}S.$$

Note that the flow matrix and the projectors do not depend on the specific choice of T and S, furthermore it is easily seen that $A^{\mathrm{diff}}\Pi = A^{\mathrm{diff}} = \Pi A^{\mathrm{diff}}$ and Π is indeed idempotent and hence a projector, but the differential and impulse projectors are not idempotent in general.

The role of projectors and the flow matrix becomes clear with the following important result.

Theorem 12. *Consider the switched DAE* (13) *with regular matrix pairs* (E_i, A_i) *and corresponding flow matrices* A_i^{diff} *and projectors* $\Pi_i, \Pi_i^{\mathrm{imp}}, \Pi_i^{\mathrm{diff}}$ *for* $i \in \Sigma$. *Assume that*

$$\Pi_i^{\mathrm{imp}} B_i = 0, \quad \forall i \in \Sigma. \quad (16)$$

Then $x : \mathbb{R}_+ \to \mathbb{R}^n$ *is the impulse free part of any (distributional) solution of* (13) *if and only if* x *is a solution of the*

switched ODE with jumps given by

$$\begin{aligned} \dot{x}(t) &= A_i^{\text{diff}} x(t) + B_i^{\text{diff}} u(t), \qquad \forall t \in (s_{k,i}, s_{k,i+1}) \\ x(s_{k,i}^+) &= \Pi_i x(s_{k,i}^-), \\ x(0^-) &= x_0, \end{aligned} \tag{17}$$

where $B_i^{\text{diff}} := \Pi_i^{\text{diff}} B_i,\ i \in \Sigma,\ k \in \mathbb{N}$.

Proof. The (impulse-free) solution of (13) is obtained by "concatenating" the solution of each mode (14), that can be written as follows

$$\begin{aligned} x(t) = e^{A_i^{\text{diff}}(t-s_{k,i})} x(s_{k,i}^+) &+ \int_{s_{k,i}}^{t} e^{A_i^{\text{diff}}(t-s)} \Pi_i^{\text{diff}} B_i u(s) \mathrm{d}s \\ &- \sum_{i=0}^{n-1} (E_i^{\text{imp}})^i \Pi_i^{\text{imp}} B_i u(t)^{(i)} \end{aligned} \tag{18}$$

with $E_i^{\text{imp}} := \Pi_i^{\text{imp}} E_i$ and $t \in (s_{k,i}, s_{k,i+1})$. Then the proof directly follows by considering (18) combined with (16). □

Remark 13. *Theorem 12 generalizes the result in Trenn and Wirth (2012) to the inhomogeneous case with arbitrarily high index.*

Remark 14. *As a consequence of Theorem 12 and Lemma 7 we can write the solution of the switched DAE* (13) *at* t_k *in a form similar to* (12) *with* $v_i = 0\ \forall i \in \mathbb{N}$*; in particular*

$$x(t_k^-) = H_{\text{diff}}(p)^k w_0 + \sum_{i=0}^{k-1} H_{\text{diff}}(p)^{k-1-i} \mathcal{I}_{\text{diff}}(p)\{u_i\}, \tag{19}$$

where

$$H_{\text{diff}}(p) = \prod_{i=1}^{\mathfrak{q}} e^{A_i^{\text{diff}} d_i p} \Pi_i, \tag{20a}$$

$$\mathcal{I}_{\text{diff}}(p)\{u_{k-1}\} = \sum_{i=1}^{\mathfrak{q}} \prod_{j=i+1}^{\mathfrak{q}} e^{A_j^{\text{diff}} d_j p} \Pi_j \int_{c_{i-1}}^{c_i} e^{A_i^{\text{diff}}(c_i-\xi)} B_i^{\text{diff}} u_{k-1}(\xi) \mathrm{d}\xi, \tag{20b}$$

with c_i *given by* (8).

3. Averaging for switched DAEs

Averaging theory is based on the observation that a rapidly time-varying system can be viewed as a small perturbation of a simplified, time-invariant, averaged system.

Given a switched DAE (13) with periodic switching signal σ given by (6) with period $p > 0$, we want to investigate the possible existence of an averaged model that approximates the behavior of the system. For that we need to show that in the limit $p \to 0$ the solution of the averaged model converges to that of the switched system.

We propose the following *averaged model* of (13)

$$\begin{aligned} \dot{x}_{\text{av}}(t) &= A_{\text{av}} x_{\text{av}}(t) + B_{\text{av}} u(t), \qquad t \in \mathbb{R}_+ \\ x_{\text{av}}(0) &= \Pi_\cap x_0 \end{aligned} \tag{21}$$

where

$$\left.\begin{aligned} A_{\text{av}} &:= \Pi_\cap A_{\text{av}}^{\text{diff}} \Pi_\cap, & B_{\text{av}} &:= \Pi_\cap B_{\text{av}}^{\text{diff}} \\ A_{\text{av}}^{\text{diff}} &:= \sum_{i=1}^{\mathfrak{q}} d_i A_i^{\text{diff}}, & B_{\text{av}}^{\text{diff}} &:= \sum_{i=1}^{\mathfrak{q}} d_i B_i^{\text{diff}} \end{aligned}\right\} \tag{22}$$

and d_i, $i \in \Sigma$, is the duty cycle of the i-th mode as in (7) and $\Pi_\cap$ is given by (2).

Remark 15. *If* $\Pi_\cap$ *is a projector then* $\operatorname{im}\Pi_\cap$ *is* A_{av}*-invariant, in particular, all solutions of* (21) *evolve within* $\operatorname{im}\Pi_\cap$*. Furthermore, if (PA) holds* $\forall i \in \Sigma$*, then due to Lemma 2 we have*

$$x_{av}(t) = \Pi_i x_{av}(t),\ \forall t \in \mathbb{R}_+,\ \forall i \in \Sigma.$$

In the following subsections we present conditions for which the system (21)–(22) indeed represent an averaged model of (13) for the homogeneous and non homogeneous cases, respectively.

3.1. Homogeneous switched DAEs

In the following example we consider a switched DAE (13) with $u = 0$. This setup has already been investigated (Iannelli et al., 2013b,a), however the following example shows that the commutative condition on the consistency projectors formulated therein is not necessary for convergence towards the averaged model.

Example 16. *Consider the switched DAE* (13) *in the homogeneous case, i.e.* $u = 0$*, with three modes, i.e.* $\mathfrak{q} = 3$*, given by*

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 8 & -1 & 8 \\ -1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -10 & -1 & -10 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 4 & 0 \\ -4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The corresponding consistency projectors are

$$\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The consistency projectors do not pairwise commute, hence the results in Iannelli et al. (2013b,a) are not applicable. However, simulations indicate that nevertheless convergence occurs for fast switching; Figure 2 illustrates the convergence for duty cycles $(d_1, d_2, d_3) = (0.2, 0.5, 0.3)$*. The corresponding averaged model* (21) *is given by*

$$\dot{x}_{\text{av}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_{\text{av}}(t), \quad x_{\text{av}}(0) = \Pi_\cap x_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_0.$$

The above example indicates that the assumptions made in Iannelli et al. (2013b,a) are too restrictive. The following main averaging result for homogeneous switched DAEs indeed shows that the assumptions on the consistency projectors can be significantly relaxed:

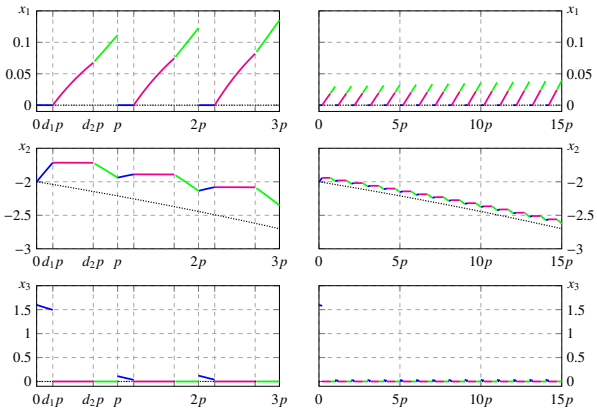

Figure 2: Evolution of state variables (first component top, second component middle, third component bottom) of Example 16 with initial value $x_0 = (0.1, -2, 1.5)^\top$ for slow switching ($p = 0.1s$, left) and fast switching ($p = 0.02s$, right). The averaging dynamics are plotted with dotted black lines, while the trajectories of the switched DAE are colored according to the active mode (mode 1 blue, mode 2 magenta, mode 3 green). Note that x_3 is not $\mathrm{O}(p)$ on $[0, p)$.

