On observability of switched differential-algebraic equations

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Abstract— **We investigate observability of switched differential algebraic equations. The article primarily focuses on a class of switched systems comprising of two modes and a switching signal with a single switching instant. We provide a necessary and sufficient condition under which it is possible to recover the value of state trajectory (globally in time) with the help of switching phenomenon, even though the constituent subsystems may not be observable. In case the switched system is not globally observable, we discuss the concept of forward observability which deals with the recovery of state trajectory after the switching. A necessary and sufficient condition that characterizes forward observability is presented.**

I. INTRODUCTION

In this paper, we consider the *switched differential algebraic equations (switched DAEs)* of the following form:

$$
\begin{aligned}
E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\
y &= C_\sigma x
\end{aligned}
\tag{1}
$$

where $\sigma : \mathbb{R} \to \{1, \cdots, P\}$ is the switching signal, P is the number of subsystems, and $E_p, A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times r}$, $C_p \in \mathbb{R}^{m \times n}$, for $p \in \{1, \cdots, P\}$. In general, a switched DAE exhibits jumps (or even impulses) in the solution, hence it cannot be expected that classical solutions exists; therefore we adopt the *piecewise-smooth distributional solution framework* introduced in [1], i.e. the state x and the external signals u and y are assumed to be piecewise-smooth distributions. We study *observability* of the switched DAE (1) where we call (1) observable when the knowledge of the external signals, σ, u and y, allow for a unique reconstruction of the state x.

DAEs arise naturally in modeling physical systems where the state variables satisfy certain algebraic constraints alongside some differential equations that govern the evolution of these state variables. It is a common practice to eliminate the algebraic constraints to arrive at a system description given by ordinary differential equations (ODEs). However, these eliminations are in general different for each subsystem of a switched system, hence a description as a switched ODE with *common* state variables is in general not possible. This problem can be overcome by studying the switched DAE (1) directly. An application of DAEs in electrical circuits with distributional solutions has also been studied in [2].

Because of their utility in modeling and control design, switched ODEs have been studied actively during the past couple of decades and several results related to their structural properties such as stability (see [3] for references), controllability [4], [5], observability [6], [7], and invertibility [8], [9] have been published. For switched DAEs however, such structural properties have not been investigated in much detail. Results on stability of switched DAEs have been published in [10], and the only ones (to the best knowledge of the authors) related to controllability and observability are reported in [11] and [12], respectively.

In the non-switched case, observability of DAEs has been studied by [13], [14]. As pointed out in [1, Thm. 5.2.5], the observability definitions from [13], [14] can be characterized by certain *pointwise* observability definition if the problem is embedded into the piecewise-smooth distributional framework. Hence, the non-switched framework discussed so far only focuses on pointwise observability. This is very different from the approach adopted in the switched framework because the switch itself might provide more information about the state trajectory. So, even if the individual subsystems are not observable pointwise in time, it may be possible to achieve global observability due to switching.

Our approach for solving the problem of observability of switched DAEs is in principal different to the existing approach of [12]. In [12], a switched DAE is considered observable if there exists at least one switching signal that makes it possible to recover the state trajectory. In our approach, we consider the switching signal to be known and fixed which makes the system time-varying. For this time-varying system, we answer the question whether it is possible to recover the state trajectory.

The first result discussed in this paper provides a complete characterization for global observability of a switched DAE with two subsystems where the switching signal is restricted to comprise of a single switching instant. The distributional framework allows us to incorporate the knowledge provided by the jump and the impulsive part of the output for obtaining information about the state trajectory. If it is not possible to recover the value of the state trajectory at all times, a weaker characterization is provided for forward observability, where we only aim to recover the state trajectory after the switching instant. Moreover, the observability conditions are given in terms of *differential* and *impulse* projectors, which present a novel concept of characterizing impulses and derivatives of state trajectories. The definition of these projectors not only makes the development of results parallel to the ODE case but also leads to conditions that are easily verifiable in terms of original system matrices.

The outline of this paper is as follows: notations and results for non-switched DAEs are presented in Section II. The main results on global and forward observability appear in Section III, followed by brief discussion on extending results to switched systems with multiple modes and general switching signals in Section IV. Conclusions and discussions on future work are given in Section V.

