Distributional solution theory for linear DAEs

Stephan Trenn[*]

Institute for Mathematics, Ilmenau University of Technology, Weimarer Str. 25, 98693 Ilmenau, DE

A solution theory for switched linear differential-algebraic equations (DAEs) is developed. To allow for non-smooth coordinate transformation, the coefficients matrices may have distributional entries. Since also distributional solutions are considered it is necessary to define a suitable multiplication for distribution. This is achieved by restricting the space of distributions to the smaller space of piecewise-smooth distributions. Solution formulae for two special DAEs, distributional ordinary differential equations (ODEs) and pure distributional DAEs, are given.

1 Introduction

Differential algebraic equations (DAEs) are equations of the form

$$Ex' = Ax + v,$$

where in general E and A are rectangular matrices. These kind of equations arise for example when studying electrical circuits, mechanical or chemical systems (see e.g. [1, I.1.3]), in particular when these models are generated automatically. In this note the standard assumptions for linear DAEs are weakend in the following sense:

(i) The coefficient matrices E and A are time variant, but not necessarily continuous, jumps are allowed.

(ii) The inhomogeneity v can exhibit jumps as well.

(iii) Inconsistent initial values are allowed.

The assumption (i) has its motivation in *switched system*, which appear where structural changes may occur, e.g. when some components of the system fail or when certain components are switched on or off. For an overview on switched systems and for further motivation, see e.g. [2]. Possible switching in an input signal yields assumption (ii). If the complete system is switched on at a certain time one cannot assume that the initial value is consistent, this yields assumption (iii). The aim of this note is to give first results of a solution theory which can deal with the above assumptions or in other word a *solution theory for switched linear DAEs*.

Distributional solutions for linear DAEs were considered already in [3] or [4], mainly to deal with inconsistent initial values. Hence a solution theory for switched linear DAEs must encompass distributions as possible solutions. Distributional solutions are also unavoidable if one allows for jumps in the inhomogeneity and no further assumptions on the "index" of the DAE is made (see e.g. [1, Remark 2.32]).

As long as the coefficient matrices E and A are constant (or at least smooth) the step from classical solutions to distributional solutions is formally unproblematic because the products Ex' and Ax are still well defined. However, if the coefficient matrices are not smooth anymore, the products Ex' and Ax are not defined because for distributions as originally introduced by Schwartz [5] only the product with smooth functions is defined. But there are even more obstacles for a distributional solutions theory for switched linear DAEs: For a mathematical analysis it is often necessary to study "equivalent" descriptions of a DAE. There are two obvious equivalence transformations for DAEs: 1) multiplication of an invertible matrix from the left, 2) coordinate change $x = Tz$ for some invertible matrix T. Applying these two transformations yields a new "equivalent" DAE

$$SETz' = (SAT - SET')z + Sv.$$

If E and A have jumps, then one may allow for jumps in S and T, too. But then T' is only well defined in a distributional sense.

Hence, a solution theory for switched DAEs should allow for coefficient matrices with distributional entries.

But it is not possible to define a multiplication for arbitrary distribution, this was already shown by Schwartz himself [5].

To resolve this problem only a subspace of distributions, *piecewise-smooth distributions*, is considered. For piecewise-smooth distributions it is possible to define a multiplication which 1) generalizes the multiplication of functions, 2) is associative, and 3) obeys the product rule of differentiation. However, this multiplication is not commutative anymore. This allows to study *distributional DAEs* of the form $Ex' = Ax + v$ where the variables x and v are piecewise-smooth distributions and the coefficient matrices E and A have piecewise-smoothly distributional entries.

There have been several other attempts to deal with distributional solutions for DAEs. As already mentioned, [3] and [4] introduced distributional solutions, but no general distributional solution theory is introduced, problems like evaluations of distributions at a certain point (which is needed to speak of initial values) are not addressed. A first rigorous distributional solution theory was given by Cobb in [6], he introduced "piecewise continuous distributions" which encompass piecewise-smooth distributions. However the space of piecewise continuous distributions is not closed under differentiation, and since Cobb seems to have overlooked this fact, some of the results in [6] might need a reformulation. The space of "impulsive smooth distributions" as defined in [7]

[*] e-mail: stephan.trenn@tu-ilmenau.de, Phone: +49 3677 69 3254

is a subspace of piecewise-smooth distributions, where jumps and Dirac impulses (and its derivatives) can only occur at time $t = 0$. Piecewise-smooth distributions were used as an underlying solution space for time-invariant higher order Rosenbrock systems in [8], "time-varying" topics like inconsistent initial values and switched systems are not addressed.