Theorem 17. *Consider the regular switched DAE* (13) *with periodic switching signal σ given by* (6) *with period $p > 0$, initial condition $x(0^-) = x_0$ and $u = 0$. Denote by $x_{\sigma,p}(t)$ the (in general discontinuous) impulse-free part of the (in general distributional) solution of* (13) *with $u = 0$ and let $x_{\mathrm{av}}(t)$ be the (smooth) solution of* (21) *with $u = 0$. Assume that (PA) holds $\forall i \in \Sigma$, then for any $\Delta > p$*

$$x_{\sigma,p}(t) - x_{\mathrm{av}}(t) = \mathrm{O}(p), \tag{23}$$

uniformly for all $t \in [p, \Delta]$.

The proof is carried out in the Appendix.

Example 18 (Example 16 revisited). *The averaged model conjectured in Example 16 can be now confirmed. It is easily seen that*

$$\operatorname{im} \Pi_\cap = \begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix} \subseteq \operatorname{im} \Pi_i \text{ and } \ker \Pi_\cap = \begin{pmatrix} * \\ 0 \\ * \end{pmatrix} \supseteq \ker \Pi_i$$

for $i = 1, 2, 3$, i.e. (PA) holds and Theorem 17 can be applied, hence the observed averaging behavior from the simulations is indeed proven.

Remark 19. *An expression for the bound corresponding to the big O-term in* (23) *can be obtained by following the three steps adopted in the proof of Theorem 17 reported in the Appendix, by using the second order Taylor remainders for the exponential matrices, see e.g. Amann and Escher (2008, Theorem 5.8), and by exploiting the bounds in Remark 6. With tedious but standard algebraic manipulation it can be shown that there exist constants $\zeta_1 > 0$, $\zeta_2 > 0$ and $\bar{p} < 1$ such that*

$$\|x_{\sigma,p}(t) - x_{\mathrm{av}}(t)\| \leq \zeta_1 e^{\zeta_2 \Delta} \|x_0\| p, \ \forall t \in [p, \Delta], \ \forall p \leq \bar{p}.$$

Note that for a fixed period p the bound in Remark 19 grows to infinity with $\Delta \to \infty$. In general, a bounded error on the whole time-axis cannot be expected, as even for the classical averaging result on switched ODEs such a bound does not exist, see the forthcoming Example 28.

A bound of the error for $\Delta \to \infty$ can be found under some additional assumptions. An interesting case is when the averaged system (21) is exponentially stable. In that case our forthcoming stability result (Theorem 27) shows that the switched system is exponentially stable too, i.e. $x_{\sigma,p}$ converges to zero as $t \to \infty$. Hence both $x_{\sigma,p}$ and x_{av} converge to zero as $t \to \infty$ then the global boundedness of $x_{\sigma,p} - x_{\mathrm{av}}$ can trivially be concluded for sufficiently small p.

The exponential stability of the averaged system is not a necessary condition for the error boundedness with $\Delta \to \infty$. For instance, assume that the consistency projectors commute with the flow-matrices and with each-other, i.e.

$$\Pi_i A_j^{\mathrm{diff}} = A_j^{\mathrm{diff}} \Pi_i, \ \forall i, j \in \Sigma, \tag{24}$$

together with (3). Then with simple algebraic manipulations on (19)–(20) one can show that

$$x_{\sigma,p}(t_k) = x_{\mathrm{av}}(t_k), \ \forall k \in \mathbb{N}.$$

Therefore, if x_{av} remains bounded, then we can conclude that (23) uniformly for all $t \in [p, \infty)$.

Remark 20. *Theorem 17 makes a statement about the homogeneous switched DAE* (13)*; however, it is also applicable to switched ODEs with jumps of the form* (17) *with $u = 0$. For this it is not necessary that A_i^{diff} and Π_i^{diff}, $i \in \Sigma$, are defined in terms of regular matrix pairs (E_i, A_i); it suffices that the following properties hold: $\Pi_i^2 = \Pi_i$, $\Pi_i A_i^{\mathrm{diff}} = A_i^{\mathrm{diff}} = A_i^{\mathrm{diff}} \Pi_i$, $i \in \Sigma$, i.e. Π_i must be projectors which are compatible with the corresponding flow matrices A_i^{diff}. Then (PA) also ensures convergence towards an averaged system for switched ODE with jumps.*

Remark 21. *The Projector Assumption (PA) means that, in contrast to the classical averaging result on switched ODEs, the averaging result for switched DAEs depends on the sequence of modes because of the presence of $\Pi_\cap$ in* (22)*. For instance, by considering in Example 16 the sequence of modes* 1, 3, 2 *instead of* 1, 2, 3*, the condition* (PA.1) *is no more satisfied and convergence towards the averaged system does not occur anymore.*

3.2. Non homogeneous switched DAEs

The following simple example shows that a straightforward generalization of the averaging result to the non homogeneous case is not possible.

Example 22. *Consider the scalar switched DAE* (13) *with*

$$(E_1, A_1, B_1) = (0, 1, 1) \text{ and } (E_2, A_2, B_2) = (0, 1, 0),$$

then $x(t) = u(t)$ in mode 1 and $x(t) = 0$ in mode 2. If the input is not zero, this means that fast switching will not result in convergence because x will jump back and forth between a non-zero value and zero. Note that $\Pi_1 = \Pi_2 = 0$, hence for $u = 0$ the assumptions of Theorem 17 are trivially satisfied.

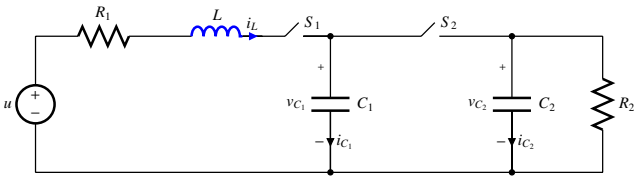

Figure 3: Electrical circuit with two capacitors, one inductor and two switches.

However, if the solutions of the switched DAE can be expressed by solutions of a switched ODE with jumps, i.e., the assumptions of Theorem 12 are satisfied, then an averaging result can be shown also in the non homogeneous case, (Mostacciuolo et al., 2015a) for the case of commuting consistency projectors.

Theorem 23. *Consider the regular switched DAE* (13) *with periodic switching signal* σ *given by* (6) *with period* $p > 0$ *and initial condition* $x(0^-) = x_0$. *Denote by* $x_{\sigma,p}$ *the (in general discontinuous) impulse-free part of the (in general distributional) solution of* (13) *and let* x_{av} *be the (smooth) solution of* (21). *If (PA) and* (16) *hold* $\forall i \in \Sigma$ *and the input* $u : \mathbb{R}_+ \to \mathbb{R}^m$ *is Lipschitz continuous, then for any* $\Delta > p$

$$x_{\sigma,p}(t) - x_{\text{av}}(t) = \mathrm{O}(p) \tag{25}$$

uniformly for all $t \in [p, \Delta]$.

The proof is carried out in the Appendix.

We will now apply the theoretical result to a model of an electrical circuit with switches as given in Figure 3.

