Chapter 0
Observability of switched linear systems
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Abstract Observability of switched linear systems has been well studied during the past decade and depending on the notion of observability, several criteria have appeared in the literature. The main difference in these approaches is how the switching signal is viewed: Is it a fixed and known function of time, is it an unknown external signal, is it the result of a discrete dynamical system (an automaton) or is it controlled and is therefore an input? We will focus on the recently introduced geometric characterization of observability which assumes knowledge of the switching signal. These geometric conditions depend on computing the exponential of the matrix and require the exact knowledge of switching times. To relieve the computational burden, some relaxed conditions that do not rely on the switching times are given; this also allows for a direct comparison of the different observability notions. Furthermore, the generalization of the geometric approach to linear switched differential-algebraic systems is possible and presented as well.

0.1 Introduction

In the context of dynamical systems, *observability* is a fundamental property that plays an important role in realization theory, state estimation, output feedback controller design, and even diagnosis and fault monitoring. Roughly speaking, observability concerns extracting information about the internal variables, called *states*, of the system using the external signals consisting of *output* measurements and the inputs. Hence (for a given input) observability is related to the study of the mapping

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from the set of state trajectories to the set of outputs, and in general a system is *observable* if this mapping is injective.

This chapter is concerned with observability of switched linear systems and we consider models given by ordinary differential equations (ODEs) with jumps or given by differential algebraic equations (DAEs).

Various notions for observability of switched systems have appeared in the literature. From the point of view of hybrid systems, the switching signal may be treated either as an unknown discrete state or as a known external signal. In the former case, observability relates to simultaneous recovery of the discrete and continuous state. Some results in this direction for continuous-time hybrid systems appear in [43, 11, 3, 14, 9, 21, 28, 17], and for discrete-time hybrid systems in [42, 2, 4, 13]. Also related is the problem of the reconstruction of the discrete mode without imposing conditions for recovery of the continuous state, and for references in this direction, see for example [10, 44, 32, 22, 12].

In this chapter we will treat the other case, i.e. we will view the discrete mode as a known switching signal. In this case, even though the individual subsystems are not observable, it is still possible to recover the state trajectory by appropriately processing the measured signals over a time interval that involves multiple switching instants. This phenomenon is of particular interest for switched systems or systems with state jumps as the notion of instantaneous observability and observability over an interval coincide for non-switched linear time invariant systems. This variant of the observability problem in switched systems has been studied most notably by [5, 48, 31, 26, 33] for switched linear systems, [19, 20] for linear impulsive systems, [27, 49, 30] for switched nonlinear systems, and [34, 35, 36] for switched linear differential-algebraic equations. In the work of [48, 31, 18, 26, 27, 25], the authors derive conditions in terms of system data under which there exists a switching signal that makes state trajectories distinguishable. In addition, in [31, 26, 27, 25] algorithms for observability reduction (i.e. transforming a state-space representation to an observable one while preserving input-output behavior) were proposed. Unfortunately, the observability concept used in the papers mentioned above does not guarantee existence of an observer. In contrast, the authors in [34, 35, 30, 36, 33] study the observability of the underlying system for a fixed switching signal and use the geometric conditions for designing observers as well.

In this chapter, we present different notions of observability depending on what role the switching signal plays in reconstruction of the state trajectories, i.e. whether the switching signal is fixed, or whether it is viewed as an input. We then derive geometric conditions for each of these notions and in the process draw connections with the existing work. The adopted approach has the advantage that it allows us to treat ordinary differential equations and differential algebraic equations with similar tools, although the later class of systems requires more sophisticated treatment because of a non-standard solution framework.

The outline of this chapter is as follows: We first present the system classes we consider and give different definitions of observability. After a detailed discussion of these definitions we first study the single switch case in Section 0.4; the switched ODE case and the switched DAE case are treated separately. The latter includes a short discussion about the implicit jump rule (the consistency projector) and the distributional solution framework. We illustrate the observability characterization for switched DAEs with a single switch by a detailed example. In Section 0.5, we build on the single switch results to obtain a characterization of observability for a general switching signal. Furthermore, we formulate necessary conditions and sufficient conditions for observability which do not depend on the specific switching times but only on the mode sequence. Finally, we present conditions for observability, where the switching signal is not fixed a priori and compare these results with the former observability conditions.

The following notational conventions are used within this chapter. For a function $f : \mathbb{R} \to \mathbb{R}$ we denote the restriction to an interval I by $f_I : \mathbb{R} \to \mathbb{R}$ given by $f_I(t) = f(t)$ if $t \in I$ and $f_I(t) = 0$, otherwise; the same notation is also used for distributions (see Appendix 0.6.3). If for a function $f : \mathbb{R} \to \mathbb{R}$ the right- and left-sided limit at some $t \in \mathbb{R}$ exist, we denote these by $f(t^+)$ and $f(t^-)$; furthermore, we assume in general right-continuity, i.e. $f(t^+) = f(t)$. For a matrix M we denote the null space as $\ker M$ and the linear space spanned by the columns of M by $\operatorname{im} M$. For two matrices M and N with the same number of columns, we denote by $[M/N]$ the matrix resulting from stacking M over N. For a switched system, $\mathscr{P}$ denotes the set of possible parameters and we assume that the switching signal $\sigma : \mathbb{R} \to \mathscr{P}$ is right-continuous and the left-sided limit $\sigma(t^-)$ exists for all $t \in \mathbb{R}$ (i.e. we exclude Zeno-behaviour). The switching times of σ are denoted by t_i, $i \in \mathbb{N}$ and the duration of the i-th mode is $\tau_i := t_{i+1} - t_i$.

0.2 System classes

We consider switched systems with possible jumps, i.e. we consider systems of the form

$$\begin{aligned}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), && t \notin \{\, t_i \mid i \in \mathbb{N} \,\}, && \text{(0.1a)}\\
x(t_i^+) &= G_{\sigma(t_i^+)}x(t_i^-) + H_{\sigma(t_i^+)}v_i, && i \in \mathbb{N}, && \text{(0.1b)}\\
y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), && t \in \mathbb{R}, && \text{(0.1c)}
\end{aligned}$$

where $\sigma : \mathbb{R} \to \mathscr{P} := \{0, 1, 2, \ldots, \mathtt{p}\}$, $\mathtt{p} \in \mathbb{N} \cup \{\infty\}$, is the switching signal with non-accumulating switching times $t_1 < t_2 < t_3 < \ldots$; $A_p, B_p, C_p, D_p, G_p, H_p$, $p \in \mathscr{P}$, are matrices of appropriate size describing the dynamics and the jump at each mode, $u : \mathbb{R} \to \mathbb{R}^{\mathtt{u}}$ is the input to the continuous dynamics, $v : \mathbb{N} \to \mathbb{R}^{\mathtt{v}}$ is the input to the discrete dynamics, $x : \mathbb{R} \to \mathbb{R}^{\mathtt{n}}$ is the state trajectory, and $y : \mathbb{R} \to \mathbb{R}^{\mathtt{y}}$ is the measured output. Note that we assume the state space dimension to be the same for all modes; allowing different state space dimensions as in, e.g., [26] is partly a topic for future research.

In the case that $G_p = I$ and $H_p = 0$ for all $p \in \mathscr{P}$ we call (0.1) a *classical* switched system. We will also study switched systems of the form

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \tag{0.2a}$$
$$y = C_\sigma x + D_\sigma u \tag{0.2b}$$

which we call *switched differential algebraic equations (switched DAEs)*. In this context we also call (0.1) a switched ordinary differential equation (switched ODE) with or without jumps. Note that although a switched DAE does not have an explicit jump rule, the solutions nevertheless have jumps in general (for details see Section 0.4.2.1). In general, one should be aware, that the solution theory of (0.2) is a bit more involved compared to the switched systems of the form (0.1) (see Section 0.4.2.2); nevertheless the observability properties of (0.2) and (0.1) are very much alike (see Section 0.5, where the main results hold in an identical form for both system classes).

Remark 0.2.1. Usually, in order to avoid notational inconveniences, it is assumed that the switching signal σ has infinitely many switching times. For classical switched systems or switched DAEs this is not a problem as arbitrarily many artificial switching times can be added where the modes do not change. However for systems of the form (0.1) this is not possible in general because any introduction of an additional switching time introduces a jump in the state even if the mode doesn't change. In particular, the usual semi-group property of switched systems does not hold any more. If $\mathscr{P}$ is finite, one could add for each $p \in \mathscr{P}$ one new mode $\mathtt{p}+1+p$ given by $A_{\mathtt{p}+1+p} = A_p, B_{\mathtt{p}+1+p} = B_p, C_{\mathtt{p}+1+p} = C_p, D_{\mathtt{p}+1+p} = D_p$ and $G_{\mathtt{p}+1+p} = I$, $H_{\mathtt{p}+1+p} = 0$. Then it is always possible to introduce arbitrarily many switching times; however, the new switched system now also allows trajectories which do *not* jump when a switch occurs and hence is not equivalent to the original system (0.1) (unless one makes restrictions on the allowed switching sequences). Another way out is to consider instead of $G_{\sigma(t_i)}$ jump maps $G_{\sigma(t_i-),\sigma(t_i)}$ which also depend on the mode before the switch. In this framework, one could simply set $G_{pp} = I$ and $G_{pq} = G_q$ for all $p \neq q$. Furthermore, this framework has also the advantage that different state space dimensions can be handled, but as mentioned above this is not in the scope of this chapter. $\triangleleft$

Finally, it should be noted that it makes a conceptionally important difference whether the time-dependent switching signal is seen as a (fixed, but arbitrary) part of the system description or is seen as an additional input to system (0.1). In the former case, (0.1) is a linear—albeit time-varying—system, whereas in the latter case the system is nonlinear. In particular, when one speaks of observability of (0.1) it is important to distinguish these two viewpoints clearly; resulting in the different notions of "observability for a specific switching signal" and "controlled observability" (see Definition 0.3.1 for precise meaning). In this chapter the focus is mainly on the first viewpoint.

0.3 Observability definitions

We first review the observability notions for non-switched linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx + Du. \tag{0.3}$$

The system (0.3) is called observable if, and only if, knowledge of the external signals (u, y) implies knowledge of the internal signal x. More precisely, if (x^1, u^1, y^1) and (x^2, u^2, y^2) are two solutions of (0.3) and $(u^1, y^1) = (u^2, y^2)$ then it must follow that $x^1 = x^2$. Note that the equality is global (in time), however it is well known that this is equivalent to local observability where one only considers an arbitrarily small interval (with nonempty interior). Furthermore, knowledge of $x(t_0)$ for some $t_0 \in \mathbb{R}$ implies knowledge of x on the whole time axis with fixed inputs, hence observability for (0.3) can often be reduced to the question: Is it possible to reconstruct the initial value $x(0)$ via the (local) knowledge of u and y?

It is easily seen that, due to linearity, observability of (0.3) does not depend on the input u. Hence the matrices B and D play no role and the question of observability can be further reduced to the question of zero-distinguishability, i.e. does $y \equiv 0$ imply $x \equiv 0$? Taking derivatives of the output it is easily seen that

$$y \equiv 0 \quad \Leftrightarrow \quad y^{(i)}(0) = 0 \;\; \forall i \in \mathbb{N} \quad \Leftrightarrow \quad x(0) \in \ker O_{(A,C)}$$

where $O_{(A,C)} := [C/CA/CA^2/\cdots/CA^{n-1}]$ is the Kalman observability matrix. In particular, (0.3) is observable if, and only if, $\ker O_{(A,C)} = \{0\}$.