II. PRELIMINARIES

A. Properties and definitions for regular matrix pairs

In the following, we collect important properties and definitions for matrix pairs (E, A). We only consider *regular* matrix pairs, i.e. for which the polynomial $\det(sE - A)$ is not the zero polynomial. A very useful characterization of regularity is the following well-known result.

Proposition 1 (Regularity and quasi-Weierstrass form): A matrix pair $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n \times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{2}$$

where $J \in \mathbb{R}^{n_1 \times n_1}$, $0 \leq n_1 \leq n$, is some matrix and $N \in \mathbb{R}^{n_2 \times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix. $\triangleleft$

In view of [15], we call the decomposition (2) *quasi-Weierstrass form*. An easy way to calculate the transformation matrices S and T for (2) is to use the following so-called *Wong sequences* [15], [16][1]:

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad i = 0, 1, \cdots$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \qquad i = 0, 1, \cdots$$

The Wong sequences are nested and get stationary after finitely many steps. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \quad \mathcal{W}^* := \bigcup_i \mathcal{W}_i.$$

For any full rank matrices V, W with $\text{im}\, V = \mathcal{V}^*$ and $\text{im}\, W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (2) holds.

Based on the Wong-sequences we define the following "projectors".

Definition 2 (Consistency, differential and impulse projectors): Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (2). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential projector* is given by

$$\Pi^{\text{diff}}_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

[1]The definition of the subspace sequence $\mathcal{V}_i$ can be traced back to [17], however the use of both subsequences $\mathcal{V}_i$ and $\mathcal{W}_i$ together in the context of matrix pairs (E, A) seems to be studied first by Wong in [16].

and the *impulse projector* is given by

$$\Pi^{\text{imp}}_{(E,A)} = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S. \qquad \triangleleft$$

Note that only the consistency projector is a projector in the usual sense (i.e. $\Pi_{(E,A)}$ is an idempotent matrix). The differential and impulse projectors are not projectors in the usual sense, because, in general, $\Pi^{\text{diff}}_{(E,A)} \Pi^{\text{diff}}_{(E,A)} \neq \Pi^{\text{diff}}_{(E,A)}$ and the same holds for $\Pi^{\text{imp}}_{(E,A)}$. Let

$$\mathfrak{C}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \mid \exists x \in \mathcal{C}^1 : E\dot{x} = Ax \wedge x(0) = x_0 \right\}$$

be the *consistency space* of the DAE $E\dot{x} = Ax$, where $\mathcal{C}^1$ is the space of differentiable functions $x : \mathbb{R} \to \mathbb{R}^n$. Then the following observations hold [15]:

1) All solutions $x \in \mathcal{C}^1$ of $E\dot{x} = Ax$ evolve within $\mathfrak{C}_{(E,A)}$,
2) $\mathfrak{C}_{(E,A)} = \mathcal{V}^*$, i.e. the first Wong-sequence converges to the consistency space,
3) $\text{im}\, \Pi_{(E,A)} = \mathcal{V}^* = \mathfrak{C}_{(E,A)}$, hence the consistency projector maps onto the consistency space.

The following lemma motivates the name of the differential projector.

Lemma 3: Consider the DAE $E\dot{x} = Ax$ with regular matrix pair (E, A). Then any solution $x \in \mathcal{C}^1$ of $E\dot{x} = Ax$ fulfills

$$\dot{x} = \Pi^{\text{diff}}_{(E,A)} Ax =: A^{\text{diff}} x.$$

Proof: Let the variables in the quasi-Weierstrass form (2) be denoted by v and w, i.e. $x = T \left(\begin{smallmatrix} v \\ w \end{smallmatrix} \right)$. Using the fact that all solutions evolve within the consistency space, we obtain $w = \dot{w} = 0$, and hence

$$\dot{x} = T \begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = T \begin{pmatrix} Jv \\ 0 \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$
$$= T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1} T \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T^{-1} T \begin{pmatrix} v \\ w \end{pmatrix}$$
$$= \Pi_{(E,A)} TSAx = \Pi^{\text{diff}}_{(E,A)} Ax$$

∎

For studying impulsive solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from [18] as the solution space, that is, we seek a solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ to the following initial-trajectory problem (ITP):

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax)_{[0,\infty)}, \end{aligned} \tag{3}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_{\mathcal{I}}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In [18], it is shown that the ITP (3) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. It is also shown there, that the ITP for the *pure DAE* $N\dot{w} = w$, where $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix, has the unique solution

$$w = \sum_{i=0}^{n_2-1} \left(N_{[0,\infty)} \tfrac{\text{d}}{\text{dt}} \right)^i (w^0_{(-\infty,0)}).$$