Example 24. *The electrical circuit in Figure 3 can be modeled by a non homogeneous switched DAE* (13) *with* $x = [v_{C_1},\ v_{C_2},\ i_L]^\top$ *and*

$$E_1 = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{R_2} & 0 \\ -1 & 0 & -R_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} C_1 & C_2 & 0 \\ 0 & 0 & L \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\frac{1}{R_2} & 1 \\ -1 & 0 & -R_1 \\ 1 & -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$E_3 = \begin{bmatrix} C_1 & C_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -\frac{1}{R_2} & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$
$$E_4 = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{R_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The correspondence between the modes $1, 2, 3, 4$ *and the switches* S_1, S_2 *is indicated in Table 1.*

Table 1: Modes of the electrical circuit.

mode	1	2	3	4
S_1	closed	closed	open	open
S_2	open	closed	closed	open

The corresponding consistency and impulse projectors are given by

$$\Pi_1 = I, \quad \Pi_2 = \begin{bmatrix} \rho_1 & \rho_2 & 0 \\ \rho_1 & \rho_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Pi_3 = \begin{bmatrix} \rho_1 & \rho_2 & 0 \\ \rho_1 & \rho_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

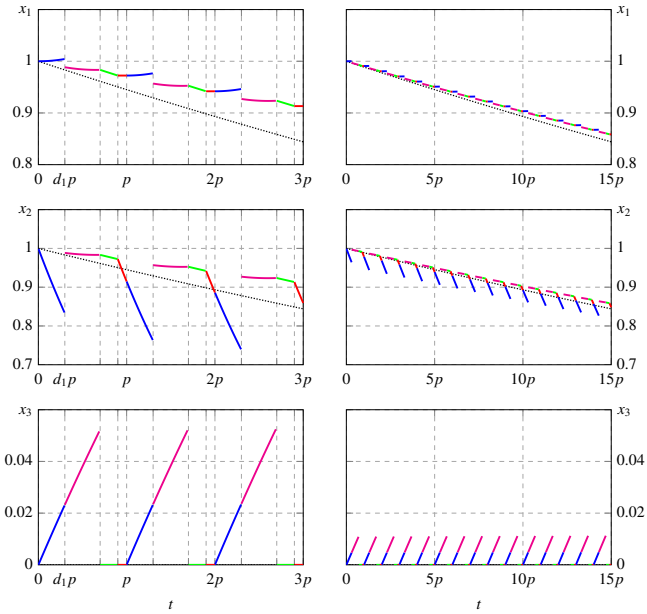

Figure 4: Evolution of the state variables (first component top, second component middle, third component bottom) for slow switching ($p = 0.1s$, left) and fast switching ($p = 0.02s$, right). The averaging dynamics are plotted with dotted black lines, while the trajectories of the switched DAE are colored according to the active mode (mode 1 blue, mode 2 magenta, mode 3 green, mode 4 red).

$$\Pi_1^{\text{imp}} = 0, \quad \Pi_2^{\text{imp}} = \begin{bmatrix} 0 & 0 & \rho_2 \\ 0 & 0 & -\rho_1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_3^{\text{imp}} = \begin{bmatrix} 0 & \rho_2 & 0 \\ 0 & -\rho_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Pi_4^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\rho_1 := \frac{C_1}{C_1+C_2}$ *and* $\rho_2 := \frac{C_2}{C_1+C_2}$. *It is easily seen that the consistency projectors commute, hence (PA) is satisfied, and furthermore* $\Pi_i^{\text{imp}} B_i = 0$, *so Theorem 23 is applicable. The corresponding averaged system* (21) *for the duty cycles* $(d_1, d_2, d_3, d_4) = (0.3, 0.4, 0.2, 0.1)$ *is given by*

$$A_{av} = \begin{bmatrix} -\frac{\rho_1^2}{R_2C_1} & -\frac{\rho_2^2}{R_2C_1} & 0 \\ -\frac{\rho_1^2}{R_2C_1} & -\frac{\rho_2^2}{R_2C_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{av} = 0, \quad \Pi_\cap = \begin{bmatrix} \rho_1 & \rho_2 & 0 \\ \rho_1 & \rho_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the resulting averaged system is homogeneous no matter what the (positive) duty cycles and the physical parameters are. Figure 4 illustrates the convergence for the following parameters: $C_1 = 80.36\,\text{mF}$, $C_2 = 8.2\,\text{mF}$, $L = 5\,\text{H}$, $R_2 = 20\,\Omega$, $R_1 = 10\,\Omega$ *and* $u = 5\,\text{V}$, *the initial value is* $x_0 = (1,\ 1,\ 0)^\top$.

The Lipschitz assumption on the input u in Theorem 23 can be relaxed in a particular case as shown by the following result.

Proposition 25. *Consider a non homogeneous switched DAE* (13) *where (PA) and* (16) *hold. If additionally*

$$B_i^{\text{diff}} = B_j^{\text{diff}}, \quad \forall i, j \in \Sigma \tag{26}$$

then the averaging result (25) *is satisfied.*

The proof is carried out in the Appendix.

Remark 26. *In the case of a switched DAE with two modes, the averaging results proved in Theorem 17, Theorem 23 and Proposition 25 hold even if* $\Pi_\cap$ *is a projector but assumption* (PA.2) *doesn't hold.*

4. Stability via fast switching

The averaging result in Theorem 17 can be used for the stability analysis of the homogeneous switched DAE

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \in \mathbb{R}_+, \tag{27}$$

with σ given by (6). As already pointed out in Remark 19, the convergence towards the averaged system is only true on any compact interval, hence it is not immediately clear what the convergence behavior for $t \to \infty$ is. This problem can be resolved in case the averaged system is exponentially stable:

Theorem 27. *Consider the regular switched DAE* (27) *with periodic switching signal σ given by* (6) *and initial condition $x(0^-) = x_0$. If the corresponding homogeneous averaged system* (21) *is exponentially stable for some duty cycle, then there exists a sufficiently small switching period $p^* > 0$, such that the switched system* (27) *is exponentially stable.*

Proof. Due to the exponential stability of the averaged system we can choose a fixed time instant $T > 0$, such that

$$\|x_{\mathrm{av}}(T)\| \leq \frac{1}{2}\|x_{\mathrm{av}}(T/2)\|, \tag{28}$$

for all initial conditions $x_0 \in \mathbb{R}^n$ in (21). Let

$$c := \min\left\{ \left\|e^{A_{\mathrm{av}}T/2}\Pi_\cap x_0\right\| \;\middle|\; \|\Pi_\cap x_0\| = 1 \right\} > 0,$$

where positivity follows from the continuity of the map $z \mapsto e^{A_{\mathrm{av}}T/2}z$ and triviality of the kernel of the matrix $e^{A_{\mathrm{av}}T/2}$. Because of (23) we can choose $\overline{p} > 0$ sufficiently small such that

$$\|x_{\mathrm{av}}(T) - x_{\sigma,p}(T^-)\| \leq \frac{c}{8} \leq \frac{1}{8}\|x_{\mathrm{av}}(T/2)\| \tag{29}$$

$$\|x_{\sigma,p}(T/2^-) - x_{\mathrm{av}}(T/2)\| \leq \frac{c}{8} \leq \frac{1}{8}\|x_{\mathrm{av}}(T/2)\|, \tag{30}$$

for all $p \in (0, \overline{p})$ and all solutions of (27) and (21) where we consider, without loss of generality, initial conditions x_0 satisfying $\|\Pi_\cap x_0\| = 1$.

Combining (29) with (28), and by using the reverse triangle inequality, we obtain

$$\begin{aligned}\|x_{\sigma,p}(T^-)\| &\leq \|x_{\mathrm{av}}(T)\| + \frac{1}{8}\|x_{\mathrm{av}}(T/2)\| \\ &\leq \frac{1}{2}\|x_{\mathrm{av}}(T/2)\| + \frac{1}{8}\|x_{\mathrm{av}}(T/2)\|,\end{aligned} \tag{31}$$

and (30) together with the reverse triangle inequality, implies

$$\|x_{\sigma,p}(T/2^-)\| \geq \|x_{\mathrm{av}}(T/2)\| - \frac{1}{8}\|x_{\mathrm{av}}(T/2)\|. \tag{32}$$