When considering observability for switched systems it is first obvious that local and global observability need not coincide anymore. Furthermore, in the presence of jumps it is not true in general that knowledge of $x(0)$ implies knowledge of x on the whole time axis. Hence we arrive at different notions of observability:

Definition 0.3.1 (Observability). Consider the switched system (0.1) (resp. (0.2)).

- We call (0.1) (resp. (0.2)) *strongly observable* if, and only if, for all solutions $(\sigma^1, u^1, v^1, y^1, x^1), (\sigma^2, u^2, v^2, y^2, x^2)$ the following implication holds:

$$(\sigma^1, u^1, v^1, y^1) \equiv (\sigma^2, u^2, v^2, y^2) \quad \Rightarrow \quad x^1 \equiv x^2. \tag{0.4}$$

- We call (0.1) (resp. (0.2)) *observable for the switching signal* σ if, and only if, the implication (0.4) holds for all solutions with $\sigma \equiv \sigma^1 \equiv \sigma^2$.
- We call (0.1) (resp. (0.2)) *controlled observable* if, and only if, there exist (σ, u, v) such that for all solutions $(\sigma, u, v, y^1, x^1), (\sigma, u, v, y^2, x^2)$ the following implication holds

$$y^1 \equiv y^2 \quad \Rightarrow \quad x^1 \equiv x^2.$$

- We call (0.1) (resp. (0.2)) *weakly controlled observable* if, and only if, any two distinct initial states $x_0^1, x_0^2 \in \mathbb{R}^n$ are distinguishable for some σ, i.e. for all $x_0^1, x_0^2 \in \mathbb{R}^n$ there exists (σ, u, v) such that the corresponding solutions (σ, u, v, y^1, x^1) and (σ, u, v, y^2, x^2) with $x^1(0^-) = x_0^1$ and $x^2(0^-) = x_0^2$ satisfy

$$y^1 \not\equiv y^2.$$

- We call (0.1) (resp. (0.2)) *forward observable for the switching signal* σ if, and only if, for all solutions $(\sigma, u^1, v^1, y^1, x^1), (\sigma, u^2, v^2, y^2, x^2)$ there exists $T \in \mathbb{R}$ such that the following implication holds

$$(u^1, v^1, y^1) \equiv (u^2, v^2, y^2) \quad \Rightarrow \quad x^1{}_{(T,\infty)} \equiv x^2{}_{(T,\infty)}. \qquad \lhd$$

Remark 0.3.2. Some remarks on the different concepts of observability follow.

(i) Clearly, strong observability of (0.1) (resp. (0.2)) implies observability for each individual switching signal. In particular, it implies observability of each mode (just chose the constant switching signal). It is easily seen that observability of each mode also implies strong observability of the switched system. Hence we already obtain the following equivalence: (0.1) (resp. (0.2)) is strongly observable if, and only if, each mode is observable in the classical sense. For this reason strong observability is not such an interesting concept for switched systems and not considered here any further in detail.

(ii) Due to linearity, the observability notions do not depend on u and v (c.f. the forthcoming Proposition 0.3.3), hence observability for a particular switching signal implies controlled observability. On the other hand, controlled observability implies observability for the corresponding switching signal.

(iii) Controlled observability implies existence of a single switching signal which distinguishes any two initial values; whereas weakly controlled observability implies that for any pair of initial values there exists a switching signal (that may depend on these specific initial values) which distinguishes these initial values. Hence controlled observability implies weakly controlled observability, but the converse cannot be expected in general. However, for a classical switched system which is weakly controlled observable one can construct a single switching signal which is able to distinguish any two initial values, hence controlled observability and weakly controlled observability are equivalent in this case, see Theorem 0.5.9. For switched ODEs with jumps or for switched DAEs this issue is not resolved yet. To the best of our knowledge, the term controlled observability was first used in [18].

(iv) If any mode of the switched system is observable, then it follows that the switched system is controlled observable (just take the corresponding constant switching signal). The converse is not true, for counter examples see [24, 31].

(v) It doesn't make much sense to define forward observability without fixing the switching signal, since on the one hand the time T from which onwards the state can be reconstructed depends on σ. For example, switching just faster will in general make T arbitrarily small. On the other hand, forward observability for all switching signals implies forward observability of each mode. For non-switched systems forward observability is the same as observability. Since observability of each mode implies strong observability of the switched system (0.1) (resp. (0.2)) it follows that forward observability without fixing a switching signal and strong observability are identical.

(vi) If the jump matrices G_p in (0.1) are invertible for all $p \in \mathscr{P}$, then solutions can be extended uniquely also back in time, hence in this case forward observability and observability (for a fixed switching signal) are equivalent. $\triangleleft$

As already mentioned in the above remark, the notion of observability does not depend on the specific inputs u and v (but it does depend on the "input" σ), hence in the following we will only consider (0.1) with $B_p = 0$ and $H_p = 0$ for all $p \in \mathscr{P}$. Furthermore, also due to linearity it suffices to consider the output $y \equiv 0$ and check whether this output can be produced by a nonzero state. These observations are summarized in the following proposition.

Proposition 0.3.3 (Observability independent of inputs, c.f. [33]). *Consider the switched system* (0.1) *without inputs:*

$$\left.\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t), & & t \notin \{\, t_i \mid i \in \mathbb{N} \,\},\\ x(t_i) &= G_{\sigma(t_i)}x(t_i^-), & & i \in \mathbb{N},\\ y(t) &= C_{\sigma(t)}x(t), & & t \in \mathbb{R}, \end{aligned}\right\} \tag{0.5}$$

- (0.1) *is observable for the switching signal* σ *if, and only if, for all corresponding solutions* (x,y) *of* (0.5) *it holds that*
$$y \equiv 0 \quad \Rightarrow \quad x \equiv 0. \tag{0.6}$$
- (0.1) *is controlled observable if, and only if, there exists* σ *such that* (0.1) *is observable for* σ.
- (0.1) *is weakly controlled observable if, and only if, for every* $x_0 \in \mathbb{R}^n \setminus \{0\}$ *there exists a switching signal* σ *such that for the corresponding solution* (x,y) *of* (0.5) *with* $x(0^-) = x_0$ *it hold that*
$$y \not\equiv 0.$$
- (0.1) *is forward observable for the switching signal* σ *if, and only if, for all solutions* (x,y) *of* (0.5) *the following implication holds for some* $T \in \mathbb{R}$
$$y \equiv 0 \quad \Rightarrow \quad x_{(T,\infty)} \equiv 0.$$

Corresponding results concerning the switched DAE without an input:

$$E_\sigma \dot{x} = A_\sigma x, \quad y = C_\sigma x \tag{0.7}$$

hold as well (even when considering distributional inputs, c.f. [34, Prop. 7]).

From a practical point of view we only consider switching signals which are constant prior to some time (say $t = 0$); in fact, we already made this assumption implicitly when we denoted the switching times of σ by $t_1 < t_2 < t_3 < \dots$ earlier. Hence the trajectories of the switched system (0.1) or (0.5) on $(-\infty, 0)$ are similar to a system without switches. In particular, if the inputs σ, u and v are known, then $x(0^-)$ uniquely determines the whole trajectory x. Therefore, it make sense to define the unobservability space as follows.

Definition 0.3.4 (Unobservable space). Consider (0.5) (resp. (0.7)). For a switching signal σ being constant on $(-\infty,0)$, let $\mathscr{M}_{0^-}^{\sigma} \subseteq \mathbb{R}^n$ be such that for all solutions (x,y) of (0.5) it holds that

$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \mathscr{M}_{0^-}^{\sigma}. \qquad \triangleleft$$

Definition 0.3.5 (T-unobservable space). Consider (0.5) (resp. (0.7)). For some $T \geq 0$, let $\mathscr{N}_{T^+}^{\sigma} \subseteq \mathbb{R}^n$ be the smallest set for which the following implication holds for all solutions (x,y) of (0.5) (resp. (0.7))

$$y \equiv 0 \quad \Rightarrow \quad x(T^+) \in \mathscr{N}_{T^+}^{\sigma}. \qquad \triangleleft$$

It is easily seen, that $\mathscr{M}_{0-}^{\sigma}$ and $\mathscr{N}_{T^+}^{\sigma}$ are linear subspaces of $\mathbb{R}^n$. The different observability notions can now be characterized in terms of the unobservable spaces as follows:

Corollary 0.3.6 (Observability and the unobservable space). *Consider* (0.1) *(resp.* (0.7)*) and let*

$$\Sigma := \left\{ \sigma : \mathbb{R} \to \mathscr{P} \;\middle|\; \begin{array}{l} \sigma \textit{ is constant on } (-\infty,0) \textit{ and has} \\ \textit{finitely many switches in every finite interval} \end{array} \right\}$$

denote the space of feasible switching signals.

- (0.1) *(resp.* (0.7)*) is observable for* $\sigma \in \Sigma$ *if, and only if,* $\mathscr{M}_{0^-}^{\sigma} = \{0\}$.
- (0.1) *(resp.* (0.7)*) is controlled observable if, and only if, there exists* $\sigma \in \Sigma$ *such that* $\mathscr{M}_{0^-}^{\sigma} = \{0\}$.
- (0.1) *(resp.* (0.7)*) is weakly controlled observable if, and only if,* $\bigcap_{\sigma\in\Sigma} \mathscr{M}_{0^-}^{\sigma} = \{0\}$.
- (0.1) *(resp.* (0.7)*) is forward observable for* σ *if, and only if,* $\mathscr{N}_{T^+}^{\sigma} = \{0\}$ *for some* $T \geq 0$.

For non-switched ODEs (0.3) we have already established that the unobservable space $\mathscr{M}$ is given by

$$\mathscr{M} = \ker O_{(A,C)} = \ker[C/CA/\cdots/CA^{n-1}].$$

Since we have also established that (0.1) is strongly observable if, and only if, each mode is observable, we have the following characterization.

Corollary 0.3.7. *The switched system* (0.1) *is strongly observable if, and only if,* $\ker O_{(A_p,C_p)} = \{0\}$ *for all* $p \in \mathscr{P}$.

Characterizing the other observability notions is more complicated and a crucial step towards this characterization is the consideration of the simplest nontrivial switching signal first.

0.4 Characterization of observability: The single switch case

In this section we restrict ourselves to the following simple switching signal

$$\sigma_1 : \mathbb{R} \to \{0,1\}, \quad t \mapsto \begin{cases} 0, & t < 0, \\ 1, & t \geq 0, \end{cases} \tag{0.8}$$

i.e. we consider switched systems (0.1) with exactly one switch (which occurs at $t = 0$).

The characterization of observability with this switching signal cannot be used to characterize (weakly) controlled observability as even in the case of two modes more than one switch might be necessary to achieve observability (see the forthcoming Example 0.5.2). Note furthermore, that for the single switch case the observability characterization significantly differs between switched ODEs and switched DAEs; this is the reason these two cases are treated separately. However, when studying the case of general switching signals in the forthcoming Section 0.5 the approach can again be unified by introducing the notion of local unobservable spaces which are based on the analysis in this section.