Using the calculus of piecewise-smooth distributions, the expression for the impulsive part of w at $t = 0$, denoted by $w[0]$, is obtained as follows:

$$w[0] = -\sum_{i=0}^{n_2-2} N^{i+1} w^0(0-)\delta_0^{(i)} = \sum_{i=0}^{n_2-2} N^{i+1}\Delta_0(w)\delta_0^{(i)},$$

where $\delta_0^{(i)}$ denotes the i-th derivative of the Dirac-impulse at zero and $\Delta_0(w) := w(0+) - w(0-)$. To express the impulsive part of the distributional solution x of the ITP (3), we need the impulse projector:

Lemma 4 (Impulses): Consider the ITP (3) with regular matrix pair (E, A) and corresponding impulse projector $\Pi_{(E,A)}^{\mathrm{imp}}$ with rank $n_2 \in \mathbb{N}$. Let $E^{\mathrm{imp}} := \Pi_{(E,A)}^{\mathrm{imp}}E$, then any solution $x \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ of (3) fulfills

$$x[0] = \sum_{i=0}^{n_2-2} (E^{\mathrm{imp}})^{i+1}\Delta_0(x)\delta_0^{(i)}.$$

Proof: First note that all solutions $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n_1}$ of the ITP for the ODE $\dot{v} = Jv$ fulfill $v[0] = 0$ and $\Delta_0(v) = 0$, hence

$$x[0] = T\begin{pmatrix} v[0] \\ w[0] \end{pmatrix} = T\sum_{i=0}^{n_2-2} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & N^{i+1} \end{bmatrix}\begin{pmatrix} \Delta_0(v) \\ \Delta_0(w) \end{pmatrix}\delta_0^{(i)}$$

$$= \sum_{i=0}^{n_2-2} T\begin{bmatrix} 0 & 0 \\ 0 & N^{i+1} \end{bmatrix}T^{-1}\Delta_0(x)\delta_0^{(i)}$$

$$= \sum_{i=0}^{n_2-2} (\Pi_{(E,A)}^{\mathrm{imp}}E)^{i+1}\Delta_0(x)\delta_0^{(i)},$$

where the last equality follows from the fact that $T\begin{bmatrix} 0 & 0 \\ 0 & N^i \end{bmatrix}T^{-1} = \left(T\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}SS^{-1}\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}T^{-1}\right)^i$, and $E = S^{-1}\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}T^{-1}$. ∎

Since the consistency projector specifies the jump from $x(0-)$ to $x(0+)$ for any solution $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ of the ITP (3), we have the following corollary.

Corollary 5: From the notation in Lemma 4 and the corresponding consistency projector $\Pi_{(E,A)}$, it follows that

$$x[0] = \sum_{i=0}^{n-1} (E^{\mathrm{imp}})^{i+1}(\Pi_{(E,A)} - I)x(0-)\delta_0^{(i)}. \quad \lhd$$

III. OBSERVABILITY OF SWITCHED DAE

The concepts introduced in the previous section are now utilized to obtain necessary and sufficient conditions for global and forward observability. In order to use the piecewise-smooth distributional solution framework and to avoid technical difficulties in general, we only consider switching signals from the following class, for $P \in \mathbb{N}$,

$$\Sigma_P := \left\{ \sigma : \mathbb{R} \to \{1, \cdots, P\} \,\middle|\, \begin{array}{l} \sigma \text{ is right continuous with a} \\ \text{locally finite number of jumps} \end{array} \right\},$$

i.e. we exclude an accumulation of switching times. In fact, for our main results the switching signal considered is further restricted to the simplest non-trivial switching given by

$$\sigma(t) = 1 \text{ for } t < 0 \quad \text{and} \quad \sigma(t) = 2 \text{ for } t \geq 0. \quad (4)$$

However the definition of observability (Definition 6), some preliminary results (Proposition 7) and further discussions (Section IV) consider the general switching signal $\sigma \in \Sigma_P$.

A. Global Observability

For switched systems, instead of pointwise observability, we adopt the notion of global obervability in order to extract information from the switching.