Altogether, we arrive at

$$\|x_{\sigma,p}(T^-)\| \leq \frac{5}{7}\|x_{\sigma,p}(T/2^-)\|, \tag{33}$$

i.e. we have shown that for all initial conditions there is a reduction of at least 5/7 of the norm of the state on a time interval of length $T/2$ and for all sufficiently small switching periods p. Without restriction, we can choose a $p^* = T/(2\theta)$ for sufficiently large $\theta \in \mathbb{N}$. Consider the solution of (27) as a concatenation of transition matrices defined as

$$\Phi_{p^*,i} := e^{A_i^{\mathrm{diff}} d_i p^*}\Pi_i,$$

then let us introduce for $t_1 > t_0 \geq 0$ the state transition matrix $\Phi_{\sigma,p}^{t_0^- \to t_1^-} \in \mathbb{R}^{n\times n}$ which maps any (possibly inconsistent) initial value $x_0 \in \mathbb{R}^n$ at t_0^- to the value of $x(t_1^-)$, in particular,

$$x_{\sigma,p^*}(t_1^+) = \Phi_{\sigma,p^*}^{t_0\to t_1} x_{\sigma,p^*}(t_0^-),$$

for all solutions of (27) and all $t_1 > t_0 \geq 0$. From (33) it follows that

$$\left\|\Phi_{\sigma,p^*}^{T/2\to T}\right\| \leq 5/7.$$

From $T/2 = \theta p^*$ for $\theta \in \mathbb{N}$ and the periodicity of the switching signal it follows that

$$\Phi_{\sigma,p^*}^{kT/2\to(k+1)T/2} = \Phi_{\sigma,p^*}^{T/2\to T} \quad \forall k \in \mathbb{N}\setminus\{0\},$$

in particular, by considering that $T/2$ is a multiple of the switching period p^*,

$$x_{\sigma,p^*}(kT/2^-) = (\Phi_{\sigma,p^*}^{T/2\to T})^{k-1}\Phi_{\sigma,p^*}^{0\to T/2}x_0,$$

and hence

$$\|x_{\sigma,p^*}(kT/2^-)\| \leq \left(\frac{5}{7}\right)^{k-1}\left\|\Phi_{\sigma,p^*}^{0\to T/2}\right\|\|x_0\|. \tag{34}$$

From (30), by applying the reverse triangle inequality we have that

$$\|\Phi_{\sigma,p}^{0\to T/2}\|\|x_0\| \leq \|x_{\mathrm{av}}(T/2)\| + \alpha p$$

with a suitable constant $\alpha > 0$. Hence, considering (1) we can conclude that

$$\Phi_{\sigma,p}^{0\to T/2} = e^{A_{\mathrm{av}}T/2} + \mathrm{O}(p) = \mathrm{O}(1). \tag{35}$$

Furthermore, for $\tau \in (0, T/2)$ we have that

$$x_{\sigma,p^*}(kT/2 + \tau^-) = \Phi_{\sigma,p^*}^{T/2\to T/2+\tau}x_{\sigma,p^*}(kT/2^-), \tag{36}$$

where

$$\Phi_{\sigma,p}^{T/2\to T/2+\tau} = e^{A_{\mathrm{av}}\tau} + \mathrm{O}(p) = \mathrm{O}(1). \tag{37}$$

Considering the time instant $t = kT/2 + \tau$ and combining (34) with (36), we have that

$$\|x_{\sigma,p^*}(t^-)\| \leq \left(\frac{5}{7}\right)^{k-1}\left\|\Phi_{\sigma,p^*}^{T/2\to T/2+\tau}\right\|\left\|\Phi_{\sigma,p^*}^{0\to T/2}\right\|\|x_0\|.$$

By considering (35), (37), $k = 2(t-\tau)/T$ and $\mu = \left(\frac{5}{7}\right)^2$ where $\mu \in (0,1)$, we have that for sufficiently small $p^* = T/(2\theta)$, there exists constant $C > 0$ such that $\forall\, t > 0$

$$\|x_{\sigma,p^*}(t^-)\| \leq C\mu^{\lfloor t/T\rfloor}\|x_0\|,$$

which implies exponential stability of the switched system (27). □

Note that *exponential* (or equivalently *asymptotic*) stability of the averaged system (21) is a crucial assumption in Theorem 27; mere stability of the averaged system is not sufficient to conclude stability of the switched system as the following switched ODE example shows.

Example 28. *Consider a simple switched DAE given by*

$$E_1 = I, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$
$$E_2 = I, \quad A_2 = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}.$$

For a duty cycle $d_1 = d_2 = 0.5$ the dynamic averaged matrix is a zero-matrix, i.e. $A_{\mathrm{av}} = 0$, then the averaged model is stable but not asymptotically. The solution of the switched system is given by

$$x_1(t_k) = x_{10}$$
$$x_2(t_k) = x_{20} + 2k(e^{p/2} - 1)x_{10}.$$

Then

$$|x_2(t_k) - x_{2,\mathrm{av}}(t_k)| = 2k(e^{p/2} - 1)x_{10}.$$

For fixed p and growing k this difference between the second state variable of the switched and the averaged models grows unbounded, hence the switched system is not stable.

Example 28 shows that the stability of the averaged model is not sufficient for having a bound of the error between the switched and the averaged states when $t \to \infty$ (see Remark 19 and the consideration reported below it).

5. Partial averaging

The averaging result in Theorem 23 allows to approximate a switched DAE by means of a smooth averaged system. If (PA) is not satisfied, it might be possible to partition the state variable such that the averaging result holds only for a part of the state. The resulting partial averaged model is still a switched system but simpler than the original one.

Therefore, we assume that the state space is partitioned in a suitable way and that the corresponding consistency projectors and flow matrices have the following structure:

$$A_i^{\mathrm{diff}} = \begin{bmatrix} A_{11,i}^{\mathrm{diff}} & 0 \\ A_{21,i}^{\mathrm{diff}} & A_{22,i}^{\mathrm{diff}} \end{bmatrix}, \; B_i^{\mathrm{diff}} = \begin{bmatrix} B_{1,i}^{\mathrm{diff}} \\ B_{2,i}^{\mathrm{diff}} \end{bmatrix}, \; \Pi_i = \begin{bmatrix} \Pi_{11,i} & 0 \\ \Pi_{21,i} & \Pi_{22,i} \end{bmatrix}, \tag{38}$$

where $A_{11,i}, \Pi_{11,i} \in \mathbb{R}^{\alpha\times\alpha}$ with $0 < \alpha < n$ being independent of mode $i \in \Sigma$. We furthermore assume that convergence towards an averaged system occurs in the first part of the state space, in view of Theorem 12 and Remark 20 it suffices to assume that (16) holds and that (PA) holds for the projectors $\Pi_{11,i}$. We then propose the following partial averaged system:

$$\dot{x}_{\mathrm{pav}}(t) = A_{\mathrm{pav},i}^{\mathrm{diff}} x_{\mathrm{pav}}(t) + B_{\mathrm{pav},i}^{\mathrm{diff}} u(t), \; t \in (s_{k,i}, s_{k,i+1}) \tag{39a}$$
$$x_{\mathrm{pav}}(s_{k,i}^+) = \Pi_i^* x_{\mathrm{pav}}(s_{k,i}^-), \tag{39b}$$
$$x_{\mathrm{pav}}(0-) = \Pi_\cap^* x_0, \tag{39c}$$

with switching times $s_{k,i}$ as in (7) and where

$$A_{\mathrm{pav}_i}^{\mathrm{diff}} := \begin{bmatrix} A_{\mathrm{pav}} & 0 \\ A_{21,i}^{\mathrm{diff}} & A_{22,i}^{\mathrm{diff}} \end{bmatrix}, \qquad B_{\mathrm{pav},i}^{\mathrm{diff}} := \begin{bmatrix} B_{\mathrm{pav}} \\ B_{2,i}^{\mathrm{diff}} \end{bmatrix},$$
$$\Pi_i^* := \begin{bmatrix} I_\alpha & 0 \\ \Pi_{21_i} & \Pi_{22,i} \end{bmatrix}, \qquad \Pi_\cap^* := \begin{bmatrix} \Pi_{11,\cap} & 0 \\ \Pi_{21,i} & \Pi_{22,i} \end{bmatrix}, \tag{40}$$

with

$$A_{\mathrm{pav}} := \Pi_{11,\cap} \sum_{i=1}^{\mathfrak{q}} d_i A_{11,i}^{\mathrm{diff}} \Pi_{11,\cap}, \; B_{\mathrm{pav}} := \Pi_{11,\cap} \sum_{i=1}^{\mathfrak{q}} d_i B_{1,i}^{\mathrm{diff}}, \tag{41}$$

and $\Pi_{11,\cap} := \prod_{i=1}^{\mathfrak{q}} \Pi_{11_i}$.