0.4.1 Observability for the single-switch case for (0.1)

For notational convenience let $O_p := O_{(A_p,C_p)}$, $p \in \mathscr{P}$. It is clear that for all solutions (x,y) of the switched systems (0.5) with switching signal (0.8) the following two implications hold:

$$\begin{aligned} y_{(-\infty,0)} \equiv 0 \quad &\Leftrightarrow \quad x(0^-) \in \ker O_0, \\ y_{(0,\infty)} \equiv 0 \quad &\Leftrightarrow \quad x(0^+) \in \ker O_1. \end{aligned}$$

Since $x(0^+) = G_1 x(0^-)$ the above two equivalences become

$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \ker O_0 \cap G_1^{-1}(\ker O_1) = \ker O_0 \cap \ker O_1 G_1$$

Hence we have arrived at the following characterization of observability.

Lemma 0.4.1 (Observability for σ_1: ODE case). *Consider the switched system* (0.1) *with switching signal σ_1 given in* (0.8). *Then the unobservable space for* (0.5) *is*

$$\boxed{\mathscr{M}_{0^-}^{\sigma_1} = \ker O_0 \cap \ker O_1 G_1.}$$

In particular, (0.1) *is observable for σ_1 if, and only if,*

$$\ker O_0 \cap \ker O_1 G_1 = \{0\}. \tag{0.9}$$

Remark 0.4.2 (Order of switching relevant). If there is no jump at the switch, i.e. $G_1 = I$ then the switched system (0.1) with a single switch is observable if, and only if, the intersection of the individual unobservable subspaces is trivial. In this case the order of the switching sequence doesn't matter, c.f. [43]. In the presence of jumps, it is however important whether the system jumps from mode 0 to mode 1 or vice versa. $\triangleleft$

Since there is no switch after time zero, it is easily seen that, for $T \geq 0$, $\mathcal{N}^{\sigma_1}_{T^+} = e^{A_1 T}\mathcal{N}^{\sigma_1}_{0+}$, hence forward observability is characterized by $\mathcal{N}^{\sigma_1}_{0^+} = \{0\}$. Clearly (taking into account some basic facts from linear algebra, see Appendix 0.6.1)

$$y \equiv 0 \quad \Rightarrow \quad x(0^+) \in G_1\mathcal{M}^{\sigma_1}_{0^-} = G_1(\ker O_0 \cap \ker O_1 G_1) = G_1 \ker O_0 \cap \ker O_1,$$

hence

$$\mathcal{N}^{\sigma_1}_{0^+} \subseteq G_1 \ker O_0 \cap \ker O_1.$$

In fact, we have equality as for every $x_0^+ \in G_1 \ker O_0 \cap \ker O_1 = G_1\mathcal{M}^{\sigma_1}_{0^-}$ there exists at least one $x_0^- \in \mathcal{M}^{\sigma_1}_{0^-}$ with $x_0^+ = G_1 x_0^-$ which shows that $\mathcal{N}^{\sigma_1}_{0^+}$ cannot be chosen smaller. So we have arrived at the following characterization for forward observability for the single switch case.

Lemma 0.4.3 (Forward observability for σ_1: ODE case). *Consider the switched system* (0.1) *with switching signal* σ_1 *given in* (0.8) *and* $T \geq 0$. *Then the* T*-unobservable space for* (0.5) *is given by*

$$\boxed{\mathcal{N}^{\sigma_1}_{T^+} = e^{A_1 T}\mathcal{N}^{\sigma_1}_{0^+} = e^{A_1 T}(G_1 \ker O_0 \cap \ker O_1).}$$

In particular, (0.1) *is forward observable if, and only if,*

$$G_1 \ker O_0 \cap \ker O_1 = \{0\}. \tag{0.10}$$

0.4.2 The single switch result for switched DAEs

We would like to generalize the above results to switched DAEs (0.2). Due to Proposition 0.3.3 we can restrict our attention to the homogeneous switched DAE (0.7) and the observability question: Does an observed zero output implies a zero initial state?

Before continuing the discussion on observability of switched DAEs we have to first clarify what we mean by a solution of (0.2) or (0.7). Since each mode is given by a DAE the description of the mode itself contains algebraic constraints, enforcing the solutions to evolve within a certain subspace of $\mathbb{R}^n$. In particular, not for all initial values a solution exists. This is a problem at any switching time, because the value of the state just before the switch is in general not *consistent* with the algebraic constraints of the mode after the switch. So even when there is no explicit jump map given, the solutions have to jump (or there is no solution at all). However these jumps are not arbitrary as the following derivation shows:

0.4.2.1 The consistency projector

Consider a single DAE

$$E\dot{x} = Ax \tag{0.11}$$

where the matrix pair (E,A) is regular, i.e. $\det(sE-A) \not\equiv 0$. It is a classical result [45, 15] that regularity of a matrix pair is characterized by the existence of the Weierstrass canonical form, i.e. there exists invertible matrices S and T such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{0.12}$$

where J and N are in Jordan canonical form and N is nilpotent with nilpotency index $\nu \in \mathbb{N}$. The latter is called the *index* of the matrix pair (E,A). In the following it is not necessary to assume that J and N are in Jordan canonical form, following [6] we therefore call (0.12) the *quasi Weierstrass form (QWF)* of the matrix pair (E,A). An easy way to obtain the transformation matrices S and T via the Wong-sequences is given in the Appendix 0.6.2.

The relevance of the QWF for the solutions of the DAE (0.11) is stated as follows:

$$x \text{ solves (0.11)} \quad \Leftrightarrow \quad \begin{pmatrix} v \\ w \end{pmatrix} = T^{-1}x \text{ solves } \begin{cases} \dot{v} = Jv \\ N\dot{w} = w \end{cases}$$

i.e. the DAE (0.11) decouples into the standard ODE $\dot{v} = Jv$ and a pure DAE $N\dot{w} = w$. For the ODE any initial value is consistent. The pure DAE on the other hand only has the trivial solution:

$$N\dot{w} = w \quad \Rightarrow \quad N^2\ddot{w} = N\dot{w} = w \quad \Rightarrow \; \ldots \; \Rightarrow \quad 0 = N^\nu w^{(\nu)} = \ldots = N\dot{w} = w.$$

In particular, the only consistent initial value is zero.

Now assume that the matrix pair (E,A) is in QWF (0.12) and the corresponding DAE is switched on at time $t=0$ with initial values $v(0^-) = v_0$ and $w(0^-) = w_0$ prior to the switch. The initial value v_0 for the ODE $\dot{v} = Jv$ is consistent, hence no jump occurs in this component. The initial value w_0 for the pure DAE $N\dot{w} = w$ is in general not consistent and since zero is the only solution of $N\dot{w} = w$ there will be a jump from any w_0 to zero. Altogether the *only plausible* jump in the QWF-coordinates is given by the map

$$\begin{pmatrix} v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} v_0 \\ 0 \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0^-) \\ w(0^-) \end{pmatrix}$$

Translating this back to the original coordinates via $x = T\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ we arrive at the jump rule

$$x(0) = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1} x(0^-)$$

and the corresponding *consistency projector*

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

It is not difficult to see that the consistency projector does not depend on the specific choice of the (non-unique) transformation matrix T. In view of the Wong-sequences approach (see Appendix 0.6.2) with limits $\mathscr{V}^*$ and $\mathscr{W}^*$, it can be seen that $\Pi_{(E,A)}$ is a projection on $\mathscr{V}^*$ along $\mathscr{W}^*$. The above analysis also shows that the *consistency space*

$$\mathfrak{C}_{(E,A)} := \{\, x(t) \mid t \in \mathbb{R},\ x \text{ is a classical (i.e. differentiable) solution of (0.11)} \,\}$$

is exactly the image of the consistency projector.

0.4.2.2 Distributional solutions of a switched DAE

The presence of jumps in the solution of the switched DAE (0.2a) results in mathematical problems as these jumps are differentiated in the expression $E_\sigma \dot{x}$. One way out is to consider the switched DAE only on the open intervals between the switching times (in the case of a single-switch, the switched DAE is supposed to hold only on the intervals $(-\infty, 0)$ and $(0, \infty)$). The forthcoming Corollary 0.4.7 shows that in this case it is possible to rewrite the homogeneous switched DAE (0.7) as a homogeneous switched ODE with jumps (0.5), c.f. [41]; and the characterizations of observability for (0.1) carry over without change to the switched DAE case.

However, simple examples based on electrical circuits (see e.g. [40, Example 2]) show that derivatives of the jumps play an important role and it will turn out that they will also play a crucial role in the observability characterization.

The derivative of a jump is not well-defined for usual functions, but when considering *distributions* (or generalized functions) the derivative of a jump is the well known *Dirac impulse* (or Dirac delta). In the Appendix 0.6.3 we give a short introduction to the theory of distributions, but it should be noted that enlarging the solution space of (0.2a) to the space of distributions does *not* resolve the problems without further adjustment. The problem is that the multiplication of a piecewise-constant coefficient matrix ($E_{\sigma(\cdot)}$ or $A_{\sigma(\cdot)}$) with a distribution ($\dot{x}$ or x) is not well-defined. Even worse, it can be shown (see e.g. [39]) that it is in fact *impossible* to define such a product for general distributions. A possible way out of this dilemma is the consideration of the smaller space of *piecewise-smooth distributions* denoted by $\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty}$ as introduced in [38, 37]. The formal definition of $\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty}$ is given in the Appendix 0.6.3; for the understanding of the following it suffices to keep in mind that any piecewise-smooth distribution $D \in \mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty}$ can be decomposed into a piecewise-smooth function f and a purely impulsive part, denoted by $D[\cdot]$, i.e.

$$D \in \mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty} \quad \Leftrightarrow \quad D = f_{\mathbb{D}} + D[\cdot],$$

where $f_{\mathbb{D}}$ denotes the distribution induced by the function f. In particular, it is possible to evaluate a piecewise-smooth distribution at a certain point $t \in \mathbb{R}$ in the three following ways:

- Right-evaluation: $D(t^+) := f(t^+) = f(t)$,
- Left-evaluation: $D(t^-) := f(t^-)$,
- Impulse-evaluation: $D[t]$.

For example, for the Dirac impulse δ we have for all $t \in \mathbb{R} \setminus \{0\}$

$$\delta(t-) = 0 = \delta(t+) \quad \text{and } \delta[t] = 0$$

and for $t = 0$

$$\delta(0-) = 0 = \delta(0+) \quad \text{and } \delta[0] = \delta.$$

Definition 0.4.4 (Distributional solutions). We call (x,u,y) (or just x if the context is clear) a *distributional solution* of the switched DAE (0.2) if, and only if, $(x,u,y) \in (\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty})^{\mathtt{n}} \times (\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty})^{\mathtt{u}} \times (\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty})^{\mathtt{y}}$ and (0.2) holds as an equation within $\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty}$.

For classical (i.e. piecewise differentiable) solutions of (0.1) we have used the following equivalence to obtain observability characterizations for the single switch case:

$$y \equiv 0 \quad \Leftrightarrow \quad y_{(-\infty,0)} \equiv 0 \ \wedge \ y_{(0,\infty)} \equiv 0.$$

But this equivalence is not true anymore for switched DAEs (0.2) in a distributional framework and must be replaced by the following equivalence:

$$y \equiv 0 \quad \Leftrightarrow \quad y_{(-\infty,0)} \equiv 0 \ \wedge \ y[0] = 0 \ \wedge \ y_{(0,\infty)} \equiv 0, \tag{0.13}$$

which takes into account the impulsive part of y as well.

