Impulse free solutions for switched differential algebraic equations

Stephan Trenn

Institute of Mathematics, Ilmenau University of Technology, Weimerarer Str. 25, 98693 Ilmenau, Germany

Abstract

Linear switched differential algebraic equations (switched DAEs) are studied. First, a suitable solution space is introduced, the space of so called piecewise-smooth distributions. Secondly, sufficient conditions are given which ensure that all solutions of the switched DAE are impulse and/or jump free. These conditions are easy to check and are expressed directly in the systems original data. As an example a simple electrical circuit with a switch is analyzed.

Key words: switched differential algebraic equations, impulses, piecewise-smooth distributions
2000 MSC: 34A09

1. Introduction

In this paper switched differential algebraic equations (switched DAEs) of the form

$$E_\sigma \dot{x} = A_\sigma x \tag{1}$$

will be studied, here $\sigma : \mathbb{R} \to \{1, \dots, N\}$, $N \in \mathbb{N}$, is a switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, are constant coefficient matrices for each parameter $p \in \{1, \dots, N\}$. Switched DAEs occur, for example, in modeling electrical circuits with switches or when modeling possible faults in systems where each (faulty and non-faulty) configuration is described by a "classical DAE" $E\dot{x} = Ax$ with constant matrices $E, A \in \mathbb{R}^{n\times n}$. There is a wide range of

Email addresses: stephan.trenn@tu-ilmenau.de (Stephan Trenn)
URL: http://www.tu-ilmenau.de/fakmn/trenn.html (Stephan Trenn)

Preprint submitted to *January 23, 2009*

literature on classical DAEs, see for example the textbooks [1, 2, 3, 4, 5] and the references therein, and there is also much literature on switched differential equations, see for example the textbook [6] and the references therein. However there seems to be no literature on switched DAEs of the form (1). The main reason for this might be that it is not clear what the right solution space should be: Some differentiability is needed (otherwise $\dot{x}$ in (1) is not defined) but if one considers the space of absolutely continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ then most switched DAEs of the form (1) will have no other solutions than the trivial solution. Already very simple examples show that "solutions" of switched DAEs (1) might have jumps or even derivatives of jumps, i.e. Dirac impulses. The obvious step to consider distributions or generalized functions as solutions does not work either, because the space of distributions is so big that many properties which are needed to work with equation (1) are lost. For example, distributions can only be multiplied with smooth functions, but in (1) the variables x and $\dot{x}$ are multiplied with matrix functions which are not even continuous.
To overcome these problems, the switched DAE (1) will be considered as a distributional DAE as recently introduced in [7], in particular solutions are *piecewise-smooth distributions*.

The aim of this paper is to give easy to check conditions which ensures that all solutions of the switched DAE (1) are impulse free. In electrical circuits, impulses occur as sparks and often lead to the destruction of some components, therefore it is important to analyze circuits with respect to the ability to produce impulses. Furthermore, switches might be induced by faults, hence the switching signal is not known and therefore the results of this paper will be independent of the switching signal. In addition, a simple condition will be given, which ensures that jumps do not occur in the state variables, i.e. a condition that guaranties that all solutions are actually classical solutions.

The conditions for impulse and/or jump freeness of the solutions of (1) are formulated in terms of so called consistency projectors. It is possible to construct these projectors directly in terms of the matrices (E_p, A_p), $p = 1, \dots, N$; it is not necessary to explicitly calculate some normal form (see Definition 7 together with Theorem 6).

Throughout the paper the following two assumptions will be used.

S1 The switching signal $\sigma : \mathbb{R} \to \{1, 2, \dots, N\}$ is piecewise-constant with a locally finite set of jump points and right-continuous.

S2 Each matrix pair (E_p, A_p), $p = 1, \ldots, N$, is regular, i.e. $\det(sE_p - A_p) \in \mathbb{C}[s] \setminus \{0\}$.

These assumption ensure that solutions for (1) exist in the sense of [7] and are uniquely determined by their past (see Theorem 2).