Definition 6 (Global observability): The switched DAE (1) with some fixed switching signal $\sigma \in \Sigma_P$, $P \in \mathbb{N}$, is called *(globally) observable* if, and only if, for every pair of inputs and outputs $(y, u) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{m+r}$ there exists at most one $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ which solves (1). $\lhd$

The following proposition is going to be helpful in developing the main result.

Proposition 7 (Observability of zero): The switched DAE (1) is observable if, and only if, $y \equiv 0$ and $u \equiv 0$ implies $x \equiv 0$.

Proof: Necessity is obvious. Assume now that (1) is not observable, hence there exists an external signal (y, u) for which there exist different solutions $x_1, x_2 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ of (1). By linearity, it follows that $x = x_1 - x_2 \neq 0$ solves $E_\sigma\dot{x} = A_\sigma x$ and $C_\sigma x = C_\sigma x_1 - C_\sigma x_2 = y - y = 0$, hence $y \equiv 0$ and $u \equiv 0$ does not imply $x \equiv 0$. ∎

The above result justifies that we can ignore the input when studying observability of (1); hence in what follows, the following homogeneous switched DAE is considered:

$$E_\sigma\dot{x} = A_\sigma x, \quad y = C_\sigma x. \quad (5)$$

Furthermore, we restrict our attention in the remainder of the section to the special switching signal given by (4), i.e. we only consider one switch from some initial subsystem given by $(C_-, E_-, A_-) := (C_1, E_1, A_1)$ – active before the switch – to some other subsystem given by $(C_+, E_+, A_+) := (C_2, E_2, A_2)$ that is active after the switch. Denote the corresponding consistency projectors by Π_-, Π_+ and analogues for the differential and impulse projectors. Let $\mathfrak{C}_\pm := \mathfrak{C}_{(E_\pm, A_\pm)}$ be the consistency spaces of the corresponding subsystems, then $y \equiv 0$, in particular $y^{(i)}(0\pm) = 0$ for all $i \in \mathbb{N}$, together with Lemma 3 implies

$$x(0-) \in \mathfrak{C}_- \cap \bigcap_{i\in\mathbb{N}} \ker C_-(\Pi_-^{\mathrm{diff}}A_-)^i,$$

$$x(0+) \in \mathfrak{C}_+ \cap \bigcap_{i\in\mathbb{N}} \ker C_+(\Pi_+^{\mathrm{diff}}A_+)^i.$$

Define the observability matrices

$$O_\pm := [C_\pm / C_\pm A_\pm^{\mathrm{diff}} / C_\pm (A_\pm^{\mathrm{diff}})^2 / \cdots / C_\pm (A_\pm^{\mathrm{diff}})^{n-1}], \quad (6)$$

where $A_\pm^{\mathrm{diff}} := \Pi_\pm^{\mathrm{diff}}A_\pm$ and $[M_1/M_2] := \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ for any two matrices M_1, M_2 of suitable sizes. Invoking the Cayley-Hamilton-Theorem, see e.g. [19, Thm. X.2.3], the above conditions can be rewritten as

$$x(0-) \in \mathfrak{C}_- \cap \ker O_- \quad \text{and} \quad x(0+) \in \mathfrak{C}_+ \cap \ker O_+$$

Invoking regularity of the matrix pairs $(E_\pm, A_\pm)$, a sufficient condition for observability of (5) is that $\mathfrak{C}_- \cap \ker O_- = \{0\}$, but the following simple example shows that this condition is not necessary.

Example 8: Consider $(C_-, E_-, A_-) = (0, 1, 0)$ and $(C_+, E_+, A_+) = (1, 1, 0)$ which reads as $\dot{x} \equiv 0$ with

output $y \equiv 0$ on $(-\infty, 0)$ and $y \equiv x$ on $[0, \infty)$. Although $\mathfrak{C}_- \cap \ker O_- = \mathbb{R}$, the switched DAE is observable. $\triangleleft$

On the other hand, the condition $\mathfrak{C}_+ \cap \ker O_+ = \{0\}$ is not sufficient for observability, because in general $x(0+) = 0$ does not imply $x(0-) = 0$. A characterization of observability has to take into account the possible jumps from $x(0-)$ to $x(0+)$ as well as the induced impulses $x[0]$. Using the additional information $y[0] = 0$ and $y(0+) = 0$ we can find stronger sufficient conditions for observability. These and the above sufficient conditions can be summarized as follows:

1) In general, $x(0-) \in \mathfrak{C}_-$, hence $\mathfrak{C}_- = \{0\}$ is sufficient for observability.
2) If $y^{(i)}(0-) = 0$ for all $i \in \mathbb{N}$, then $x(0-) \in \ker O_-$, hence $\ker O_- = \{0\}$ is sufficient for observability.
3) If $y^{(i)}(0+) = 0$ for all $i \in \mathbb{N}$, then $x(0+) \in \ker O_+$, together with $x(0+) = \Pi_+ x(0-)$, implies that $x(0-) \in \Pi_+^{-1} \ker O_+ = \ker(O_+ \Pi_+)$, hence $\ker(O_+ \Pi_+) = \{0\}$ is sufficient for observability.
4) If $y[0] = 0$, then Lemma 4 implies that $\Delta_0 x \in \ker[C_+ E_+^{\mathrm{imp}} / C_+ (E_+^{\mathrm{imp}})^2 / \cdots / C_+ (E_+^{\mathrm{imp}})^{n_2-1}]$ and Corollary 5 yields $x(0-) \in \ker[C_+ E_+^{\mathrm{imp}} / C_+ (E_+^{\mathrm{imp}})^2 / \cdots / C_+ (E_+^{\mathrm{imp}})^{n_2-1}](\Pi_+ - I)$; a sufficient condition for observability is therefore that the latter kernel is trivial.

Of course, the condition that the intersection of the above mentioned four "unobservable" subspaces for $x(0-)$ be trivial is another sufficient condition encompassing all four from above. Actually it turns out, that this condition is also necessary.

Theorem 9 (Characterization of observability): Consider the switched DAE (1) with the switching signal given by (4). Use the notation $O_\pm$ as given by (6), let $O_+^- := O_+ \Pi_+$,

$$O_+^{\mathrm{imp}} := [C_+ E_+^{\mathrm{imp}} / C_+ (E_+^{\mathrm{imp}})^2 / \cdots / C_+ (E_+^{\mathrm{imp}})^{n-1}],$$

where $E_+^{\mathrm{imp}} := \Pi_+^{\mathrm{imp}} E_+$, and let $O_+^{\mathrm{imp}-} := O_+^{\mathrm{imp}} (\Pi_+ - I)$. Then (1) is observable if, and only if,

$$\boxed{\{0\} = \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathrm{imp}-}.} \tag{7}$$

Proof: Because of Proposition 7, it suffices to consider (5) with zero output. The sufficiency part was already shown above, we now prove the necessity part.

Let $0 \neq x_0 \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathrm{imp}-}$, then by regularity of the switched DAE (5) there exists a unique, non-trivial solution $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ of (5) with $x(0-) = x_0$. From $x(0-) \in \ker O_-$ it follows that $y^{(i)}(0-) = 0$ for all $i \in \mathbb{N}$, and hence by analyticity of y on $(-\infty, 0)$ it follows that $y \equiv 0$ on $(-\infty, 0)$. Corollary 5 and $x(0-) \in \ker O_+^{\mathrm{imp}-}$ imply that $y[0] = 0$. Finally, $x(0-) \in \ker O_+^-$ implies that $x(0+) = \Pi_+ x(0-) \in \ker O_+$, hence $y^{(i)}(0+) = 0$ for all $i \in \mathbb{N}$ and hence $y \equiv 0$ on $(0, \infty)$. Altogether this shows that there exists a nontrivial solution x with zero output, hence the switched DAE (5) is not observable. $\blacksquare$

The following corollary is an immediate consequence of Theorem 9 (in particular the necessity part of the proof).

Corollary 10: The subspace $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathrm{imp}-}$ is the *unobservable subspace*, i.e. for every solution x of (5) it holds that $x(0-) \in \mathcal{M}$ if, and only if, the corresponding output is zero. $\triangleleft$

Remark 11 (The switched ODE special case): If the system in (1) is a switched ODE with $E_\pm = I_{n \times n}$, then $\mathfrak{C}_\pm = \mathbb{R}^n, \Pi_\pm = I_{n \times n}, \ker O_+^{\mathrm{imp}-} = \mathbb{R}^n$, and the condition (7) reduces to

$$\{0\} = \ker O_- \cap \ker O_+.$$

This result also appears in [6] as a sufficient condition for observability of switched ODEs. However, for the class of switching signals considered in this paper, this condition is also necessary. $\triangleleft$