The special structure of the switched DAE (and the corresponding switched ODE with jumps) implies that the first part of the state can be viewed as an input to the second part of the state. Hence one would expect that the second state components behave similar for the switched DAE and for partial averaged system, because they are both "driven" by similar inputs (at least for small periods p). However, this intuition is not true in general as the following example shows.

Example 29. *Consider the following matrix pairs (E_i, A_i) with $i = 1, 2$*

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{4} & 3 \\ 3 & -\frac{1}{4} & 1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 7 & 0 \\ -3 & 1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 5 & -2 & 0 \\ 6 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix}.$$

The corresponding flow-matrices and the consistency projectors have a triangular structure (38) *with $\alpha = 1$. In particular, the projectors $\Pi_{11,1}$ and $\Pi_{11,2}$ commute, hence (PA) is satisfied.*

Figure 5 illustrates the solution behavior of the switched DAE in comparison to the partial averaged system. As expected due the assumptions made, the first state variable converges to the smooth part of the partial averaged system. Also the second state variable seems to converge to the discontinues solution of the partial averaged system. However, the third variable does not converge. As highlighted in Figure 5 the absolute distance does not decrease for a decreasing switching period. Even worse, the third state variable grows unbounded for $p \to 0$. This phenomena is due the fact that the set of consistency projectors is not product bounded, cf. Trenn and Wirth (2012).

The previous example indicates that some further assumptions are necessary. The following main result on partial averaging provides sufficient conditions for convergence.

Theorem 30. *Consider the regular switched DAE* (13) *with periodic switching signal σ with period $p > 0$ given by* (6) *and initial condition $x(0^-) = x_0$. Assume that the following conditions hold.*

(i) *The matrix pairs (E_i, A_i) are regular and* (16) *holds $\forall i \in \Sigma$.*
(ii) *The corresponding consistency projectors Π_i, flow matrices A_i^{diff} and B_i^{diff} are in the form of* (38).
(iii) $\forall i \in \Sigma$

$$\operatorname{im} \Pi_{11,\cap} \subseteq \operatorname{im} \Pi_{11,i}, \quad \ker \Pi_{11,\cap} \supseteq \ker \Pi_{11,i}. \tag{42a}$$

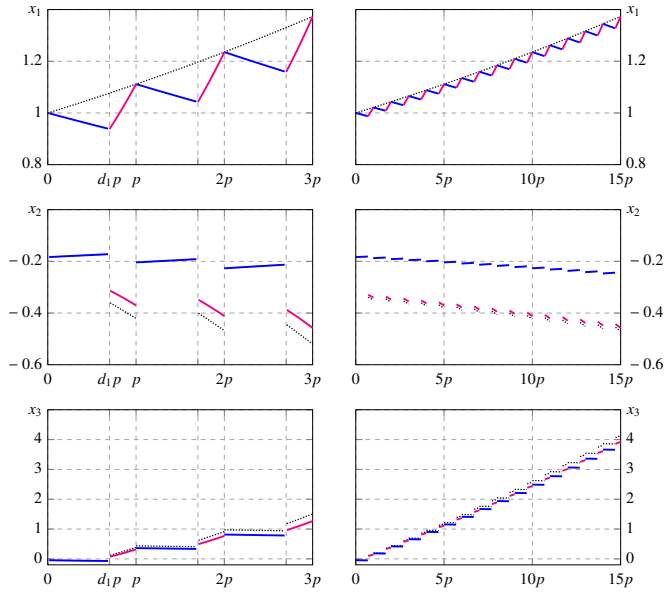

Figure 5: Evolution of the state variables (first component top, second component middle, third component bottom) of Example 29 for slow switching ($p = 0.1s$, left) and fast switching ($p = 0.02s$, right). The averaging dynamics are plotted with dotted black lines, while the trajectories of the switched DAE are colored according to the active mode (mode 1 blue, mode 2 magenta).

(iv) *The matrix* $\prod_i^{\mathrm{q}} \Pi_{22,i}$ *is a projector.*

(v) $\Pi_{22,i}\Pi_{21,i-1} = 0,\ \forall i \in \Sigma$ *with* $\Pi_{21,0} := \Pi_{21,\mathrm{q}}$.

Denote by $x_{\sigma,p}(t)$ *the (in general discontinuous) impulse-free part of the (in general distributional) solution of* (13) *and let* $x_{\mathrm{pav}}(t)$ *be the solution of the switched partial averaged system* (39). *Then for any* $\Delta > 0$

$$x_{\sigma,p}(t) - x_{\mathrm{pav}}(t) = \mathrm{O}(p), \tag{43}$$

uniformly for all $t \in [p, \Delta]$.

Proof. By decomposing $x_{\mathrm{pav}}(t) = [z_{\mathrm{pav}}(t)\ y_{\mathrm{pav}}(t)]^\top$ and $x_{\sigma,p}(t) = [z_\alpha(t)\ y(t)]^\top$, we can define the error variables $w_y = y - y_{\mathrm{pav}}$ and $w_z = z - z_{\mathrm{pav}}$ and can consider the corresponding error dynamics which are given by a switched ODE with jumps. Then the proof is a straightforward combination of the Remark 20 extended to the case of non homogeneous systems and Lemma 8. □

We conclude by discussion a variation of Example 24.

Example 31. *Consider again the electrical circuit of Figure 3 where, for simplicity, the inductor is replaced by a short circuit. In contrast to Example 24, we now consider as state variables the currents and the voltages of the two capacitors, i.e.* $x = [v_{C_1}, v_{C_2}, i_{C_1}, i_{C_2}]^\top$. *However, with this choice of variables,* $\Pi_i^{\mathrm{imp}} B_i \neq 0$ *for* $i = 1, 2$, *i.e. it is not possible to express the solutions of the switched DAE by solutions of a switched ODE with jumps. Nevertheless, it is possible to apply the (partial) averaging result by assuming that the input is constant. Because then we can reinterpret the input as a state variable with governing equation* $\dot{u} = 0$. *This results in the new state* $x = [v_{C_1}, v_{C_2}, u, i_{C_1}, i_{C_2}]^\top$ *and the following matrices:*

$$E_i = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad i = 1, \dots, 4,$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & R_1 & 0 \\ 0 & 1 & 0 & 0 & -R_2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & \frac{R_1}{R_2} & -1 & R_1 & R_1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 1 & 1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -R_2 \end{bmatrix},$$

and $B_i = 0$, $i = 1, \dots, 4$. *Consider the constants* ρ_1 *and* ρ_2 *defined in Example 24 and let* $\rho_3 := \frac{\rho_1\rho_2(R_1+R_2)}{R_1R_2}$, *then the consistency projectors are*

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{R_1} & 0 & \frac{1}{R_1} & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 & 0 \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} \rho_1 & \rho_2 & 0 & 0 & 0 \\ \rho_1 & \rho_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{\rho_3 C_1}{C_2} & -\rho_3 & \frac{\rho_1}{R_1} & 0 & 0 \\ -\rho_3 & -\frac{\rho_3 C_2}{C_1} & \frac{\rho_2}{R_1} & 0 & 0 \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} \rho_1 & \rho_2 & 0 & 0 & 0 \\ \rho_1 & \rho_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{\rho_1^2}{R_2} & -\frac{\rho_1\rho_2}{R_2} & 0 & 0 & 0 \\ -\frac{\rho_1\rho_2}{R_2} & -\frac{\rho_2^2}{R_2} & 0 & 0 & 0 \end{bmatrix}, \qquad \Pi_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 & 0 \end{bmatrix}.$$

We can see that (PA) is not satisfied and an averaging result as stated in Theorem 17 does not hold. However, the consistency projectors (as well as the A^{diff}*-matrices) can be partitioned according to* (38) *with* $\alpha = 3$ *and it can be verified that all assumptions of Theorem 30 are satisfied, i.e. convergence towards the partial averaged system* (39) *is guaranteed. Simulations for duty cycles* $(d_1, d_2, d_3, d_4) = (0.3, 0.4, 0.2, 0.1)$, *initial value* $x_0 = (1,\ 1,\ 5,\ 0,\ 0)^\top$ *and the same physical parameters as in Example 24 (apart from L) are shown in Figure 6.*

6. Conclusion

In this paper we have analyzed the averaging technique applied to switched linear DAEs; an averaged model has been formulated, for which convergence of solutions is shown. In Theorem 17 and Theorem 23 the averaging result is obtained by making assumptions on the image and on the kernel of the consistency projectors. If the averaged model is exponentially stable this averaging result is utilized to conclude that exists a periodic switching signal such that the switched DAE is s exponentially stable.