The paper is structured as follows. The next section summarizes the necessary theoretical background which is needed to formulate and prove the main results in Section 3. In particular, piecewise-smooth distributions are defined and existence of unique solutions for switched DAEs is shown. Furthermore, some specific properties of solutions of (1) are presented and consistency projectors are defined. An important relation between the consistency projectors and the solutions of (1) is shown (Theorem 8). The main results consists of the three Assumptions A1, A2 and A3, each of which is a sufficient condition for a certain impulse/jump freeness of solutions of (1) under arbitrary switching (Theorems 10, 12 and 14). Finally in Section 4 a simple circuit with a switch is analyzed and it is checked whether all solutions are impulse and/or jump free.

2. Preliminaries

2.1. Distributional solution theory for switched DAEs

Basic knowledge of distribution theory as introduced in [8] is assumed and is only briefly summarized in the following. The space of *distributions* is given by

$$\mathbb{D} := \{ \ D : \mathcal{C}_0^\infty \to \mathbb{R} \ | \ D \text{ is linear and continuous } \},$$

where $\mathcal{C}_0^\infty$ is the space of smooth[1] functions $\varphi : \mathbb{R} \to \mathbb{R}$ with bounded support. Note that the space $\mathcal{C}_0^\infty$ has to be equipped with a special topology (see e.g. [9, IV.12]), otherwise "continuity" is not well defined. The space of locally integrable functions $\mathcal{L}_{1,\text{loc}}$ is injectively imbedded into the space of distributions via the homomorphism

$$\mathcal{L}_{1,\text{loc}} \ni f \mapsto f_{\mathbb{D}} := \left(\varphi \mapsto \int_{\mathbb{R}} f\varphi \right) \in \mathbb{D}.$$

[1] arbitrarily often differentiable

The *Dirac impulse* at $t \in \mathbb{R}$ is given by

$$\delta_t : \mathcal{C}^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t).$$

Distributions are arbitrarily often differentiable and the derivative of $D \in \mathbb{D}$ is given by

$$D' := (\varphi \mapsto -D(\varphi')) \in \mathbb{D}.$$

It is easy to see that if f is a differentiable function, then this definition coincides with the classical derivative, i.e. $(f')_{\mathbb{D}} = (f_{\mathbb{D}})'$. Furthermore, the Dirac impulse is the distributional derivative of the unit step function, i.e. $\delta_t = \big((\mathbb{1}_{[t,\infty)})_{\mathbb{D}}\big)'$. The k-th derivative, $k \in \mathbb{N}$, of a distribution $D \in \mathbb{D}$ is denoted by $D^{(k)}$.

Definition 1 (Piecewise-smooth distributions, [7])**.** *Let $\mathcal{C}^\infty_{\text{pw}}$ be the space of piecewise-smooth functions, given by all functions $\alpha : \mathbb{R} \to \mathbb{R}$ so that there exist a locally finite strictly ordered set $\{\ t_i \in \mathbb{R} \mid i \in \mathbb{Z}\ \}$ and a family of smooth functions $(\alpha_i)_{i\in\mathbb{Z}}$ such that $\alpha = \alpha_i$ on $[t_i, t_{i+1})$ for all $i \in \mathbb{Z}$.*

A distribution $D \in \mathbb{D}$ is called piecewise-smooth *if, and only if, there exist a piecewise-smooth function $f \in \mathcal{C}^\infty_{\text{pw}}$ and a locally finite set $T \subseteq \mathbb{R}$ such that*

$$D = f_{\mathbb{D}} + \sum_{t\in T} D_t,$$

where, for each $t \in T$, the distribution D_t has support $\{t\}$, i.e. there exists $N \in \mathbb{N}$ and $a_0, a_1, \ldots, a_N \in \mathbb{R}$ such that

$$D_t = a_0\delta_t + a_1\delta_t' + \ldots + a_N\delta_t^{(N)}.$$

The space of all piecewise-smooth distributions is denoted by $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$.
The impulsive part of $D = f_{\mathbb{D}} + \sum_{t\in T} D_t$ is $D[\cdot] := \sum_{t\in T} D_t$ and can be evaluated at any $t \in \mathbb{R}$: $D[t] = D_t$ if $t \in T$ and $D[t] = 0$ otherwise. Furthermore, D allows for a left- and rightsided evaluation at any $t \in \mathbb{R}$: $D(t+) = f(t)$ and $D(t-) = \lim_{\varepsilon\searrow 0} f(t-\varepsilon)$.