There also have been numerous approaches to define a multiplication for distributions (but not in the context of DAEs). König [9] enlarged the space of distributions to define a multiplication, Fuchssteiner [10] introduced the space of "almost bounded" distributions, see also [11], (which can be identified with the space of piecewise-smooth distributions of finite order) and he defined an associative multiplication which ensures that the product rule for differentiation is fulfilled. This non-commutative multiplication is identical to the multiplication defined in this note for piecewise-smooth distributions, although the approach is very different. In [12] a commutative but non-associative multiplication was defined for another subspace of distributions. Finally there are several textbooks on the topic of multiplications of distributions [13, 14, 15].

Due to space limitation the proofs of the given results cannot be included in this note.

2 Piecewise smooth distributions

Let $\mathcal{C}_0^\infty$ be the space of smooth[1] functions $f:\mathbb{R}\to\mathbb{R}$ with bounded support and let it be equipped with the topology given in [16, IV.12], this function space is the space of *test functions*. A *distribution* is a linear and continuous operator $D:\mathcal{C}_0^\infty\to\mathbb{R}$. The space of all distributions is denoted by $\mathbb{D}$. The space of locally integrable functions can injectively embedded into the space of distributions, a locally integrable function f is mapped to the distribution

$$f_{\mathbb{D}}:\varphi\mapsto\int_{\mathbb{R}} f\varphi.$$

The *support* of a distribution D is the complement of the largest open set $O\subseteq\mathbb{R}$ for which $D(\varphi)=0$ for all test functions φ whose support is contained in O. The space of all distributions whose support is contained in some set $M\subseteq\mathbb{R}$ is denoted by $\mathbb{D}_M$. The distributional derivative $\frac{\mathrm{d}_{\mathbb{D}}}{\mathrm{d}t}:\mathbb{D}\to\mathbb{D}$ is given by $\frac{\mathrm{d}_{\mathbb{D}}}{\mathrm{d}t}D(\varphi)=-D(\varphi')$, the term $\frac{\mathrm{d}_{\mathbb{D}}}{\mathrm{d}t}D$ is often abbreviated as D' and $D^{(n)}:=(\frac{\mathrm{d}_{\mathbb{D}}}{\mathrm{d}t})^n D$ for $n\in\mathbb{N}$. The *Dirac impulse* δ_t at some time $t\in\mathbb{R}$ is given by $\delta_t(\varphi)=\varphi(t)$. It is well known (e.g. [16, Satz 32.1]) that every distribution D with point support, i.e. $D\in\mathbb{D}_{\{t\}}$ for some $t\in\mathbb{R}$, can be written as

$$D=\sum_{i=0}^{n}\alpha_i\delta_t^{(i)},\tag{1}$$

for some $\alpha_0,\ldots,\alpha_n\in\mathbb{R}$, $n\in\mathbb{N}$.

Definition 2.1 A function $f:\mathbb{R}\to\mathbb{R}$ is called *piecewise-smooth* if, and only if, there exists an ordered set $S=\{\, s_i\in\mathbb{R}\mid i\in\mathbb{Z}\,\}$ with $\lim_{i\to\pm\infty}s_i=\pm\infty$ and a family of smooth functions $\{\, f_i\in\mathcal{C}^\infty\mid i\in\mathbb{N}\,\}$ such that

$$f=\sum_{i\in\mathbb{Z}}\mathbb{1}_{[s_i,s_{i+1})}f_i,$$

where $\mathbb{1}_M:\mathbb{R}\to\{0,1\}$ is the indicator function of the set $M\subseteq\mathbb{R}$.