0.4.2.3 The differential and impulse projectors

In order to utilize the equivalence (0.13) we need to find convenient representations of the three terms on the right-hand side of (0.13). To this end we define the so called differential and impulsive "projectors" as follows.

Definition 0.4.5 (Differential and impulse projector, [34]). Consider a regular matrix pair (E,A) with QWF (0.12) and corresponding transformation matrices S and T. The *differential projector* is

$$\Pi^{\mathrm{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

and the *impulse projector* is

$$\Pi^{\mathrm{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S,$$

where the block sizes correspond to the block sizes in the QWF. Furthermore, define

$$A^{\mathrm{diff}} := \Pi^{\mathrm{diff}}_{(E,A)} A \quad \text{and} \quad E^{\mathrm{imp}} := \Pi^{\mathrm{imp}}_{(E,A)} E. \qquad \triangleleft$$

Note that the differential and impulse projector are not projectors in the usual sense as they are not idempotent in general, but they play a similar role as projectors in the explicit solution formula for inhomogeneous DAEs [39].

The significance of the matrix A^{diff} becomes clear in the following lemma:

Lemma 0.4.6 (Role of A^{diff}, [34]). *Consider a regular matrix pair (E,A) with corresponding matrix A^{diff}. Then every solution x of the DAE $E\dot{x} = Ax$ also solves*

$$\dot{x} = A^{\mathrm{diff}} x.$$

The converse is also true, but only if $x(0) \in \operatorname{im} \Pi_{(E,A)}$, i.e. $x(0)$ is consistent.

A simple consequence of this result is the following result which establishes the strong similarity of a switched DAE with a switched ODE with jumps:

Corollary 0.4.7 (Switched DAE interpreted as switched ODE with jumps, [41]). *Consider a switched DAE* (0.2a) *with switching signal $\sigma \in \Sigma$ and with switching times $t_1 < t_2 < t_3 \ldots$, where each matrix pair (E_p, A_p), $p \in \mathscr{P}$, is regular with corresponding consistency projector Π_p and matrix A^{diff}_p. Then the impulse-free part $\bar{x}$ given by $\bar{x}_{\mathbb{D}} := x - x[\cdot]$ of any (distributional) solution x of* (0.2a) *is the solution of the switched ODE with jumps:*

$$\begin{aligned} \dot{\bar{x}} &= A^{\mathrm{diff}}_{\sigma(t)} \bar{x}, && t \notin \{\, t_i \mid i \in \mathbb{N} \,\} \\ \bar{x}(t_i^+) &= \Pi_{\sigma(t_i)} \bar{x}(t_i^-), && i \in \mathbb{N} \\ \bar{x}(t_1^-) &\in \operatorname{im} \Pi_{\sigma(t_1^-)}. \end{aligned}$$

Note that $A^{\mathrm{diff}}_p \Pi_p = A^{\mathrm{diff}}_p = \Pi_p A^{\mathrm{diff}}_p$ for all p, hence Remark 0.2.1 is not relevant here, because arbitrarily many "trivial" switches can be introduced without altering the solutions.

To understand the role of the impulse projector it is first necessary to consider an initial trajectory problem (ITP) for the DAE given by (E,A):

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)}, \\ (E\dot{x})_{[0,\infty)} &= (Ax)_{[0,\infty)}, \end{aligned} \tag{0.14}$$

where $x^0 \in (\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty})^n$ is an arbitrary initial trajectory. In [37] it is shown that for regular matrix pairs (E,A) there exists a unique solution $x \in (\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty})^n$ for any ITP (0.14). In particular,

$$x(0^+) = \Pi_{(E,A)} x^0(0^-)$$

and the impulsive part is uniquely given as follows:

Lemma 0.4.8 (Impulsive part of ITP solution, [34]). *Consider the ITP* (0.14) *for a regular matrix pair* (E,A) *and corresponding matrix* E^{imp}*. Then the unique solution* $x \in (\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty})^n$ *fulfills*

$$x[0] = -\sum_{i=0}^{n-2} (E^{\mathrm{imp}})^{i+1} x^0(0^-)\delta^{(i)},$$

where $\delta^{(i)}$ *denotes the i-th derivative of the Dirac impulse* δ*.*

Remark 0.4.9. In [34] the formula for $x[0]$ is a bit more complicated, but is identical to the one presented here, once it is realized that $E^{\mathrm{imp}}\Pi_{(E,A)} = 0$. Furthermore, the upper limit of the sum can be reduced to $\nu - 2 \leq n-2$, where ν is the index of (E,A), because $(E^{\mathrm{imp}})^k = 0$ for all $k \geq \nu$. However, in the context of switched DAE we use the formula with $n-2$ instead of $\nu-2$ because the dimension n of x does not depend on the specific mode whereas the index ν might depend on the mode. $\triangleleft$

0.4.2.4 Observability condition

We are now ready to state the generalization of Lemmas 0.4.1 and 0.4.3 to switched DAEs.

Theorem 0.4.10 (Observability for σ_1: DAE case, [34]). *Consider the switched DAE* (0.2) *with regular matrix pairs* (E_p,A_p)*,* $p \in \mathscr{P}$*, and corresponding consistency projectors* Π_p *and matrices* A_p^{diff}*,* E_p^{imp}*. Let*

$$\begin{aligned} O_p^{\mathrm{diff}} &:= [C_p / C_p A_p^{\mathrm{diff}} / C_p (A_p^{\mathrm{diff}})^2 / \cdots / C_p (A_p^{\mathrm{diff}})^{n-1}],\\ O_p^{\mathrm{imp}} &:= [C_p E_p^{\mathrm{imp}} / C_p (E_p^{\mathrm{imp}})^2 / \cdots / C_p (E_p^{\mathrm{imp}})^{n-2}], \end{aligned}$$

then for the single switch switching signal σ_1 *given by* (0.8) *we have*

$$\boxed{\mathscr{M}_{0^-}^{\sigma_1} = \operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \ker O_1^{\mathrm{diff}}\Pi_1 \cap \ker O_1^{\mathrm{imp}}}$$

and, for any $T \geq 0$*,*

$$\boxed{\mathscr{N}_{T^+}^{\sigma_1} = e^{A_1^{\mathrm{diff}}T}\mathscr{N}_{0^+}^{\sigma_1} = e^{A_1^{\mathrm{diff}}T}\Pi_1\mathscr{M}_{0^-}^{\sigma_1},}$$

where $\mathscr{M}_{0^-}^{\sigma_1}$ *and* $\mathscr{N}_{T^+}^{\sigma_1}$ *are as in Definitions 0.3.4 and 0.3.5, respectively. In particular, the switched DAE* (0.2) *is observable for* σ_1 *if, and only if,*

$$\operatorname{im}\Pi_0 \cap \ker O_0^{\mathrm{diff}} \cap \ker O_1^{\mathrm{diff}}\Pi_1 \cap \ker O_1^{\mathrm{imp}} = \{0\}$$

and forward observable if, and only if,

$$\Pi_1 (\operatorname{im} \Pi_0 \cap \ker O_0^{\text{diff}} \cap \ker O_1^{\text{diff}} \Pi_1 \cap \ker O_1^{\text{imp}}) = \{0\}.$$

Remark 0.4.11. Each of the four subspaces involved in the intersection defining $\mathscr{M}_{0^-}^{\sigma_1}$ has an intuitive meaning. Recall that $\mathscr{M}_{0^-}^{\sigma_1}$ consists of all initial values just before $t = 0$ yielding a zero output. Hence we can derive the following inclusions:

- $\mathscr{M}_{0^-}^{\sigma_1} \subseteq \operatorname{im} \Pi_0$ because $x(0^-)$ must be consistent with the mode before the switch.
- $\mathscr{M}_{0^-}^{\sigma_1} \subseteq \ker O_0^{\text{diff}}$ because $y_{(-\infty,0)} \equiv 0$ implies, invoking Lemma 0.4.6, that $x(0^-) \in \ker O_0^{\text{diff}}$.
- $\mathscr{M}_{0^-}^{\sigma_1} \subseteq \ker O_1^{\text{diff}} \Pi_1$ because $y_{(0,\infty)} \equiv 0$ implies that $x(0^+) = \Pi_1 x(0^-) \in \ker O_1^{\text{diff}}$, which in turn implies that $x(0^-) \in \Pi_1^{-1}(\ker O_1^{\text{diff}}) = \ker O_1^{\text{diff}} \Pi_1$.
- $\mathscr{M}_{0^-}^{\sigma_1} \subseteq \ker O_1^{\text{imp}}$ because $y[0] = 0$ implies, due to Lemma 0.4.8, that $x(0^-) \in \ker O_1^{\text{imp}}$.

This already shows that $\mathscr{M}_{0^-}^{\sigma_1} \subseteq \operatorname{im} \Pi_0 \cap \ker O_0^{\text{diff}} \cap \ker O_1^{\text{diff}} \Pi_1 \cap \ker O_1^{\text{imp}}$. The converse inclusion follows from the observation that for any x_0 in the intersection there exists a unique solution x with initial condition $x(0^-) = x_0$ and this solution produces a zero output. $\triangleleft$

We conclude this section with an example which shows that each of the four subspaces in the representation of $\mathscr{M}_{0^-}^{\sigma_1}$ as in Theorem 0.4.10 plays a crucial role for observability, i.e. the intersection of only three of the four subspace will in general not yield the trivial subspace.

Example 0.4.12 ([34]). Let the switched DAE (0.2) be given by

$$\left.\begin{aligned} \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix} \dot{x} &= \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 1\\0\\1\\1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0&0&0&1 \end{bmatrix} x \end{aligned}\right\} \quad t \in (-\infty, 0),$$

$$\left.\begin{aligned} \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{bmatrix} \dot{x} &= \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} x + \begin{bmatrix} 0\\1\\1\\0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0&1&0&1 \end{bmatrix} x \end{aligned}\right\} \quad t \in [0, \infty).$$

Neither subsystem is observable in the classical sense. But it is possible to determine the exact value of the state trajectory with the switching signal (0.8). The consistency, differential, and impulse projectors as well as O_p^{diff}, O_p^{imp} for each of the two subsystems are:

$$\Pi_0 = \Pi_0^{\text{diff}} = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{bmatrix}, \qquad \Pi_1 = \Pi_1^{\text{diff}} = \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix},$$

$$\Pi_0^{\text{imp}} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix}, \qquad \Pi_1^{\text{imp}} = \begin{bmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{bmatrix},$$

$$O_0^{\text{diff}} = \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \qquad O_1^{\text{diff}} = \begin{bmatrix} 0&1&0&1\\ 0&1&0&0\\ 0&1&0&0\\ 0&1&0&0 \end{bmatrix},$$

$$O_0^{\text{imp}} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \qquad O_1^{\text{imp}} = \begin{bmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix},$$

and the four subspaces from Theorem 0.4.10 are:

$$\begin{aligned} \operatorname{im}\Pi_0 &= \operatorname{span}\{e_1, e_2, 0, e_4\}, \\ \ker O_0^{\text{diff}} &= \operatorname{span}\{e_1, e_2, e_3, 0\}, \\ \ker O_1^{\text{diff}}\Pi_1 &= \operatorname{span}\{e_1, 0, e_3, e_4\}, \\ \ker O_1^{\text{imp}} &= \operatorname{span}\{0, e_2, e_3, e_4\}. \end{aligned}$$

where $e_i \in \mathbb{R}^4$, $i = 1,2,3,4$, is the corresponding natural basis vector.