We also considered the case in which the state variables present jumps that are independent from the switching period. This state variables cannot be represented in a continuous way but we can still use an averaged model for the remaining state variables.

Our averaging results for non homogeneous switched DAEs are based on the analysis of an equivalent switched ODE with jumps (Theorem 12). This equivalence is only valid for some structural assumptions on the B-matrices and for the full averaging result this assumptions seem justified. However, for the

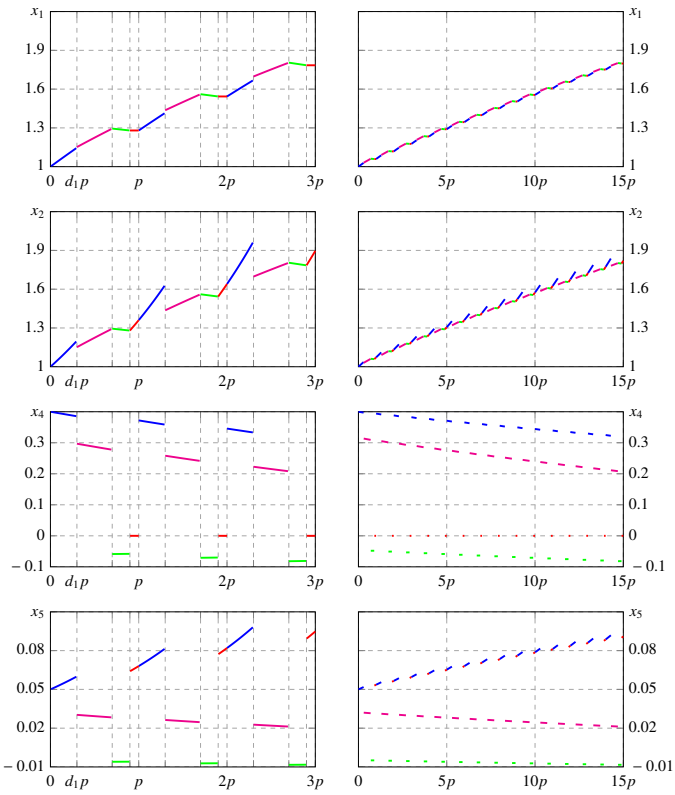

Figure 6: Evolution of the state variables (first component top, fifth component bottom) of Example 31 for slow switching ($p = 0.1s$, left) and fast switching ($p = 0.02s$, right). The trajectories of the switched DAE are colored according to the active mode (mode 1 blue, mode 2 magenta, mode 3 green, mode 4 red).

partial averaging result, Example 31 indicates that the equivalence to a switched ODE with jumps is too restrictive. Finding less restrictive assumptions which ensure a partial averaging model is still an open question.

It seems that Theorem 17, Theorem 23 and Theorem 30, may be extended to switching signals with non-constant duty cycles d_i, $i \in \Sigma$. This would result in a time-dependent averaged model, in analogy with the result of the averaging theory for switched ODE, see Pedicini et al. (2012b); this is a topic of future research.

Appendix

Proof of Lemma 7.

The solution of the switched ODE on the interval $(s_{k-1,\mathsf{q}}, t_k)$ evaluated at t_k^- is given by

$$w(t_k^-) = e^{A_\mathsf{q} d_\mathsf{q} p} w(s_{k-1,\mathsf{q}}^+) + \int_{s_{k-1,\mathsf{q}}}^{t_k} e^{A_\mathsf{q}(t_k-\xi)} B_\mathsf{q} u(\xi) \mathrm{d}\xi. \tag{44}$$

Furthermore

$$w(s_{k-1,\mathsf{q}}^+) = \Pi_\mathsf{q} w(s_{k-1,\mathsf{q}}^-) + Q_\mathsf{q} v_{s_{k-1,\mathsf{q}}}, \tag{45}$$

where $w(s_{k-1,\mathsf{q}}^-)$ is the solution on the interval $(s_{k-1,\mathsf{q}-1}, s_{k-1,\mathsf{q}})$ evaluated at $s_{k-1,\mathsf{q}}^-$.

Substituting the solution $w(s_{k-1,\mathsf{q}}^-)$ in (45) and then in (44), and by iterating for all q modes one obtains the linear discrete time system

$$w(t_k^-) = H(p) w(t_{k-1}^-) + N(p) v_{k-1} + \mathcal{I}(p)\{u_{k-1}\}, \tag{46}$$

with solution (12), where

$$H(p) = \prod_{i=1}^{\mathsf{q}} e^{A_i d_i p} \Pi_i \tag{47a}$$

$$\mathcal{I}(p)\{u_{k-1}\} = \sum_{i=1}^{\mathsf{q}} \prod_{j=i+1}^{\mathsf{q}} e^{A_j d_j p} \Pi_j \int_{c_{i-1}}^{c_i} e^{A_i(c_i-\xi)} B_i u_{k-1}(\xi) \mathrm{d}\xi \tag{47b}$$

$$N(p) = \Big[\prod_{i=1}^{\mathsf{q}} (e^{A_i d_i p} \Pi_i) Q_1 \ \prod_{i=2}^{\mathsf{q}} (e^{A_i d_i p} \Pi_i) Q_2 \ \dots e^{A_\mathsf{q} d_\mathsf{q} p} Q_\mathsf{q} \Big], \tag{47c}$$

and c_i are given by (8) with $i \in \Sigma$. □

Proof of Lemma 8.

The solution of (10) is given by (12) where $w_0 = 0$. Let $\ell(p)$ be the number of consecutive periods of length p inside $[0, \Delta]$, i.e., $\Delta - p < p\ell(p) \le \Delta$. Note that $p\ell(p) = \mathrm{O}(1)$. Taking into account that $\Pi_1^2 = \Pi_1$, the expression (12) can be rewritten as

$$w(t_k^-) = \sum_{i=0}^{k-2} H(p)^{k-1-i} (\Pi_1 N(p) v_i + \Pi_1 \mathcal{I}(p)\{u_i\}) + N(p) v_{k-1} + \mathcal{I}(p)\{u_{k-1}\},$$

for $k = 2, \dots, \ell(p)$ and the same expression without the first sum for $k = 1$. By using (1) in (47a) and (47c) we obtain

$$H(p) = \Pi_\cap + \mathrm{O}(p),$$
$$N(p) = \Big[\Pi_\mathsf{q} \Pi_{\mathsf{q}-1} \cdots \Pi_2 Q_1 + \mathrm{O}(p) \quad \dots \quad Q_\mathsf{q} + \mathrm{O}(p) \Big],$$

and by invoking the assumption (iv), $N(p) = \mathrm{O}(p)$. Furthermore, invoking (iii) and (4),

$$H(p)^{k-1} = \mathrm{O}(1), \quad k = 1, \dots, \ell(p).$$

Finally, taking into account the general bound $\| \int_a^b f \| \le (b-a) \|f\|_\infty$ and using (1) in (47b), it follows

$$\mathcal{I}(p)\{u_i\} = \mathrm{O}(p) \|u_i\|_\infty = \mathrm{O}(p^2), \quad i = 0, \dots, \ell(p) - 1,$$

where we also used (i).
Hence it follows, together with assumptions (i) and (ii),

$$w(t_k^-) = (k-1)\mathrm{O}(p^2) + \mathrm{O}(p^2),$$

for $k = 1, \dots, \ell(p)$. Since $\ell(p)\mathrm{O}(p^2) = \mathrm{O}(p)$ from the equation above we obtain

$$w(t_k^-) = \mathrm{O}(p) \text{ as well as } w(t_k^+) = \mathrm{O}(p), \tag{48}$$

for $k = 1, \dots, \ell(p)$. It remains to be shown that $w(t) = \mathrm{O}(p)$ for $t \in (t_k, t_{k+1})$ with $k = 1, \dots, \ell(p)$. The solution of (10) for any $\tau \in [s_{k,i}, s_{k,i+1})$ and for any $i \in \Sigma$ can be written as follows

$$w(\tau) = e^{A_i(\tau - s_{k,i})} w(s_{k,i}^+) + \int_{s_{k,i}}^{\tau} e^{A_i(\tau-\xi)} B_i u(\xi) \mathrm{d}\xi.$$