Remark 12 (Order of subsystems important): The condition (7) is not symmetric, i.e. observability of the switched system (1) with the switching signal (4) does not, in general, imply observability of (1) with the reversed mode sequence. This is in stark contrast to results on switched ODEs, which are in general symmetric [6]. The underlying reason for this difference is the presence of jumps in the solutions of switched DAEs. Consider for example $(C_-, E_-, A_-) = (1, 0, 1)$ and $(C_+, E_+, A_+) = (0, 1, 0)$ which reads as $y \equiv x \equiv 0$ on $(-\infty, 0)$ and $\dot{x} = 0$ with $y \equiv 0$ on $[0, \infty)$. Hence the unique solution is given by $x \equiv 0$, which makes the switched DAE trivially observable. The converse switching signal, i.e. switching from (C_+, E_+, A_+) to (C_-, E_-, A_-), yields an unobservable switched DAE because the jump at zero "destroys" all information from the past. $\triangleleft$

Example 13: Consider a switched DAE with two modes:

$$\Gamma_- : \begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x \end{cases}$$

$$\Gamma_+ : \begin{cases} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} x \end{cases}$$

Neither subsystem is observable in the classical sense. But we show that because of switching, it is possible to determine the exact value of the state trajectory. The consistency, differential, and impulse projector for each of these subsystems are: $\Pi_- = \Pi_-^{\mathrm{diff}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\Pi_-^{\mathrm{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\Pi_+ = \Pi_+^{\mathrm{diff}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\Pi_+^{\mathrm{imp}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $O_- = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$O_+ = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, and the subspaces indicated in Theorem 9 are:

$$\mathfrak{C}_- = \mathrm{span}\{e_1, e_2, e_4\}, \qquad \ker O_- = \mathrm{span}\{e_1, e_2, e_3\},$$

$$\ker O_+^- = \mathrm{span}\{e_1, e_3, e_4\}, \quad \ker O_+^{\mathrm{imp}-} = \mathrm{span}\{e_2, e_3, e_4\}.$$

where $e_i \in \mathbb{R}^4$, $i = 1, 2, 3, 4$, is the corresponding natural basis vector.

Clearly, $\mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathrm{imp}-} = \{0\}$ and the switched system is observable according to Theorem 9. Note that each of the four subspaces $\mathfrak{C}_-$, $\ker O_-$, $\ker O_+^-$ and $\ker O_+^{\mathrm{imp}-}$ is necessary to obtain a trivial intersection. If even one of them is not taken into account, then the intersection would be nontrivial. In fact, each of the subspaces restricts exactly one state variable. In view of Remark 12, note that the switched system with subsystem Γ_+ active on $(-\infty, 0)$ and Γ_- active on $[0, \infty)$, is not observable because (with the corresponding notation)

$$\{0\} \neq \mathfrak{C}_+ \cap \ker O_+ \cap \ker O_-^+ \cap \ker O_-^{\mathrm{imp}+}.$$

As an illustration of constructing state trajectories from the knowledge of the output and the input, let us consider an input[2], $u(t) = e^{2t} + \delta_{-1} + \delta_0$, and assume that the following output is produced by the system with $\sigma(t)$ specified in (4):

$$y(t) = \begin{cases} -1, & t \in (-\infty, -1) \\ 0, & t \in [-1, 0) \\ e^t + e^{2t} + \delta_0, & t \in [0, \infty) \end{cases}$$

The closed form solution for the state variables, parameterized by $a, b, c \in \mathbb{R}$, is given as follows:

$$x_1(t) = \begin{cases} -e^{t+1} + (a-1)e^t + e^{2t}, & t \in (-\infty, -1), \\ (a-1)e^t + e^{2t}, & t \in [-1, 0), \\ 0, & t \in [0, \infty), \end{cases}$$

$$x_2(t) = \begin{cases} e^t b, & t \in (-\infty, 0), \\ e^t + e^{2t} + (b-1)e^t, & t \in [0, \infty), \end{cases}$$

$$x_3(t) = \begin{cases} -e^{2t} - \delta_{-1}, & t \in (-\infty, 0), \\ -e^t + e^{2t}, & t \in [0, \infty), \end{cases}$$

$$x_4(t) = \begin{cases} \frac{1}{2}e^{2t}c - 1, & t \in (-\infty, -1), \\ \frac{1}{2}e^{2t}c, & t \in [-1, 0), \\ -a\delta_0, & t \in [0, \infty). \end{cases}$$