Clearly, $\operatorname{im}\Pi_0 \cap \ker O_0^{\text{diff}} \cap \ker O_1^{\text{diff}}\Pi_1 \cap \ker O_1^{\text{imp}} = \{0\}$ and the switched system is observable for σ_1 according to Theorem 0.4.10. Furthermore, each of the four subspaces $\mathfrak{C}_0$, $\ker O_0^{\text{diff}}$, $\ker O_1^{\text{diff}}\Pi_1$ and $\ker O_1^{\text{imp}}$ is necessary to obtain a trivial intersection. In fact, each subspace restricts exactly one state variable. In view of Remark 0.4.2, note that the switched system with a reversed mode sequence is not observable because

$$\operatorname{im}\Pi_1 \cap \ker O_1^{\text{diff}} \cap \ker O_0^{\text{diff}}\Pi_0 \cap \ker O_0^{\text{imp}} = \operatorname{span}\{e_3\} \neq \{0\}.$$

As an illustration of constructing state trajectories from the knowledge of the output and the input, let us consider an input[1], $u(t) = e^{2t} + \delta_{-1} + \delta_0$, and assume that the following output is produced by the system with σ_1 specified in (0.8):

[1] Note that, for simplicity, we are misusing the notation by writing $u(t) = e^{2t} + \delta_{-1} + \delta_0$ because u is a piecewise-smooth distribution and therefore only the evaluations $u(t-)$, $u(t+)$, $u[t]$ are well defined. The correct way of writing would be to write $\hat{u}(t) = e^{2t}$ and $u = \hat{u}_{\mathbb{D}} + \delta_{-1} + \delta_0$.

$$y(t) = \begin{cases} -1, & t \in (-\infty, -1), \\ 0, & t \in [-1, 0), \\ e^t + e^{2t} + \delta_0, & t \in [0, \infty). \end{cases}$$

The closed form solution for the state variables, parameterized by $a, b, c \in \mathbb{R}$, is given as follows:

$$x_1(t) = \begin{cases} -e^{t+1} + (a-1)e^t + e^{2t}, & t \in (-\infty, -1), \\ (a-1)e^t + e^{2t}, & t \in [-1, 0), \\ 0, & t \in [0, \infty), \end{cases}$$
$$x_2(t) = \begin{cases} e^t b, & t \in (-\infty, 0), \\ e^t + e^{2t} + (b-1)e^t, & t \in [0, \infty), \end{cases}$$
$$x_3(t) = \begin{cases} -e^{2t} - \delta_{-1}, & t \in (-\infty, 0), \\ -e^t + e^{2t}, & t \in [0, \infty), \end{cases}$$
$$x_4(t) = \begin{cases} \frac{1}{2}e^{2t}c - 1, & t \in (-\infty, -1), \\ \frac{1}{2}e^{2t}c, & t \in [-1, 0), \\ -a\delta_0, & t \in [0, \infty). \end{cases}$$

First note that $x_3(0^-) = -1$, which corresponds to the fact that in the homogeneous case the consistency space $\operatorname{im}\Pi_0$ restricts $x_3(0^-)$ to be zero. Since $\ker O_0^{\text{diff}}$ restricts $x_4(0^-)$, we would expect that $y(0^-)$, $\dot{y}(0^-)$, $\ldots$, determine $x_4(0^-)$. In fact, $0 = y(0^-) = x_4(0^-)$. The space $\ker O_1^{\text{diff}}\Pi_1$ restricts $x_2(0^-)$, and hence by using the values for $y^{(i)}(0^+)$, we are able to reconstruct $x_2(0^-)$: $2 = y(0^+) = x_2(0^+) + x_4(0^+) = 1 + b = 1 + x_2(0^-)$, i.e. $x_2(0^-) = 1$. Finally, $\ker O_1^{\text{imp}}$ restricts $x_1(0^-)$, therefore, the information from the impulse of y at zero can be used to determine $x_1(0^-)$: $\delta_0 = y[0] = x_2[0] + x_4[0] = -a\delta_0$, hence $-1 = a = x_1(0^-)$. Altogether, we were able to determine $x(0^-)$ which together with the knowledge of u and the regularity of the matrix pairs $(E_0, A_0), (E_1, A_1)$ makes it possible to uniquely reconstruct the whole state x.

0.5 Observability for general switching signals

We now consider a general switching signal $\sigma \in \Sigma$. Let

$$\Sigma_{\mathbb{N}} := \left\{ \sigma \in \Sigma \;\middle|\; \sigma\big|_{[t_i, t_{i+1})} = i \quad \text{for } i = 0, 1, 2, \ldots \right\} \tag{0.15}$$

where $0 = t_1 < t_2 < t_3 < \ldots$ are the switching times of σ (note however Remark 0.2.1 when the switching signal has only finitely many switches) and $t_0 := -\infty$. When considering a fixed switching signal we can, without loss of generality, restrict our attention to switching signals of the class $\Sigma_{\mathbb{N}}$ by a suitable relabeling of the modes.

0.5.1 Observability characterization

In this subsection we present a characterization of observability of system (0.1) and (0.2) with a switching signal $\sigma \in \Sigma_{\mathbb{N}}$. Instead of treating both the cases separately, our goal is to present a single result that generalizes to both system classes. Towards this end, we introduce some notation that alternates its meaning depending on the system class under consideration. For switched ODEs (0.1), we let, for $i \in \mathbb{N}$,

$$F_i := A_i, \qquad J_i := G_i \tag{0.16a}$$

denote the flow matrix and the jump matrix for subsystem $i \in \mathscr{P}$, respectively. The local unobservable space at the i-th switch for system (0.1) is given by:

$$\mathscr{M}^i := \ker O_{i-1} \cap \ker O_i G_i. \tag{0.16b}$$

Analogously, for switched DAEs (0.2), we let:

$$F_i := A_i^{\text{diff}} \qquad J_i := \Pi_i \tag{0.17a}$$

and the corresponding local unobservable space at the i-th switch is:

$$\mathscr{M}^i := \operatorname{im} \Pi_{i-1} \cap \ker O_{i-1}^{\text{diff}} \cap \ker O_i^{\text{diff}} \Pi_i \cap \ker O_i^{\text{imp}}, \tag{0.17b}$$

with the notation as in Theorem 0.4.10.

According to Lemma 0.4.1 and Theorem 0.4.10, $\mathscr{M}^i$ is the unobservable space if there would be only a single switch from mode $i-1$ to mode i. Note that this local unobservable space does not depend on the actual switching time t_i.

We now combine the local unobservable spaces of the first m switches as follows:

$$\boxed{\begin{aligned} \mathscr{M}_m^m &:= \mathscr{M}^m, \\ \mathscr{M}_i^m &:= \mathscr{M}^i \cap J_i^{-1}(e^{-F_i \tau_i} \mathscr{M}_{i+1}^m), \quad m > i \geq 1, \end{aligned}} \qquad \begin{aligned} &(0.18\text{a}) \\ &(0.18\text{b}) \end{aligned}$$

where $\tau_i := t_{i+1} - t_i$.

The intuition behind the sequence (0.18) is as follows: Starting at the m-th switch we go backward in time and combine the local knowledge from each of the previous switches to obtain knowledge of the initial value $x(0^-)$. In fact, the local unobservable space at the m-th switch is moved by the flow of the $(m-1)$-st mode (via the negative exponential) to obtain the unobservable space at t_{m-1}^+ taking into account the knowledge of the m-th switch only. With the preimage of the jump map J_{m-1} this information is translated to the information at t_{m-1}^-. Combining this with the local unobservable space at the $(m-1)$-st switch given by $\mathscr{M}^{m-1}$, we get a refined unobservable space at the $(m-1)$-st switch taking into account the information obtained from the switches at t_{m-1} and t_m. In particular $y_{(t_{m-2},t_{m+1})} \equiv 0$ implies $x(t_{m-1}^-) \in \mathscr{M}_{m-1}^m$. Repeating this argument we arrive at the following implication, for all $i \in \mathbb{N}$:

$$y_{(t_{i-1},t_{m+1})} \equiv 0 \quad \Rightarrow \quad x(t_i^-) \in \mathscr{M}_i^m.$$

In fact, we can formulate a stronger result using the following notation:

$$\Sigma_m := \{ \ \sigma_m \mid \exists \sigma \in \Sigma_{\mathbb{N}} : \sigma_m := \sigma \text{ on } (-\infty, t_m) \text{ and } \sigma_m(t) = m \text{ on } [t_m, \infty) \ \}. \quad (0.19)$$

Theorem 0.5.1 (Geometric observability characterization, [35, 33]). *Consider the switched system* (0.1) *(respectively* (0.2)*) with switching signal* $\sigma \in \Sigma_{\mathbb{N}}$ *and corresponding* $\sigma_m \in \Sigma_m$ *for some* $m \in \mathbb{N}$*. Then the unobservable spaces* $\mathscr{M}_{0^-}^{\sigma}$ *and* $\mathscr{M}_{0^-}^{\sigma_m}$ *as in Definition 0.3.4 fulfill*

$$\boxed{\mathscr{M}_{0^-}^{\sigma} \subseteq \mathscr{M}_{0^-}^{\sigma_m} = \mathscr{M}_1^m,}$$

where $\mathscr{M}_1^m$ *is obtained using* (0.18) *together with* (0.16) *(resp.* (0.17)*). In particular* (0.1) *(resp.* (0.2)*) is observable for* σ *if, and only if, there exists* $m \in \mathbb{N}$ *such that*

$$\mathscr{M}_1^m = \{0\}. \quad (0.20)$$

Example 0.5.2. Consider the switched system (0.1) characterized by:

$$A_0 = A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$C_0 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

with $G_i = I$, $H_i = 0$, $B_i = 0$, $D_i = 0$ for $i = 0,1,2$. It is noted that none of the pairs (A_i, C_i) is observable. Consider the switching signal σ with the mode sequence $0 \to 1 \to 2$ with switching times t_1, t_2 such that $\tau_1 := t_2 - t_1 \neq k\pi$ for any $k \in \mathbb{N}$. Clearly, $\mathscr{M}_2^2 := \operatorname{span}\{e_2\}$ and

$$\mathscr{M}_1^2 = \operatorname{span}\{e_2\} \cap e^{-A_1 \tau_1} \operatorname{span}\{e_2\} = \operatorname{span}\{e_2\} \cap \operatorname{span}\left\{ \left(\begin{smallmatrix} \sin \tau_1 \\ \cos \tau_1 \end{smallmatrix} \right) \right\}.$$

Thus, $\mathscr{M}_{0^-}^{\sigma} = \mathscr{M}_1^2 = \{0\}$ and system (0.1) is observable, provided $\tau_1 \neq k\pi$. Note that this switched system cannot be made observable with only a single switch. $\triangleleft$

0.5.2 Removing dependency from switching times

The observability condition (0.20) given in Theorem 0.5.1 is for a fixed switching signal σ given by (0.15), in particular we fixed the mode sequence. It is entirely possible that for the same mode sequence the system is observable for certain switching times and unobservable for others. So it would be more useful to know whether the observability property holds for all switching signals with the same switching sequence. It can be shown that if there is a switching signal that satisfies $\mathscr{M}_{0^-}^{\sigma} = \{0\}$, then the set of switching signals, with the same mode sequence, for which $\mathscr{M}_{0^-}^{\sigma} \neq \{0\}$, is nowhere dense. The result is formally stated as follows:

Theorem 0.5.3 (Genericity of observability, [33, Thm. 2]). *If for $m \in \mathbb{N}$ and some σ_m given by* (0.19) *the switched system* (0.1) *(resp.* (0.2)*) is observable then the set*

$$\Sigma^o := \{\ \sigma \in \Sigma_{\mathbb{N}} \mid (0.1)\ \textit{(resp. } (0.2)\textit{) is observable for } \sigma_m \textit{ given by } (0.19)\ \}$$

is open and dense in the set of all switching signals in $\Sigma_{\mathbb{N}}$, where the topology on $\Sigma_{\mathbb{N}}$ is given in the Appendix 0.6.4.