Considering the Taylor expression (1) with $s = \tau - s_{k,i}$ and by applying (i) we have

$$w(\tau) = (I + \mathrm{O}(p))w(s_{k,i}^+) + \mathrm{O}(p^2) = w(s_{k,i}^+) + \mathrm{O}(p). \quad (49)$$

By concatenating (49) for increasing values of $i \in \Sigma$ and by using (48) together with

$$w(s_{k,i}^+) = \Pi_i w(s_{k,i}^-) + Q_i v_{s_{k,i}} = \Pi_i w(s_{k,i}^-) + \mathrm{O}(p),$$

$\forall i \in \Sigma$; we obtain that $w(\tau) = \mathrm{O}(p)$ $\forall \tau \in [t_k, t_{k+1})$ and $k = 1, \ldots, \ell(p)$, which completes the proof. □

Proof of Theorem 17

The proof proceeds in three steps.

Step 1: We show that (23) holds for $t = t_1 = p$.

Invoking Remark 13, the impulse-free part of the solution of (13) and the solution of (21) at t_1 can be written as

$$x_{\sigma,p}(t_1^+) = \Pi_1 x_{\sigma,p}(t_1^-) = \Pi_1 H_{\mathrm{diff}}(p) x_0$$
$$x_{\mathrm{av}}(t_1) = H_{\mathrm{av}}(p)\Pi_\cap x_0,$$

where $H_{\mathrm{av}}(p) = e^{A_{\mathrm{av}}p}$.

By taking into account the Taylor approximation (1), we have

$$H_{\mathrm{diff}}(p) = \Pi_\cap + \tilde{A}p + \mathrm{O}(p^2) = \Pi_\cap + \mathrm{O}(p) \quad (50a)$$
$$H_{\mathrm{av}}(p) = I + A_{\mathrm{av}}p + \mathrm{O}(p^2) = I + \mathrm{O}(p), \quad (50b)$$

where

$$\tilde{A} := A_{\mathrm{q}}^{\mathrm{diff}}\Pi_\cap d_{\mathrm{q}} + \Pi_{\mathrm{q}} A_{\mathrm{q\text{-}1}}^{\mathrm{diff}} \Pi_{\mathrm{q\text{-}1}} \cdots \Pi_1 d_{\mathrm{q}-1} + \ldots$$
$$+ \Pi_{\mathrm{q}}\Pi_{\mathrm{q\text{-}1}} \cdots \Pi_2 A_2^{\mathrm{diff}} \Pi_1 d_2 + \Pi_\cap A_1^{\mathrm{diff}} d_1.$$

Then

$$x_{\sigma,p}(t_1^+) - x_{\mathrm{av}}(t_1) = (\Pi_1(\Pi_\cap + \mathrm{O}(p)) - (I + \mathrm{O}(p))\Pi_\cap)x_0$$
$$= (\Pi_1\Pi_\cap - \Pi_\cap)x_0 + \mathrm{O}(p) = \mathrm{O}(p). \quad (51)$$

where we used $\Pi_1\Pi_\cap x_0 = \Pi_\cap x_0$ because of (PA.1).

Step 2: We show that (23) holds for time instants multiples of the the period p, i.e. for any $\{t_k\}_{k=2}^{\ell(p)}$ where $\ell(p)$ is the integer such that $\Delta - p < p\ell(p) \le \Delta$. Clearly $p\ell(p) = \mathrm{O}(1)$.
By applying the Taylor approximation (1) to the solution of the impulse-free part of the switched system, we have

$$x_{\sigma,p}(t_k^+) = \Pi_1 H_{\mathrm{diff}}^k(p) x_0 = (\Pi_\cap + \tilde{A}p + \mathrm{O}(p^2))^k x_0$$
$$= (\Pi_\cap + \mathrm{O}(p))(\Pi_\cap + \tilde{A}p + \mathrm{O}(p^2))^{k-2}(\Pi_\cap + \mathrm{O}(p))x_0$$

for $k = 2, \ldots, \ell(p)$. Taking into account (PA.1) together with Lemma 2, and by applying (4) we obtain

$$x_{\sigma,p}(t_k^+) = \Pi_\cap(\Pi_\cap + \tilde{A}p + \mathrm{O}(p^2))^{k-2}\Pi_\cap x_0 + \mathrm{O}(p).$$

Invoking (1) and (4) we can express the solution of the averaged system as

$$x_{\mathrm{av}}(t_k) = H_{\mathrm{av}}(p)^k \Pi_\cap x_0$$
$$= \Pi_\cap(\Pi_\cap + A_{\mathrm{av}}p + \mathrm{O}(p^2))^{k-2}\Pi_\cap x_0 + \mathrm{O}(p).$$

Hence, invoking $\Pi_\cap \tilde{A}\Pi_\cap = \Pi_\cap A_{\mathrm{av}}\Pi_\cap$ and (5), we arrive at

$$x_{\sigma,p}(t_k^+) - x_{\mathrm{av}}(t_k) = \Pi_\cap\Big((\Pi_\cap + \tilde{A}p + \mathrm{O}(p^2))^{k-2}$$
$$-(\Pi_\cap + A_{\mathrm{av}}p + \mathrm{O}(p^2))^{k-2}\Big)\Pi_\cap x_0 + \mathrm{O}(p) = \mathrm{O}(p), \quad (52)$$

for $k = 2, \ldots, \ell(p)$.

Step 3: We show that (23) holds for time instants different from multiples of the period p.

The solution of (13) and (21) for any $\tau \in [s_{k,i}, s_{k,i+1})$ with $i \in \Sigma$ and $k \in \mathbb{N}$, can be written respectively as

$$x_{\sigma,p}(\tau) = e^{A_i^{\mathrm{diff}}(\tau - s_{k,i})} x_{\sigma,p}(s_{k,i}^+) \quad (53a)$$
$$x_{\mathrm{av}}(\tau) = e^{A_{\mathrm{av}}(\tau - s_{k,i})} x_{\mathrm{av}}(s_{k,i}). \quad (53b)$$

Considering (1) with $s = \tau - s_{k,i}$ we have

$$x_{\sigma,p}(\tau) - x_{\mathrm{av}}(\tau) = x_{\sigma,p}(s_{k,i}^+) - x_{\mathrm{av}}(s_{k,i}) + \mathrm{O}(p). \quad (54)$$

Taking into account Remark 15 we can write

$$x_{\sigma,p}(s_{k,i}^+) - x_{\mathrm{av}}(s_{k,i}) = \Pi_i x_{\sigma,p}(s_{k,i}^-) - x_{\mathrm{av}}(s_{k,i})$$
$$= \Pi_i(x_{\sigma,p}(s_{k,i}^-) - x_{\mathrm{av}}(s_{k,i})). \quad (55)$$

Then by concatenating (54) for increasing values of $i \in \Sigma$ and $k = 1, \ldots, \ell(p)$, and by using (51) and (52) it follows that (23) holds $\forall t \in [p, \Delta]$. □

The proof of Theorem 23 is based on the following Lemma.

Lemma 32. *Consider the operator $\mathcal{I}_{\mathrm{diff}}(p)$ given in* (20b) *and let*

$$\mathcal{I}_{\mathrm{av}}(p)\{u\} := \int_0^p e^{A_{\mathrm{av}}(p-\xi)} B_{\mathrm{av}} u(\xi) \mathrm{d}\xi.$$

Assume that (PA.2) *holds and that $u : \mathbb{R}_+ \to \mathbb{R}^m$ is Lipschitz continuous. Then*

$$\Pi_\cap \mathcal{I}_{\mathrm{diff}}(p)\{u\} - \mathcal{I}_{\mathrm{av}}(p)\{u\} = \mathrm{O}(p^2), \quad (56)$$

Proof.