Note that the space of piecewise-smooth distribution is the smallest space containing all piecewise-smooth functions (interpreted as distributions) and is closed under differentiation. It is shown in [7] that piecewise-smooth functions can be multiplied with each other and that the product rule for derivatives is valid. In particular the product of a piecewise-smooth function (interpreted as a piecewise-smooth distribution) with another piecewise-smooth distribution is possible. In the following define, for $\alpha \in \mathcal{C}^\infty_{\text{pw}}$ and $x \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$,

$$\alpha x := \alpha_{\mathbb{D}} x \tag{2}$$

and analogously for matrix-vector products. Furthermore, it is possible to restrict piecewise-smooth distributions to intervals, the restriction of some piecewise-smooth distribution $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ to some interval $M \subseteq \mathbb{R}$ is denoted by D_M, for details see [7].

Theorem 2 (Existence and uniqueness of distributional solutions)**.** *Consider the switched DAE* (1) *with Assumptions S1 and S2. Then for every initial trajectory $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ and every initial time $t_0 \in \mathbb{R}$ there exists a unique $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ with*

$$\begin{aligned} x_{(-\infty,t_0)} &= {x^0}_{(-\infty,t_0)} \\ (E_\sigma \dot{x})_{[t_0,\infty)} &= (A_\sigma x)_{[t_0,\infty)} \end{aligned}$$

Proof. Assumption S1 ensures that $E_\sigma, A_\sigma \in (\mathcal{C}^\infty_{\text{pw}})^{n\times n}$, hence, together with (2), the switched DAE (1) is indeed a distributional DAE as defined in [7]. Assumption S2 ensures that for each matrix pair (E_p, A_p), $p = 1, \ldots, N$, there exists invertible matrices $S_p, T_p \in \mathbb{R}^{n\times n}$ such that $(S_p E_p T_p, S_p A_p T_p) = \left(\left[\begin{smallmatrix} I & \\ & N_p \end{smallmatrix}\right], \left[\begin{smallmatrix} J_p & \\ & I \end{smallmatrix}\right]\right)$, where $J_p \in \mathbb{R}^{n_p\times n_p}$, $n_p \in \mathbb{N}$, is some matrix, $N_p \in \mathbb{R}^{(n-n_p)\times(n-n_p)}$ is a strictly lower (and in particular nilpotent) matrix and I is an identity matrix of appropriate size, [10], see also [5, Thm. 2.7]. Hence, with $S = S_\sigma$ and $T = T_\sigma$, [7, Cor. 25] completes the proof. qed

Remark 3 (ITP solutions and consistent solutions)**.** *Note that, strictly speaking, Theorem 2 does not deal with solutions of the switched DAE* (1) *because equation* (1) *is only valid on the interval* $[t_0, \infty)$ *and x is not a* global *solution. Therefore, in the following "solutions" as in Theorem 2 are called* solutions of the initial trajectory problem (ITP solutions) *for* (1) *with initial trajectory x^0 and initial time t_0. Each (global) solution of* (1) *will be called in the following* consistent *solution. Clearly, each consistent solution is also an ITP solution.*

Lemma 4 (Explicit local solution)**.** *Let Assumptions S1 and S2 hold and let $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ be some ITP solution of* (1) *with initial time $t_0 \in \mathbb{R}$. Furthermore, let $s, t \in \mathbb{R}$ with $t_0 \leq s < t$ be such that the switching signal*

σ is constant on $[s,t)$. Then there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$, a matrix $J \in \mathbb{R}^{n_1\times n_1}$, $n_1 \in \mathbb{N}$, and $v_0 \in \mathbb{R}^{n_1}$ such that

$$x_{(s,t)} = T\begin{pmatrix} \left(\tau \mapsto e^{J(\tau-s)}v_0\right)_{\mathbb{D}} \\ 0 \end{pmatrix}_{(s,t)} \tag{3}$$