A distribution D is called *piecewise-smooth* if, and only if, there exists a piecewise-smooth function f, a locally finite set $T\subseteq\mathbb{R}$, and a family of distribution $\{\, D_t\in\mathbb{D}_{\{t\}}\mid t\in T\,\}$ with point support such that

$$D=f_{\mathbb{D}}+\sum_{t\in T}D_t.\tag{2}$$

The *regular part* of D is $D_{\text{reg}}:=f$ and the *impulse part* of D is $D[\cdot]:=\sum_{t\in T}D_t$, hence $D=(D_{\text{reg}})_{\mathbb{D}}+D[\cdot]$. The space of *piecewise-smooth distributions* is denoted by $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$.

The following list summarizes the important properties of piecewise-smooth distributions:

Properties 2.2

(i) *Point evaluation*: Any piecewise-smooth distribution $D=f_{\mathbb{D}}+\sum_{t\in T}D_t\in\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ can be left and right "evaluated" at any point $t\in\mathbb{R}$ by

$$D(t-):=f(t-):=\lim_{\varepsilon\to 0+}f(t-\varepsilon),$$
$$D(t+):=f(t+)=f(t),$$

Furthermore, the *impulsive part* at t is given by

$$D[t]:=\begin{cases}D_t, & \text{if } t\in T,\\ 0, & \text{otherwise.}\end{cases}$$

(ii) *Closed under differentiation and integration*: The derivative of every piecewise-smooth distribution is again piecewise-smooth. Furthermore, for every piecewise-smooth distribution F and for every $t_0\in\mathbb{R}$ there exists a unique piecewise-smooth distribution $G=\int_{t_0}F$ such that $G'=F$ and $G(t_0-)=0$.

(iii) *Multiplication*: Let, for $F\in\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ and $t\in\mathbb{R}$,

$$\delta_t F:=F(t-)\delta_t,\quad F\delta_t:=F(t+)\delta_t,$$

and, for $n\in\mathbb{N}$,

$$\delta_t^{(n+1)}F:=(\delta_t^{(n)}F)'-\delta_t^{(n)}F',$$
$$F\delta_t^{(n+1)}:=(F\delta_t^{(n)})'-F'\delta_t^{(n)}.$$

Now for $G\in\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, recalling (1) and (2), the product of F and G is defined by

$$FG=(F_{\text{reg}}G_{\text{reg}})_{\mathbb{D}}+(F_{\text{reg}})_{\mathbb{D}}G[\cdot]+F[\cdot](G_{\text{reg}})_{\mathbb{D}}.$$

Note that this definition ensures that the product rule of differentiation holds.

(iv) *Restriction*: The restriction of any piecewise-smooth distribution $D\in\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ to an interval $M=[a,b)\subseteq\mathbb{R}$ is given by

$$D_M:=(\mathbb{1}_M)_{\mathbb{D}}D,$$

in particular, using the representation (2),

$$D_M=(f_M)_{\mathbb{D}}+\sum_{t\in T}\mathbb{1}_M(t)D_t,$$

where $f_M:=\mathbb{1}_M f$.

The multiplication 2.2(iii) might be called *Fuchssteiner multiplication* because Fuchssteiner introduced a similar multiplication in [10], see also [11], although his approach is different to the one presented here. Fuchssteiner showed that this multiplication is in some sense unique.

[1] Throughout this note "smooth" means arbitrarily often differentiable.

3 Distributional DAEs

3.1 Preliminaries

Consider the *distributional DAE*

$$Ex' = Ax + v, \tag{3}$$

where $E, A \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{m\times n}$, $n, m \in \mathbb{N}$, are matrices with piecewise-smoothly distributional entries, $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^m$ is a piecewise-smoothly distributional inhomogeneity and $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ is a vector with piecewise-smoothly distributional entries. It is an open question how to characterize the "solvability" of the DAE (3), even the definition of "solvability" is not obvious.

For this reason only two special cases are studied, namely the *ODE* case

$$x' = Ax + v,$$

and the so called *pure DAE* case

$$Nx' = x + v,$$

where the regular part N_{reg} is a strictly lower triangular matrix.