First note that $x_3(0-) = -1$, which corresponds to the fact that in the homogeneous case the consistency space $\mathfrak{C}_-$ restricts $x_3(0-)$ to be zero. Since O_- restricts $x_4(0-)$, we would expect that $y(0-)$, $\dot{y}(0-)$, ..., determine $x_4(0-)$. In fact, $0 = y(0-) = x_4(0-)$. The space O_+^- restricts $x_2(0-)$, and hence by using the values for $y^{(i)}(0+)$, we are able to reconstruct $x_2(0-)$: $2 = y(0+) = x_2(0+) + x_4(0+) = 1 + b = 1 + x_2(0-)$, i.e. $x_2(0-) = 1$. Finally, $O_+^{\mathrm{imp}-}$ restricts $x_1(0-)$, therefore, the information from the impulse of y at zero can be used to determine $x_1(0-)$: $\delta_0 = y[0] = x_2[0] + x_4[0] = -a\delta_0$, hence $-1 = a = x_1(0-)$. Altogether, we were able to determine $x(0-)$ which together with the knowledge of u and the regularity of the matrix pairs $(E_\pm, A_\pm)$ makes it possible to uniquely reconstruct the whole state x. ◁

B. Forward Observability

In Theorem 9, we derived conditions which restrict $x(0-)$ to a single point. Since there are no switches over the

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interval $(-\infty, 0)$, we have a unique solution over the interval $(-\infty, 0)$. Regularity assumption guarantees that the solution is also well defined over the interval $[0, \infty)$. It is possible that (7) does not hold for a given system but $x_{(0,\infty)}$ could still be determined uniquely from the knowledge of the output (in fact, from the knowledge of $y^{(i)}(0-)$, $y[0]$ and $y^{(i)}(0+)$, $i \in \mathbb{N}$). This motivates the following definition.

Definition 14 (Forward observability): The switched DAE (1) with the switching signal given by (4) is called *forward observable* if, and only if, for every pair of triplets (x_1, u_1, y_1), $(x_2, u_2, y_2) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n+r+m}$ which solve (1), the implication $(u_1, y_1) = (u_2, y_2) \Rightarrow x_{1_{(0,\infty)}} = x_{2_{(0,\infty)}}$ holds. ◁

Remark 15 (Backward observability): Note that backward observability (analogously defined as forward observability above) is equivalent to global observability. The reason is that, by regularity of the corresponding matrix pairs, knowledge of $x(0-)$ yields knowledge of $x(0+)$. In particular, backward observability implies forward observability, but the converse is not true in general. ◁

As an illustration of a system which is forward observable but not globally observable, consider the following example:

Example 16: Let $(E_-, A_-, C_-) = (I_{2\times 2}, I_{2\times 2}, \begin{bmatrix} 1 & 0 \end{bmatrix})$, and $(E_+, A_+, C_+) = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, I_{2\times 2}, \begin{bmatrix} 0 & 1 \end{bmatrix})$. Condition (7) does not hold, so $x(0-)$ can not be determined from the output. But $x(0+) = \begin{bmatrix} y(0-) & 0 \end{bmatrix}^\top$ is completely specified by the output. Consequently, $x_{(0,\infty)}$ can be determined uniquely for this switched system. ◁

Proposition 17 (Forward observability of zero): The switched DAE (1) is forward observable if, and only if, $y \equiv 0$ and $u \equiv 0$ implies $x_{(0,\infty)} \equiv 0$. ◁

The proof is analogous to that of Proposition 7.

The following result is derived from Theorem 9 and gives a characterization for systems that are forward observable.

Corollary 18: Consider the switched DAE (1) with the switching signal given by (4). Then (1) is forward observable if, and only if,

$$\Pi_+(\mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathrm{imp}-}) = \{0\}. \quad (8)$$

Remark 19: For subspaces $\mathcal{R}_1$, $\mathcal{R}_2$, and a linear map Π, $\Pi(\mathcal{R}_1 \cap \mathcal{R}_2) = \Pi(\mathcal{R}_1) \cap \Pi(\mathcal{R}_2)$ if, and only if

$$(\mathcal{R}_1 + \mathcal{R}_2) \cap \ker \Pi = \mathcal{R}_1 \cap \ker \Pi + \mathcal{R}_2 \cap \ker \Pi.$$