Applying the Taylor approximation of the exponential matrix (1) to $\mathcal{I}_{\mathrm{diff}}(p)$ and $\mathcal{I}_{\mathrm{av}}(p)$ we obtain

$$\mathcal{I}_{\mathrm{diff}}(p)\{u\} = \sum_{i=1}^{\mathrm{q}} \Bigg[(\Pi_{\mathrm{q}}\Pi_{\mathrm{q}-1} \cdots \Pi_{i+1} + \mathrm{O}(p)) \times$$
$$\times \int_{c_{i-1}}^{c_i} (I + \mathrm{O}(p)) B_i^{\mathrm{diff}} u(\xi) \mathrm{d}\xi\Bigg] \quad (57a)$$
$$\mathcal{I}_{\mathrm{av}}(p)\{u\} = \int_0^p (I + \mathrm{O}(p)) B_{\mathrm{av}} u(\xi) \mathrm{d}\xi, \quad (57b)$$

where we used that $\mathrm{O}(c_i - \xi)$ can be substituted by $\mathrm{O}(p)$ since $(c_i - \xi) \le p$ for all $i \in \Sigma$.
Furthermore taking into account that $\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = f(\alpha)$, with $\alpha \in [a, b]$ and (PA.2) we have

$$\Pi_\cap \mathcal{I}_{\mathrm{diff}}(p)\{u\} = \sum_{i=1}^{\mathrm{q}} \Pi_\cap B_i^{\mathrm{diff}} u(\alpha_i) d_i p + \mathrm{O}(p^2) \quad (58a)$$
$$\mathcal{I}_{\mathrm{av}}(p)\{u\} = B_{\mathrm{av}} u(\alpha_{\mathrm{q}+1}) p + \mathrm{O}(p^2), \quad (58b)$$

where $\alpha_i \in [c_{i-1}, c_i]$ and $\alpha_{\mathsf{q}+1} \in [0, p]$. Hence, considering

$$\begin{aligned}
&\Big\| \Big(\sum_{i=1}^{\mathsf{q}} \Pi_\cap d_i B_i^{\text{diff}} u(\alpha_i)\, p \Big) - B_{\text{av}} u(\alpha_{\mathsf{q}+1})\, p \Big\| \\
&\le \sum_{i=1}^{\mathsf{q}} \big\| \Pi_\cap d_i B_i^{\text{diff}} u(\alpha_i)\, p - \Pi_\cap d_i B_i^{\text{diff}} u(\alpha_{\mathsf{q}+1})\, p \big\| \\
&\le \sum_{i=1}^{\mathsf{q}} \big\| \Pi_\cap B_i^{\text{diff}} \big\|\, L \, \big\| \alpha_i - \alpha_{\mathsf{q}+1} \big\|\, d_i\, p \\
&\le \sum_{i=1}^{\mathsf{q}} \big\| \Pi_\cap B_i^{\text{diff}} \big\|\, L\, d_i\, p^2,
\end{aligned}$$

where $L > 0$ is the Lipschitz-constant of u. By combining the last inequality with (58) we obtain that (56) holds. □

Proof of Theorem 23

First recall, that the solutions of (13) are given by (19) and it is easily seen that the solutions of (21) are given by

$$x_{\text{av}}(t_k^-) = H_{\text{av}}(p)^k \Pi_\cap x_0 + \sum_{i=0}^{k-1} H_{\text{av}}(p)^{k-1-i} \mathcal{I}_{\text{av}}(p)\{u_i\},$$

where $H_{\text{av}}(p) = e^{A_{\text{av}} p}$, $\mathcal{I}_{\text{av}}(p)$ is given as in Lemma 32 and u_i is defined analogously as in Lemma 7. Due to assumption (PA) the averaging result (23) for the homogeneous part holds, hence

$$\forall k = 1, \ldots, \ell(p) : H_{\text{diff}}(p)^k x_0 - H_{\text{av}}(p)^k \Pi_\cap x_0 = \mathrm{O}(p). \quad (59)$$

By considering (50), taking into account (4) and (PA.1) and noting that $\mathcal{I}_{\text{diff}}(p)\{u_i\}$ and $\mathcal{I}_{\text{av}}(p)\{u_i\}$ are $\mathrm{O}(p)$ functions we obtain

$$\begin{aligned}
x_{\sigma,p}(t_k^+) - x_{\text{av}}(t_k) = \sum_{i=0}^{k-2} &(H_{\text{diff}}(p)^{k-1-i} \Pi_\cap \mathcal{I}_{\text{diff}}(p)\{u_i\} \\
&- H_{\text{av}}(p)^{k-1-i} \mathcal{I}_{\text{av}}(p)\{u_i\}) + \mathrm{O}(p). \quad (60)
\end{aligned}$$

For $j = 1, \ldots, \ell(p) - 1$ and some generic $u : [0, p] \to \mathbb{R}^m$ we have

$$\begin{aligned}
&H_{\text{diff}}(p)^j \Pi_\cap \mathcal{I}_{\text{diff}}(p)\{u\} - H_{\text{av}}(p)^j \mathcal{I}_{\text{av}}(p)\{u\} \\
&\quad = (H_{\text{diff}}(p)^j - H_{\text{av}}(p)^j \Pi_\cap) \Pi_\cap \mathcal{I}_{\text{diff}}(p)\{u\} \\
&\qquad + H_{\text{av}}(p)^j (\Pi_\cap \mathcal{I}_{\text{diff}}(p)\{u\} - \mathcal{I}_{\text{av}}(p)\{u\}) = \mathrm{O}(p^2),
\end{aligned}$$

where we used (59), Lemma 32 and (4). Plugging this into (60), we have that

$$x_{\sigma,p}(t_k^+) - x_{\text{av}}(t_k) = (k-2)\mathrm{O}(p^2) + \mathrm{O}(p) = \mathrm{O}(p) \ \forall \{t_k\}_{k=1}^{\ell(p)}.$$

Analogously as in Step 3 of the proof of Theorem 17 we can now conclude the proof. □

Proof of Proposition 25.

By using (57) and noting that the functions $\mathcal{I}_{\text{diff}}(p)\{u\}$ and $\mathcal{I}_{\text{av}}(p)\{u\}$ are $\mathrm{O}(p)$ we have

$$\begin{aligned}
&\Pi_\cap \mathcal{I}_{\text{diff}}(p)\{u\} - \mathcal{I}_{\text{av}}(p)\{u\} \\
&= \sum_{i=0}^{\mathsf{q}} \int_{c_i}^{c_{i+1}} \Pi_\cap B_i^{\text{diff}} u(\xi) \mathrm{d}\xi - \int_0^p B_{\text{av}} u(\xi) \mathrm{d}\xi + \mathrm{O}(p^2) \\
&= \sum_{i=0}^{\mathsf{q}} \int_{c_i}^{c_{i+1}} \Pi_\cap (B_i^{\text{diff}} - B_{\text{av}}) u(\xi) \mathrm{d}\xi + \mathrm{O}(p^2), \quad (61)
\end{aligned}$$

where $j = 1, \ldots, \ell(p) - 1$. Then considering that

$$\begin{aligned}
&\Pi_\cap \Big(B_i^{\text{diff}} - B_i^{\text{diff}} d_i - \sum_{h \ne i \in \Sigma} B_h^{\text{diff}} d_h \Big) \\
&\qquad = \Pi_\cap \sum_{h \ne i \in \Sigma} (B_i^{\text{diff}} - B_h^{\text{diff}}) d_h, \quad (62)
\end{aligned}$$

where we use $d_i = 1 - \sum_{h \ne i \in \Sigma} d_h$ with $i \in \Sigma$. By combining (61) and (62) with (60) and taking into account (23), the averaging result (25) holds for any $\{t_k\}_{k=1}^{\ell(p)}$. It is easy to prove that (25) also holds for all time instants different from multiples of p, hence the proof is complete. □

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