These two special cases are motivated by the well known characterization of *regularity* of classical DAEs with constant or analytical coefficient: classical DAEs are regular, if and only if, there exists a decoupling of the DAE into an ODE and a pure DAE subsystem, this decoupling is called *Weierstraß normal form*, see e.g. [1, Thm. 2.7] (constant coefficients) and [17] (analytical coefficients). Furthermore, the property of regularity is strongly connected to a certain concept of "solvability", hence for classical DAEs the solutions of the ODE case and of the pure DAE case already characterize the solutions of all "solvable" DAEs.

Invertibility of a matrix with piecewise-smoothly distributional entries is needed in the following and is also of general interest. The next lemma gives a sufficient and necessary condition for the invertibility of a matrix with entries in $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$.

Lemma 3.1 *Let $E \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$. Then E is invertible in $(\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$, $n \in \mathbb{N}$, if, and only if, E_{reg} is invertible in $(\mathcal{C}^\infty_{\mathrm{pw}})^{n\times n}$. If E is invertible, its inverse is given by*

$$E^{-1} = (E_{\mathrm{reg}}^{-1})_{\mathbb{D}} + (E_{\mathrm{reg}}^{-1})_{\mathbb{D}} E[\cdot] (E_{\mathrm{reg}}^{-1})_{\mathbb{D}}.$$

This subsection finishes with a definition of an inconsistent initial value problem (IIVP), due to space limitation it is not possible to give more details on IIVPs.

Definition 3.2 (IIVP) Consider a distributional DAE (3) and a given past trajectory x_{past} with support in $(-\infty, t_0) \subseteq \mathbb{R}$ then x is the solution of the IIVP (3), $x_{(-\infty,t_0)} = x_{\mathrm{past}}$ if, and only if, x is a solution of

$$E_{\mathrm{ivp}} x' = A_{\mathrm{ivp}} x + v_{\mathrm{ivp}},$$

where $E_{\mathrm{ivp}} := E_{[t_0,\infty)}$, $A_{\mathrm{ivp}} := I_{(-\infty,t_0)} + A_{[t_0,\infty)}$, and $v_{\mathrm{ivp}} := -x_{\mathrm{past}} + v_{[t_0,\infty)}$.

3.2 Main results for the ODE case

In this subsection the special distributional DAE (a distributional ODE)

$$x' = Ax + v \tag{4}$$

with $A \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$, $n \in \mathbb{N}$, and $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ is considered. The first result characterizes the solution of the homogeneous distributional ODE (4).

Theorem 3.3 *Consider* (4) *with $v = 0$. For any $t_0 \in \mathbb{R}$ for which $A[\cdot]_{(-\infty,t_0)} = 0$ there exists a matrix $\Phi_{t_0} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ with $\Phi_{t_0}(t_0-) = I$ such that $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ is a solution of* (4) *if, and only if,*

$$x = \Phi_{t_0} x_0$$

for some $x_0 \in \mathbb{R}^n$.
The matrix Φ_{t_0} is given by

$$\Phi_{t_0} = \lim_{i\to\infty} \Phi_{t_0,i}$$

where

$$\Phi_{t_0,0} := \phi(\cdot, t_0)_{\mathbb{D}}$$

and, for $i \in \mathbb{N}$,

$$\Phi_{t_0,i+1} := \phi(\cdot, t_0)_{\mathbb{D}} \left(I + \int_{t_0} \phi(\cdot, t_0)_{\mathbb{D}}^{-1} A[\cdot] \Phi_{t_0,i} \right),$$

here $\phi(\cdot, t_0)$ is the transition matrix of the classical ODE $\dot{x} = A_{\mathrm{reg}} x$.