Using this and the fact that $\ker \Pi_+ \subseteq \ker O_+ \Pi_+$, condition (8) can be simplified to

$$\Pi_+(\mathfrak{C}_- \cap \ker O_- \cap \ker O_+^{\mathrm{imp}-}) \cap \ker O_+ = \{0\}. \quad ◁$$

IV. SYSTEMS WITH MULTIPLE SWITCHING INSTANTS

So far, we have studied switched systems with single switching instant. For switched systems with more than two subsystems and multiple switchings, if condition (7) holds for each pair of subsystems, then the switched system is globally observable because, in this case, $x(\tau+)$ and $x(\tau-)$ are determined uniquely at the first switching instant τ; forward and backward propagation of these values with known switching signal then specifies the state trajectory globally. But this is only a sufficient condition. To obtain conditions

for observability of systems with multiple switching instants, that are both necessary and sufficient, and depend only on the switching sequence, is not so straightforward. We motivate the discussion with the help of an example.

Example 20: Consider a bimodal switched system with the following matrices: $E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Generalizing the notations defined earlier in a straightforward way, it can be verified that

$$\mathfrak{C}_1 = \text{span}\{e_2\}; \qquad \ker O_1 = \text{span}\{e_1, e_2\}; \quad \ker O_2^1 = \mathbb{R}^3;$$
$$\mathfrak{C}_2 = \text{span}\{e_1, e_2\}; \quad \ker O_2 = \text{span}\{e_1, e_2\}; \quad \ker O_1^2 = \mathbb{R}^3.$$

Similarly, $\ker O_2^{\text{imp1}} = \mathbb{R}^3$ and $\ker O_1^{\text{imp2}} = \text{span}\{e_2, e_3\}$. It follows that:

$$\mathfrak{C}_1 \cap \ker O_1 \cap \ker O_2^1 \cap \ker O_2^{\text{imp1}} = \text{span}\{e_2\} \neq \{0\}$$
$$\mathfrak{C}_2 \cap \ker O_2 \cap \ker O_1^2 \cap \ker O_1^{\text{imp2}} = \text{span}\{e_2\} \neq \{0\}$$

So condition (7) does not hold when there is a switching from Γ_1 to Γ_2. Condition (7) is also violated if there is a switching from Γ_2 to Γ_1.

Now consider the following switching signal:

$$\sigma(t) = \begin{cases} 1, & t \in (-\infty, 0) \\ 2, & t \in [0, \frac{\pi}{2}) \\ 1, & t \in [\frac{\pi}{2}, \infty) \end{cases}$$

We claim that this switching signal makes the system observable. Because of Proposition 7, it suffices to show that $y \equiv 0$ can only be produced by $x \equiv 0$. The closed form solution of the state trajectories, parameterized by a scalar a, is given as follows:

$$x_1(t) = a \sin t \cdot \mathbb{1}_{[0, \frac{\pi}{2})}$$
$$x_2(t) = ae^{2t} \cdot \mathbb{1}_{(-\infty, 0)} + a \cos t \cdot \mathbb{1}_{[0, \frac{\pi}{2})}$$
$$x_3(t) = -a\delta_{\frac{\pi}{2}}$$

where $\mathbb{1}_{\mathcal{I}}$ denotes the indicator function of the interval $\mathcal{I}$. For an identically zero output, the impulsive part of the output at second switching instant yields $a = x_2(0-) = 0$ and this makes $x(t) = 0, \forall t$. $\triangleleft$

As shown in the above example, even though the individual switchings between the subsystems do not make the system observable, a combination of multiple switches make the system observable. Even switched ODEs exhibit this phenomenon. Deriving characterizations for this general case is an interesting problem and further work is being pursued in this direction.

V. CONLUSIONS

This article addressed observability of switched DAEs with two subsystems where there is only a single switching instant. Necessary and sufficient conditions for global and forward observability are presented. These characterizations are formulated in terms of consistency projectors and the newly introduced differential and impulse projectors which are obtained by utilizing the so called Wong sequences.

As a future direction of research, a natural extension is to obtain somewhat similar characterization for the systems with more than two subsystems and the switching signals belonging to a larger class. We expect results which generically only depend on the sequences of values of the switching signal and not on the switching times. It would also be interesting to investigate the problem of state recovery when the switching signal is unknown. Finally, the construction of observers for switched DAEs could be a potential application of the results derived in this paper.

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