In other words, the above result indicates that the condition (0.20) is somewhat robust with respect to the switching times.

Next, we want to derive conditions for observability which are independent of switching times, while keeping the mode sequence fixed as in (0.15). As stated in Theorem 0.5.1, condition (0.20), which depends on the switching times, is necessary and sufficient for observability. To obtain a relaxed version of (0.20) for deriving conditions independent of switching times, one must introduce some degree of conservatism. For this reason, we only arrive at a sufficient condition and a necessary condition for a fixed mode sequence, which do not depend on switching times.

Corollary 0.5.4 (Sufficient condition for observability, [33, 35]). *For system* (0.1) *(resp.* (0.2)*) with σ given by* (0.15) *and $m \in \mathbb{N}$ define the following sequence of subspaces:*

$$\begin{aligned}
\overline{\mathscr{M}}{}^m_m &:= \mathscr{M}^m, \\
\overline{\mathscr{M}}{}^m_i &:= \mathscr{M}^i \cap J_i^{-1} \left\langle F_i \,\middle|\, \overline{\mathscr{M}}{}^m_{i+1} \right\rangle, \qquad m > i \geq 1,
\end{aligned}$$

where $F_i, J_i, \mathscr{M}^i$ are defined via (0.16) *(resp.* (0.17)*), and $\langle F \mid \mathscr{M} \rangle$ denotes the smallest F-invariant subspace which contains $\mathscr{M}$, for some matrix F and subspace $\mathscr{M}$. Then,*

$$\overline{\mathscr{M}}{}^m_i \supseteq \mathscr{M}^m_i \quad \forall i = 1, \dots, m$$

hence, system (0.1) *(resp.* (0.2)*) is observable for σ if there exists $m \in \mathbb{N}$ such that $\overline{\mathscr{M}}{}^m_1 = \{0\}$.*

It is natural to ask how much conservatism has been introduced in obtaining the sufficient condition. If the condition in Corollary 0.5.4 holds, that is $\overline{\mathscr{M}}{}^m_1 = \{0\}$, then the system (0.1), or (0.2), is observable for *all* σ with mode sequence 1 through m, regardless of the switching times t_i. To address the reverse implication, note that system (0.1) is uniformly (with respect to switching times) observable for σ_m given by (0.19), if and only if,

$$\bigcup_{\sigma_m \in \Sigma_m} \mathscr{M}^{\sigma_m}_{0^-} = \bigcup_{\tau_1, \dots, \tau_{m-1} > 0} \mathscr{M}^m_1 = \{0\}. \tag{0.21}$$

However, in order to check the above condition in practice, a difficulty arises due to the fact that the union of two subspaces is not necessarily a subspace because the resulting union is not closed under addition in general. In case, if $\bigcup_{\tau_i, \dots, \tau_{m-1} > 0} \mathscr{M}^m_i$, for each $1 \leq i \leq m-1$ is a subspace, then $\overline{\mathscr{M}}{}^m_i = \bigcup_{\tau_i, \dots, \tau_{m-1} > 0} \mathscr{M}^m_i$, and in that case $\overline{\mathscr{M}}_1^m = \{0\}$ implies observability for all σ_m regardless of switching times. The following example provides an illustration of these arguments:

Example 0.5.5. We reconsider the switched system from Example 0.5.2. It was highlighted there that for some special switching signals the system is not observable; and indeed the sufficient condition of Corollary 0.5.4 is not satisfied because $\overline{\mathscr{M}}_1^2 = \overline{\mathscr{M}}_2^2 = \operatorname{span}\{e_2\} \neq \{0\}$. Now, in the first mode of Example 0.5.2, let us replace the matrix A_1 by $\tilde{A}_1 := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. For the resulting switched system, we get again $\overline{\mathscr{M}}_1^2 = \overline{\mathscr{M}}_2^2 = \operatorname{span}\{e_2\} \neq \{0\}$. Hence the sufficient condition in Corollary 0.5.4 is violated; however, the resulting switched system is observable for all $\tau_1 > 0$ (which can be seen from Theorem 0.5.1). The source of this gap is the fact that the set

$$\bigcup_{\tau_1>0} e^{-\tilde{A}_1\tau_1}\mathscr{M}_2^2 = \bigcup_{\tau_1>0} e^{-\tilde{A}_1\tau_1}\operatorname{span}\{e_2\} = \left\{ \left(\begin{smallmatrix} -\tau_1\xi \\ \xi \end{smallmatrix}\right) \;\middle|\; \tau_1 > 0, \xi \in \mathbb{R} \right\}$$

is not a subspace and its intersection with $\mathscr{M}^1 = \operatorname{span}\{e_2\}$ is just $\{0\}$.

Having developed a sufficient condition using subspaces that contain $\mathscr{M}_i^m$ (for all switching times), we now obtain a necessary condition in terms of subspaces contained in $\mathscr{M}_i^m$.

Corollary 0.5.6 (Necessary condition for observability, [33, 35]). *For system* (0.1) *(resp.* (0.2)*) with $\sigma \in \Sigma_{\mathbb{N}}$ and $m \in \mathbb{N}$ define the following sequence of subspaces:*

$$\begin{aligned} \underline{\mathscr{M}}_m^m &:= \mathscr{M}^m, \\ \underline{\mathscr{M}}_i^m &:= \mathscr{M}^i \cap J_i^{-1}\left\langle \underline{\mathscr{M}}_{i+1}^m \,\middle|\, F_i \right\rangle, \qquad m > i \geq 1. \end{aligned}$$

where $F_i, J_i, \mathscr{M}_i$ are defined via (0.16) *(resp.* (0.17)*), and $\langle \mathscr{M} \,|\, F \rangle$ is the largest F-invariant subspace contained within $\mathscr{M}$, for some subspace $\mathscr{M}$ and matrix F. Then,*

$$\underline{\mathscr{M}}_i^m \subseteq \mathscr{M}_i^m \quad \forall i = 1, \ldots, m,$$

hence if system (0.1) *(resp.* (0.2)*) is observable for σ then there exists $m \in \mathbb{N}$ such that $\underline{\mathscr{M}}_1^m = \{0\}$.*

The natural question now to ask is whether $\underline{\mathscr{M}}_1^m = \{0\}$ always guarantees the existence of a switching signal that renders the switched system observable (i.e. whether the switched system is controlled observable under the constraint that $\sigma \in \Sigma_{\mathbb{N}}$). It can be shown that

$$\underline{\mathscr{M}}_1^m = \bigcap_{\tau_1,\ldots,\tau_{m-1}>0} \mathscr{M}_1^m, \tag{0.22}$$

from where one sees that the right-hand side may be $\{0\}$ even though there exists no $\sigma \in \Sigma_{\mathbb{N}}$ with dwell-times $\tau_1, \tau_2, \cdots, \tau_{m-1} > 0$ such that $\mathscr{M}_1^m = \{0\}$. However, in the next section, we present a connection between the necessary condition in Corollary 0.5.6 and weakly controlled observability.

Example 0.5.7. Consider the switched system (0.1) characterized by:

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$C_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

with $G_i = I$, $H_i = 0$, $B_i = 0$, $D_i = 0$ for $i = 0,1,2$. Then, for $\sigma^m \in \Sigma_m$, with $m = 2$, we obtain that $\underline{\mathscr{M}}_2^2 = \ker O_2 \cap \ker O_1 = \operatorname{span}\{e_2\}$ and $\underline{\mathscr{M}}_1^2 = \langle \operatorname{span}\{e_2\} \,|A_1\rangle = \{0\}$. However, it is seen that $\mathscr{M}_1^2 = \operatorname{span}\left\{\left(\begin{smallmatrix} -\tau_1 \\ 1 \end{smallmatrix}\right)\right\} \neq \{0\}$, so that (0.20) does not hold for any $\tau_1 > 0$, showing that the system is not observable for $\sigma^m \in \Sigma_m$ with $m = 2$ even though the necessary condition of Corollary 0.5.6 is satisfied. On the other hand, if the mode sequence is repeated at least once, then the system is observable for the new switching signal.

0.5.3 Conditions for controlled and weakly controlled observability

Since (weakly) controlled observability does not consider an a priori fixed switching signal we have to consider again a general switching signal

$$\sigma : \mathbb{R} \to \{1,2,\ldots,\mathtt{p}\}$$

instead of a switching signal given by (0.15). Here $\mathtt{p} \in \mathbb{N}$ denotes the *finite* number of different modes.

0.5.3.1 Classical switched systems

Observability of classical switched systems was studied in [31], where the authors call (0.1) observable if any initial value can be distinguished from zero (via a suitable switching signal). Due to linearity, this notion is equivalent to our notion of weakly controlled observability. We will present the algebraic and geometric conditions derived in [31] in the following.

We start by defining the following generalized observability matrices $\mathfrak{O}_i$, $i \in \mathbb{N}$ as follows

$$\mathfrak{O}_0 = [C_1/C_2/\cdots/C_{\mathtt{p}}] \tag{0.23a}$$
$$\mathfrak{O}_{i+1} = [\mathfrak{O}_0/\mathfrak{O}_i A_1/\mathfrak{O}_i A_2/\cdots/\mathfrak{O}_i A_{\mathtt{p}}],\ i \in \mathbb{N}. \tag{0.23b}$$

Note that $\mathfrak{O}_i$ is of size $\mathtt{py}\frac{\mathtt{p}^{i+1}-1}{\mathtt{p}-1} \times n$, where $\mathtt{y} \in \mathbb{N}$ denotes the output dimension. Let $\mathscr{K}_i = \ker \mathfrak{O}_i$. It is then easy to see that, for $i \in \mathbb{N}$,

$$\boxed{\mathscr{K}_i = \bigcap_{k=0}^{i} \bigcap_{p_0,\cdots,p_k} \ker C_{p_0} A_{p_1} \cdots A_{p_k}.}$$

Furthermore, $\mathscr{K}_{i+1} \subseteq \mathscr{K}_i$ and there exists $k^* < n$ such that $\mathscr{K}^* := \mathscr{K}_{k^*} = \mathscr{K}_k$ for all $k \geq k^*$. It follows that $\mathscr{K}^*$ is the largest subset of $\mathscr{K}_0 = \bigcap_{p \in \mathscr{P}} \ker C_p$ which is invariant for each A_p, $p \in \mathscr{P}$, i.e. it is the largest subspace which satisfies

$$\forall p \in \mathscr{P}: \quad \mathscr{K}^* \subseteq \ker C_p \text{ and } A_p \mathscr{K}^* \subseteq \mathscr{K}^*.$$

Weakly controlled observability can now be characterized as follows:

Theorem 0.5.8 (Weakly controlled observability for systems without jumps, [31]). *The switched system* (0.1) *without jumps is weakly controlled observable if, and only if,*

$$\mathscr{K}^* = \{0\}$$

i.e. there is no nontrivial $\left\{ A_p \;\middle|\; p \in \mathscr{P} \right\}$*-invariant subspace contained in the intersection of the null spaces of* C_p*,* $p \in \mathscr{P}$*.*

The intuition behind this result is as follows: An initial value $x_0 \in \mathbb{R}^n$ is indistinguishable from zero if, and only if, the corresponding output satisfies $y \equiv 0$ for all switching signals, i.e.

$$\{0\} = \left\{ C_{p_k} e^{A_{p_k}\tau_k} e^{A_{p_{k-1}}\tau_{k-1}} \cdots e^{A_{p_1}\tau_1} x_0 \;\middle|\; k \in \mathbb{N}, p_1, \ldots, p_k \in \mathscr{P}, \tau_1, \ldots \tau_k > 0 \right\}$$

Due to analyticity of the map $(\tau_1, \tau_2, \ldots, \tau_k) \mapsto C_{p_k} e^{A_{p_k}\tau_k} e^{A_{p_{k-1}}\tau_{k-1}} \cdots e^{A_{p_1}\tau_1} x_0$ (evaluated at zero) it follows that $x_0 \in \mathbb{R}^n$ is indistinguishable from zero if, and only if,

$$C_{p_i} A_{p_{i-1}} A_{p_{i-2}} \cdots A_{p_1} x_0 = 0 \quad \forall i \in \mathbb{N} \quad \forall p_1, p_2, \ldots, p_i \in \mathscr{P}.$$

The latter is nothing else but the condition

$$x_0 \in \ker \mathfrak{O}_i = \mathscr{K}_i, \quad \forall i \in \mathbb{N},$$

or equivalently,

$$x_0 \in \bigcap_{i \in \mathbb{N}} \mathscr{K}_i = \mathscr{K}^*.$$

Note that the condition $\mathscr{K}^* = \{0\}$ is equivalent to

$$\operatorname{rank} \mathfrak{O}_{n-1} = n,$$

which shows the similarities to the classical full rank assumption of the Kalman observability matrix for nonswitched linear ODEs. In view of Remark 0.3.2(iv), it is also clear that full rank of the observability matrix of any mode implies full rank of $\mathfrak{O}_{n-1}$

The authors of [31] also study controllability of switched systems and they establish a duality between observability and controllability [31, Cor. 4.28]. For the problem of controllability it can be shown that a single switching signal and a final time $t_f > 0$ exists such that for all initial states one finds an input u that gives $x(t_f) = 0$ ([31, Rem. 4.23], see also [16]). We can use this result to establish the following important equivalence.

Theorem 0.5.9 (Weakly controlled and controlled observability, no jumps). *If the switched system* (0.1) without jumps *is weakly controlled observable, then there exists a single switching signal σ for which any initial value can be distinguished from zero. In particular, for classical switched systems weakly controlled observability is equivalent to controlled observability.*

Proof. For the classical switched system

$$\dot{x} = A_\sigma x, \quad x(0) = x_0 \in \mathbb{R}^n, \quad y = C_\sigma x$$

consider the dual switched system

$$\dot{z} = -A_\sigma^\top z + C_\sigma^\top v, \quad z(0) = z_0 \in \mathbb{R}^n. \tag{0.24}$$

As shown in [31], controllability of (0.24) is equivalent to weakly controlled observability of the classical switched systems and implies existence of a single switching signal σ^* and $t_f > 0$ such that for all $z_0 \in \mathbb{R}^n$ there exists an input v such that the corresponding solution of (0.24) satisfies $z(t_f) = 0$. Note that for any solution x of the classical switched system and any solution z of (0.24) we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(z^\top x) = (-z^\top A_{\sigma^*} + v^\top C_{\sigma^*})x + z^\top A_{\sigma^*} x = v^\top y,$$

and therefore

$$z^\top(t_f)x(t_f) - z_0^\top x_0 = \int_0^{t_f} v^\top y.$$

Hence the assumption that there exists x_0 with vanishing output implies $z_0^\top x_0 = z^\top(t_f)x(t_f)$. Controllability of (0.24) now implies that for all $z_0 \in \mathbb{R}^n$ we can achieve $z(t_f) = 0$; hence $x_0 = 0$ must hold. This shows that for the switching signal σ^* any initial value is distinguishable from zero, i.e. the classical switched system is controlled observable. □

The above equivalence between weakly controlled observability and controlled observability is rather significant because with a simple algebraic test (independent of the switching signal) one can conclude existence of a specific switching signal which makes the switched system observable and this allows the construction of an observer (see Chapter **??** in this book or [33]).

Remark 0.5.10 (Generalized Kalman observability decomposition, [23]). Assume that $\dim \mathscr{K}^* = n - o$, and let $b_1, \ldots, b_n$ be a basis in $\mathbb{R}^n$ such that $b_{o+1}, \ldots, b_n$ span $\mathscr{K}^*$. Since $\mathscr{K}^*$ is an A_p-invariant subspace and it is contained in $\ker C_p$ for all $p \in \mathscr{P}$, in this new basis the matrices A_p, B_p, and C_p can be rewritten as

$$A_p = \begin{bmatrix} A_p^{\mathrm{O}}, & 0 \\ A_p^{'}, & A_p^{''} \end{bmatrix}, C_p = \begin{bmatrix} C_p^{\mathrm{O}}, & 0 \end{bmatrix}, B_p = \begin{bmatrix} B_p^{\mathrm{O}} \\ B_p^{'} \end{bmatrix},$$

where $A_p^{\mathrm{O}} \in \mathbb{R}^{o \times o}, B_p^{\mathrm{O}} \in \mathbb{R}^{o \times \mathtt{u}}$, and $C_p^{\mathrm{O}} \in \mathbb{R}^{\mathtt{y} \times o}$. The corresponding switched system

$$\begin{aligned}\dot{z}(t) &= A^{\mathrm{O}}_{\sigma(t)} z(t) + B^{\mathrm{O}}_{\sigma(t)} u(t)\\ y(t) &= C^{\mathrm{O}}_{\sigma(t)} z(t) + D_{\sigma(t)} u(t)\end{aligned} \tag{0.25}$$

is then weakly controlled observable and it is equivalent to the original switched system without jumps in the following sense: for every solution (x,y,σ,u) of (0.1) there exists a solution of (0.25) of the form (z,y,σ,u) and vice versa, i.e. the input-output behavior of (0.1) and (0.25) coincide. $\triangleleft$

Note that the above observability reduction method is in fact an algorithm and can be implemented, see [23] for more details on the numerical implementation.

0.5.3.2 Switched systems with jumps

Weakly controlled observability for switched ODEs with jumps was studied in [26]. The system class of so called linear hybrid systems studied therein is more general than (0.1) in two major aspects: 1) The switching signal itself is generated by a finite automaton and observability of the active mode is part of the observability definition; 2) the jump maps depend on the modes before and after the switch, this allows in particular to also study modes with different state space dimensions. For details on this framework we refer to Chapter **??** in this book. Under the assumption that the discrete output map is identity (i.e. observability of the discrete state is trivially fulfilled) and under the assumption that the jump maps only depend on the mode directly after the switch (this implies that the state space dimensions must be equal) we can obtain conditions for weakly controlled observability for (0.1).

To this end, we define for every $p \in \mathscr{P}$ and every matrix C with n columns the Kalman observability matrix of (A_p, C) by

$$O_p(C) = [C/CA_p/\dots/CA_p^{n-1}].$$

For each $i = 0,1,2,\dots$, and every $p \in \mathscr{P}$ define

$$\begin{aligned}\mathfrak{O}_{p,0} &= O_p(C_p)\\ \mathfrak{O}_{p,i+1} &= O_p\big([C_p \,/\, \mathfrak{O}_{1,i}G_1 \,/\, \mathfrak{O}_{2,i}G_2 \,/\, \cdots \,/\, \mathfrak{O}_{\mathtt{p},i}G_{\mathtt{p}}]\big)\end{aligned}$$

It is not difficult to see that $O_{p,i}$ consists of "rows" of the form $C_p A_p^{\alpha}$ for $p \in \mathscr{P}$ and $\alpha \in \{0,1,2,\dots,n-1\}$ or rows of the form

$$\boxed{C_{p_k} A_{p_k}^{\alpha_k} G_{p_k} A_{p_{k-1}}^{\alpha_{k-1}} G_{p_{k-1}} \cdots A_{p_1}^{\alpha_1} G_{p_1} A_p^{\alpha_0}}$$

where $1 \le k \le i$, $p_1,\dots,p_k \in \mathscr{P}$, $\alpha_0,\dots,\alpha_k \in \{0,1,\dots,n-1\}$. With similar arguments as in Section 0.5.3.1 it follows that

$$x_0 \in \ker \mathfrak{O}_{p,i} \quad \forall p \in \mathbb{N}\ \forall i \in \mathbb{N}$$

is equivalent to indistinguishability of x_0 from zero. Furthermore, it can be shown (taking into account that $O_p(C)$ is the largest A_p-invariant subspace contained in the kernel of C) that $\ker \mathfrak{O}_{p,i+1} \subseteq \ker \mathfrak{O}_{p,i}$. Hence we arrive at the following characterization of weakly controlled observability for switched systems (0.1).

Theorem 0.5.11 (Weakly controlled observability characterization, [26]). *The switched system* (0.1) *is weakly controlled observable if, and only if,*

$$\bigcap_{p\in\mathscr{P}} \bigcap_{i=0}^{\infty} \ker \mathfrak{O}_{p,i} = \{0\}.$$

Note that the condition of Theorem 0.5.11 can be checked numerically, see [23] for details.

Unfortunately, for the case with state jumps it is not yet clear whether weak controlled observability implies controlled observability. We conjecture that if the reset maps are invertible, then weak controlled observability implies controlled observability.

Similar to Remark 0.5.10, an observer reduction can be carried out. However, the resulting reduced switched system will have different state-space dimension and does not fit anymore into the framework considered here.

We conclude this section by presenting a possible connection between the algebraic condition for weakly controlled observability as established in Theorem 0.5.11 and the geometric observability conditions obtained in Section 0.5.2.

Conjecture 0.5.12. Consider the switched system (0.1). For $m > 0$ and the switching signal $\sigma : \mathbb{R} \to \mathscr{P}$ let $\underline{\mathscr{M}}_1^{\sigma,m}$ denote the analogue of space $\underline{\mathscr{M}}_1^m$ constructed in Corollary 0.5.6 with mode sequence $\sigma(t_1), \sigma(t_2), \ldots, \sigma(t_m)$ instead of $1, 2, \ldots, m$. Then $\ker \mathfrak{O}_{p,m}$ is the largest A_p-invariant subspace contained in $\underline{\mathscr{M}}_1^{\sigma,m}$ for all switching signals with $\sigma(0) = p$. $\triangleleft$

0.5.4 Forward observability

As mentioned earlier, observability deals with recovering the state trajectory of the system at all times. The weaker notion of forward observability is concerned with recovering the state on a certain interval of the form (T, ∞), and is particularly useful in designing observers for switched systems (c.f. Chapter **??** in this book). The geometric conditions for characterization of forward observability for system (0.1) (resp. (0.2)) can be obtained through parallel development. Towards that end, consider the following sequence of subspaces:

$$\boxed{\begin{aligned} \mathscr{N}^1 &:= J_1 \mathscr{M}^1, \\ \mathscr{N}^{i+1} &:= J_{i+1}(\mathscr{M}^{i+1} \cap e^{F_i \tau_i} \mathscr{N}^i), \quad i > 0, \end{aligned}} \qquad \begin{aligned} &(0.26a) \\ &(0.26b) \end{aligned}$$

where $F_i, J_i, \mathscr{M}^i$ are defined in (0.16) (resp. (0.17)).