At first glance it seems that the given formula for the transition matrix Φ_{t_0} is not of much practical use, however a practical calculation is possible:

Proposition 3.4 *Consider the homogeneous distributional ODE* (4) *with $v = 0$. Assume $A[\cdot]_{(-\infty,t_0)} = 0$ for some $t_0 \in \mathbb{R}$ and let $\Phi_{t_0,i}$, $i \in \mathbb{N}$, be given as in Theorem 3.3. Then for all $t \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that, for all $i \geq N$,*

$$(\Phi_{t_0,i})_{(-\infty,t)} = (\Phi_{t_0,N})_{(-\infty,t)}.$$

Furthermore, if there exists $t \in \mathbb{R}$ such that $A[\cdot]_{[t,\infty)} = 0$, then there exists $N \in \mathbb{N}$ such that

$$\Phi_{t_0,N+1} = \Phi_{t_0,N}$$

and, in particular,

$$\Phi_{t_0} = \Phi_{t_0,N}.$$

The next theorem characterizes the solutions of the distributional ODE (4) with an arbitrary inhomogeneity $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$.

Theorem 3.5 *Consider* (4). *For every $t_0 \in \mathbb{R}$ for which $A[\cdot]_{(-\infty,t_0)} = 0$, there exists a linear mapping $\Psi_{t_0} : (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ with the property $\Psi_{t_0}(v)(t_0-) = 0$ such that $x \in (\mathbb{D}_{\mathrm{pw}}\mathcal{C}^\infty)^n$ is a solution of* (4) *with any $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ if, and only if,*

$$x = \Phi_{t_0} x_0 + \Psi_{t_0}(v), \tag{5}$$

where $x_0 \in \mathbb{R}^n$ and Φ_{t_0} is the matrix from Theorem 3.3.
The linear mapping Ψ_{t_0} is given by

$$v \mapsto \Psi_{t_0}(v) = \lim_{i\to\infty} \Psi_{t_0,i}(v),$$

where

$$\Psi_{t_0,0} : (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n,$$
$$v \mapsto \phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)_{\mathbb{D}}^{-1} v$$

and, for $i \in \mathbb{N}$,

$$\Psi_{t_0,i+1} : (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n,$$
$$v \mapsto \phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)_{\mathbb{D}}^{-1} \left(v + A[\cdot] \Psi_{t_0,i}(v) \right),$$

here $\phi(\cdot, t_0)$ is the transition matrix of the classical ODE $\dot{x} = A_{\mathrm{reg}} x$.

It is now possible the conclude the following statement for initial value problems for distributional ODEs.

Corollary 3.6 *Every initial value problem* (4), $x(t_0-) = x_0 \in \mathbb{R}^n$, *with* $t_0 \in \mathbb{R}$ *such that* $A[\cdot]_{(-\infty,t_0)} = 0$ *has a unique solution given by* (5).

One might see no essential difference between the solution theory of classical ODEs and distributional DAEs, but consider the following example:

Example 3.7 All solution of $x' = -\delta_0 x$ are given by

$$x = \mathbb{1}_{(-\infty,0)} x_0, \quad x_0 \in \mathbb{R}.$$

Note that every solution x fulfils $x_{[0,\infty)} = 0$. This is not possible for classical ODEs; for those the set of all solutions evaluated at any time fills the whole state space. Furthermore, it is obvious that the initial value problem $x(t_0-) = x_0$ for $t_0 > 0$ is in general not solvable. Again, this is in strong contrast to classical ODEs, where every initial value problem is solvable.

The reason for this difference is that the *distributional transition matrix* Φ_{t_0} as defined in Theorem 3.3 is in general not invertible.

3.3 Main results for the pure DAE case

In this subsection consider the pure DAE

$$Nx' = x + v, \tag{6}$$

where $N \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n\times n}$ is such that N_{reg} is a strictly lower triangular matrix. The key "ingredient" for the solution theory for pure DAEs is the following result:

Lemma 3.8 *Let* $N \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n\times n}$, $n \in \mathbb{N}$, *be such that* N_{reg} *is a strictly lower triangular matrix and consider the linear operator*

$$N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t} : (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n,$$
$$x \mapsto \left(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t}\right)(x) = Nx'$$

and define the power $\left(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t}\right)^i : (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$, $i \in \mathbb{N}$, *of this operator by*

$$\left(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t}\right)^0 = \text{id} := x \mapsto x$$

and

$$\left(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t}\right)^{i+1}(x) = N\left(\left(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t}\right)^i(x)\right)'.$$

Then $N\frac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t}$ *is nilpotent, i.e. there exists* $\nu \in \mathbb{N}$ *such that* $(N\frac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t})^\nu = 0$.

It is now very simple to characterize all solutions of the pure DAE (6), because the pure DAE can be rewritten as a linear operator equation

$$(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t} - I)(x) = v,$$

and since by Lemma 3.8 the operator $N\frac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t}$ is nilpotent it is easy to see that the operator $(N\frac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t} - I) : (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is bijective with inverse operator

$$(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t} - I)^{-1} = -\sum_{i=0}^{\nu-1}(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t})^i,$$

where $\nu \in \mathbb{N}$ is such that $(N\frac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t})^\nu = 0$. This already yields the following theorem.

Theorem 3.9 *The pure DAE* (6) *is uniquely solvable and its solution is given by*

$$x = -\sum_{i=0}^{\nu-1}(N\tfrac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t})^i(v),$$

where $\nu \in \mathbb{N}$ *is such that* $(N\frac{\mathrm{d}_\mathbb{D}}{\mathrm{d}t})^\nu = 0$. *In particular the homogeneous pure DAE has only the trivial solution.*

Although this results looks very similar to the results for classical pure DAEs, there is a difference: ν can be bigger than n. This implies that more than $n-1$ derivatives of v can influence the result. This is illustrated in the following example.

Example 3.10 Consider the pure DAE

$$\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} x' = x + v,$$

where $\delta := \delta_0$. The solution is given by

$$x = -v - \begin{pmatrix} 0 & \delta \\ 1 & \delta' \end{pmatrix} v' - \begin{pmatrix} \delta & 0 \\ \delta' & \delta \end{pmatrix} v'' - \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} v'''$$

Acknowledgements I thank Achim Ilchmann for helpful discussions and suggestions for improvements.

References

[1] P. Kunkel and V. Mehrmann, Differential-Algebraic Equations. Analysis and Numerical Solution (EMS Publishing House, Zürich, Switzerland, 2006).
[2] D. Liberzon, Switching in Systems and Control, Systems and Control: Foundations and Applications (Birkhäuser, Boston, 2003).
[3] G. C. Verghese, B. C. Lévy, and T. Kailath, IEEE Trans. Aut. Contr. **AC-26**(4), 811–831 (1981).
[4] S. L. Campbell, Singular Systems of Differential Equations II (Pitman, New York, 1982).
[5] L. Schwartz, Théorie des Distributions (Hermann, Paris, 1957, 1959).
[6] D. J. Cobb, IEEE Trans. Aut. Contr. **AC-29**, 1076–1082 (1984).
[7] P. J. Rabier and W. C. Rheinboldt, Theoretical and Numerical Analysis of Differential-Algebraic Equations, Handbook of Numerical Analysis, Vol. VIII (Elsevier Science, Amsterdam, The Netherlands, 2002).
[8] D. Hinrichsen and D. Prätzel-Wolters, Int. J. Control **32**(5), 777–802 (1980).
[9] H. König, Math. Annalen **128**, 420–452 (1955).
[10] B. Fuchssteiner, Math. Annalen **178**, 302–314 (1968).
[11] B. Fuchssteiner, Studia Mathematica **76**, 439–453 (1984).
[12] H. F. Walter, Math. Annalen **189**, 211–221 (1970).
[13] J. F. Colombeau, Multiplication of distributions, No. 1532 in Lecture Notes in Mathematics (Springer Verlag, Berlin Heidelberg, 1992).
[14] M. Oberguggenberger, Multiplication of distributions and applications to partial differential equations, No. 259 in Pitman research notes in mathematics (Longman Scientific & Technical, Harlow, Essex, UK, 1992).
[15] R. F. Hoskins and J. J. M. Sousa Pinto, Distributions, ultradistributions and other generalized functions, Mathematics and its applications (Ellis Horwood, Chichester, West Sussex, UK, 1994).
[16] L. Jantscher, Distributionen, De Gruyter Lehrbuch (Walter de Gruyter, Berlin, New York, 1971).
[17] S. L. Campbell and L. R. Petzold, SIAM J. Alg. Discr. Meth. **4**, 517–521 (1983).