The intuition behind this sequence of subspaces is as follows: The subspace $\mathscr{N}^i$ contains all forward unobservable states at the i-th switching instant where we use all the knowledge up to the i-th switching instant. At the next switching instant we propagate forward the information from $\mathscr{N}^i$ and intersect it with the locally unobservable subspace $\mathscr{M}^{i+1}$. Using then the jump map J_{i+1} gives the next forward unobservable subspace $\mathscr{N}^{i+1}$. This procedure is significantly different to the subspace iteration in (0.18) as the iterations do not proceed backward in time. We can now characterize forward observability with the help of the subspace iteration (0.26).

Theorem 0.5.13 (Forward observability characterization, [35, 33]). *Consider the switched system* (0.1) *(respectively* (0.2)*) with switching signal σ given by* (0.15)*. Then the forward unobservable space at the $m-th$ switch of σ is given by $\mathscr{N}^m$, i.e. for σ_m defined in* (0.19)*, it holds that*

$$\boxed{\mathscr{N}_{t_m^+}^{\sigma_m} = \mathscr{N}^m,}$$

where $\mathscr{N}^m$ is obtained using (0.16) *(resp.* (0.17)*) and* (0.26)*. In particular* (0.1) *(resp.* (0.2)*) is forward observable for σ if, and only if, there exists $m \in \mathbb{N}$ such that*

$$\mathscr{N}^m = \{0\}. \tag{0.27}$$

The conditions for forward observability independent of switching times could be developed similarly as in the previous section.

Corollary 0.5.14 (Sufficient condition for forward observability). *For system* (0.1) *(resp.* (0.2)*) with σ given by* (0.15) *and $m \in \mathbb{N}$ define the following sequence of subspaces:*

$$\begin{aligned} \overline{\mathscr{N}}^1 &:= J_1 \mathscr{M}^1, \\ \overline{\mathscr{N}}^{i+1} &:= J_{i+1}\left(\mathscr{M}^{i+1} \cap \left\langle F_i \,\middle|\, \overline{\mathscr{N}}^i \right\rangle\right), \qquad i > 1, \end{aligned}$$

where $F_i, J_i, \mathscr{M}^i$ are defined via (0.16) *(resp.* (0.17)*). Then,*

$$\overline{\mathscr{N}}^i \supseteq \mathscr{N}^i \quad \forall i \geq 1,$$

hence, system (0.1) *(resp.* (0.2)*) is forward observable for σ if there exists $m \in \mathbb{N}$ such that $\overline{\mathscr{N}}^m = \{0\}$.*

Corollary 0.5.15 (Necessary condition for forward observability). *For system* (0.1) *(resp.* (0.2)*) with σ given by* (0.15) *and $m \in \mathbb{N}$ define the following sequence of subspaces:*

$$\begin{aligned} \underline{\mathscr{N}}^1 &:= J_1 \mathscr{M}^1, \\ \underline{\mathscr{N}}^{i+1} &:= J_{i+1}\left(\mathscr{M}^{i+1} \cap \left\langle \underline{\mathscr{N}}^i \,\middle|\, F_i \right\rangle\right), \qquad i > 1, \end{aligned}$$

where $F_i, J_i, \mathscr{M}^i$ are defined via (0.16) *(resp.* (0.17)*). Then,*

$$\underline{\mathcal{N}}^i \subseteq \mathcal{N}^i \quad \forall i \geq 1,$$

hence if system (0.1) *(resp.* (0.2)*) is forward observable for σ then there exists $m \in \mathbb{N}$ such that* $\underline{\mathcal{N}}^m = \{0\}$.

0.6 Appendix

0.6.1 Some basic facts concerning linear algebra

Let $\mathcal{N}, \mathcal{M}$ be some linear subspaces of $\mathbb{R}^n$ and A, B matrices of suitable size. Then the following properties are easy to verify:

(i) $A^{-1}(\ker B) = \ker BA$
(ii) $A(A^{-1}(\mathcal{N})) = \mathcal{N} \cap \operatorname{im} A$
(iii) $A(\mathcal{N} \cap \mathcal{M}) \subseteq A\mathcal{N} \cap A\mathcal{M}$ with equality if, and only if (c.f. [47, Sec. 0.4]),

$$(\mathcal{N} + \mathcal{M}) \cap \ker A = \mathcal{N} \cap \ker A + \mathcal{M} \cap \ker A;$$

the latter holds, for example, if $\ker A \subseteq \mathcal{N}$.

0.6.2 The Wong sequences and the QWF

Consider a regular matrix pair (E,A). The Wong sequences [46, 1, 6] are defined as

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), \quad i \in \mathbb{N}, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i), \quad i \in \mathbb{N}. \end{aligned}$$

It is easily seen that the the Wong sequence are nested and terminate after finitely many steps:

$$\begin{aligned} &\mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_{k^*} = \mathcal{V}_{k^*+1} = \ldots \\ &\mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{\ell^*} = \mathcal{W}_{\ell^*+1} = \ldots \end{aligned}$$

Let $\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i = \mathcal{V}_{k^*}$ and $\mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}_i = \mathcal{W}_{\ell^*}$. It can be shown [7] that (E,A) is regular if, and only if,

$$\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n = E\mathcal{V}^* \oplus A\mathcal{W}^*. \tag{0.28}$$

In that case it also holds that $k^* = \ell^* = \nu$ is the index of the matrix pair (E,A). Let V, W be full (column) rank matrices such that $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$. Because of (0.28) the matrices

$$T := [V, W], \quad S := [EV, AW]^{-1} \tag{0.29}$$

are then invertible matrices.

Theorem 0.6.1 (QWF, [6]). *Consider a regular matrix pair (E,A) and the corresponding Wong sequences and (invertible) matrices S,T as in* (0.29). *Then*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

where N is nilpotent.

Remark 0.6.2. In the case of a singular matrix pair, the Wong sequences can also be used to obtain the *quasi Kronecker form*, see [7, 8]. $\triangleleft$

0.6.3 Distribution theory

We recall the basic definitions and properties of classical distributions as formalized by Schwartz [29]. The space of *test functions* (i.e., smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support) is denoted by $\mathscr{C}_0^\infty$, the *space of distributions* is the dual space of the space of test functions, i.e.

$$\mathbb{D} := \{ D : \mathscr{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous } \}.$$

Note that continuity requires a topology on the space of test function. However, in practice, the continuity is tested via sequential continuity: A linear map $D : \mathscr{C}_0^\infty \to \mathbb{R}$ is continuous if, and only if, the sequence of real numbers $D(\varphi_n)$ converges to zero as $n \to \infty$ for any sequence $(\varphi_n)_{n\in\mathbb{N}}$ of test function fulfilling the following two properties:

(i) The support of each φ_n is contained in a common compact set.
(ii) For each $i \in \mathbb{N}$ the sequence $\varphi_n^{(i)}$ converges uniformly to the zero function as $n \to \infty$.

The main two properties of distributions are 1) that they can be interpreted as generalized functions and 2) that they are arbitrarily often differentiable. To be more precise, let $\mathscr{L}_{1,\mathrm{loc}}$ be the space of locally integrable functions, then the mapping

$$\mathscr{L}_{1,\mathrm{loc}} \to \mathbb{D},\ f \mapsto f_\mathbb{D} := \left(\varphi \mapsto \int_\mathbb{R} f\varphi \right)$$

is well defined (i.e. $f_\mathbb{D}$ is indeed a distribution) and an injective homomorphism. The simplest distribution which is not induced by a function is the *Dirac impulse* given by $\delta(\varphi) := \varphi(0)$, or, in general for $t \in \mathbb{R}$, $\delta_t(\varphi) := \varphi(t)$ for $\varphi \in \mathscr{C}_0^\infty$. For $i \geq 1$, the i-th derivative of an arbitrary distribution $D \in \mathbb{D}$ is given by

$$D^{(i)}(\varphi) := -D^{(i-1)}(\varphi'), \quad \varphi \in \mathscr{C}_0^\infty,$$

where we take $D^{(0)}(\varphi) = D(\varphi)$. Distributions can be multiplied with smooth functions:

$$(\alpha D)(\varphi) := D(\alpha\varphi), \quad \alpha \in \mathscr{C}^\infty, D \in \mathbb{D}, \varphi \in \mathscr{C}_0^\infty.$$

Let $\mathscr{C}^\infty_{\text{pw}}$ be the space of piecewise-smooth function, where $\alpha : \mathbb{R} \to \mathbb{R}$ is called piecewise-smooth when there exists a locally finite ordered set $S = \{ s_i \in \mathbb{R} \mid i \in \mathbb{Z} \}$ and smooth functions $\alpha_i \in \mathscr{C}^\infty$, $i \in \mathbb{Z}$, such that $\alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[s_i,s_{i+1})}$. The space of *piecewise-smooth distributions* is then given by

$$\mathbb{D}_{\text{pw}\mathscr{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau \;\middle|\; \begin{array}{l} f \in \mathscr{C}^\infty_{\text{pw}}, T \subseteq \mathbb{R} \text{ locally finite,} \\ \forall \tau \in T : D_\tau \in \operatorname{span}\{\delta_\tau, \delta'_\tau, \delta''_\tau, \ldots\} \end{array} \right\}.$$

The properties of $\mathbb{D}_{\text{pw}\mathscr{C}^\infty}$ and corresponding definitions are summarized in the following, where $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau \in \mathbb{D}_{\text{pw}\mathscr{C}^\infty}$ and $t \in \mathbb{R}$:

(i) Closed under differentiation: $D' \in \mathbb{D}_{\text{pw}\mathscr{C}^\infty}$.
(ii) Left- and right-evaluation: $D(t^+) := f(t)$, $D(t^-) := f(t^-)$.
(iii) Impulsive part: $D[t] := D_t$ if $t \in T$, $D[t] = 0$ otherwise.
(iv) Restriction to interval: $D_I := (f_I)_{\mathbb{D}} + \sum_{\tau\in T\cap I} D_\tau$, where $I \subseteq \mathbb{R}$ is some interval.
(v) Multiplication with piecewise-smooth function: $\alpha D := \sum_{i\in\mathbb{Z}} \alpha_i D_{[s_i,s_{i+1})}$, where $\alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[s_i,s_{i+1})}$ as above; in particular, $\alpha\delta_t = \alpha(t)\delta_t$.

For more details see [37, 38].

0.6.4 Topology on the space of switching signals

Analogously as in [33] we define for $m \in \mathbb{N}$ the (pseudo-)metric

$$d_m(\sigma, \sigma') = \sum_{i=1}^{m-1} |\tau_i - \tau'_i|, \quad \sigma, \sigma' \in \Sigma_{\mathbb{N}},$$

where $\tau_i := t_{i+1} - t_i > 0$ and $\tau'_i = t'_{i+1} - t'_i > 0$ and t_i, t'_i are the switching times of σ and σ', respectively. For each $m \in \mathbb{N}$ this metric induces a topology on $\Sigma_{\mathbb{N}}$ which is isomorphic to the usual topology of $\mathbb{R}^{m-1}$ restricted to the open positive orthant.

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Index

Index