Proof. Let $p := \sigma(s)$. Since x is an ITP solution, it follows that $(E_\sigma \dot{x})_{[t_0,\infty)} = (A_\sigma x)_{[t_0,\infty)}$ and, in particular (see also [7, Defn. 8,Prop. 10]),

$$E_p\dot{x}_{(s,t)} = A_p x_{(s,t)}.$$

By regularity of the matrix pair (E_p, A_p), there exist invertible matrices $S,T \in \mathbb{R}^{n\times n}$ such that $(SE_pT, SA_pT) = ([\begin{smallmatrix} I & \\ & N\end{smallmatrix}], [\begin{smallmatrix} J & \\ & I\end{smallmatrix}])$ where $J \in \mathbb{R}^{n_1\times n_1}$, $n_1 \in \mathbb{N}$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix. Let $\left(\begin{smallmatrix} v \\ w\end{smallmatrix}\right) := T^{-1}x$ where $v \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n_1}$ and $w \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n_2}$, then

$$\begin{aligned} \dot{v}_{(s,t)} &= Jv_{(s,t)} \\ N\dot{w}_{(s,t)} &= w_{(s,t)}. \end{aligned}$$

It remains to show that a) $v_{(s,t)} = \left((\tau \mapsto e^{J(\tau-s)}v_0)_{\mathbb{D}}\right)_{(s,t)} =: \zeta_{(s,t)}$ for some $v_0 \in \mathbb{R}^{n_1}$ and b) $w_{(s,t)} = 0$.
To show a), it suffices to show that $e := (v-\zeta)_{(s,t)} = 0$, therefore consider

$$\dot{e}_{(s,t)} = \left(\left((v-\zeta)_{(s,t)}\right)'\right)_{(s,t)} = \dot{v}_{(s,t)} - \dot{\zeta}_{(s,t)} = Jv_{(s,t)} - J\zeta_{(s,t)} = Je_{(s,t)}.$$

Note that in the above calculation the derivative together with the inner restriction will produce Dirac impulses at s and t, however, the outer restriction to the open interval (s,t) deletes these impulses. From the definition of the distributional restriction it follows (it is important here that an *open* interval is considered) that the equation $\dot{x}_{(s,t)} = Je_{(s,t)}$ is equivalent to

$$\forall \varphi \in \mathcal{C}_0^\infty \text{ with support in } (s,t): \quad \dot{e}(\varphi) = Je(\varphi),$$

now [11, 6.II.Cor.] yields that e is a constant distribution on (s,t). Choosing $v_0 := v(s+)$, it follows that $e = e_{(s,t)} = 0$ and a) is shown.
To show b), take the derivative of the equation $N\dot{w}_{(s,t)} = w_{(s,t)}$, restrict it to (s,t) and multiply it from the left with N to obtain

$$N^2\ddot{w}_{(s,t)} = N\dot{w}_{(s,t)}.$$

Note again, that the differentiation produces Dirac impulses which are deleted by the restriction to the open interval (s,t). This process can be repeated and since N is nilpotent it follows that $N^{n_2} = 0$ where $n_2 := n - n_1$, hence

$$0 = N^{n_2} {w^{(n_2)}}_{(s,t)} = N^{n_2-1} {w^{(n_2-1)}}_{(s,t)} = \ldots = N\dot{w}_{(s,t)} = w_{(s,t)}$$

and b) is shown. qed

Remark 5. *Lemma 4 only states the existence of matrices* $T \in \mathbb{R}^{n\times n}$ *and* $J \in \mathbb{R}^{n_1\times n_1}$ *such that all ITP solutions of* (1) *fulfill* (3). *However, the proof of Lemma 4 reveals that* (3) *holds for any invertible matrix* $T \in \mathbb{R}^{n\times n}$ *and any matrix* $J \in \mathbb{R}^{n_1\times n_1}$, $n_1 \in \mathbb{N}$, *for which there exists an invertible matrix* $S \in \mathbb{R}^{n\times n}$ *and a nilpotent matrix* $N \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ *such that* $(SE_{\sigma(s)}T, SA_{\sigma(s)}T) = ([\begin{smallmatrix} I & \\ & N\end{smallmatrix}], [\begin{smallmatrix} J & \\ & I\end{smallmatrix}])$.

Finally, it should be stressed that the ITP solution x in Lemma 4 is only considered on the *open* interval (s,t), in particular nothing is said about the impulsive part $x[s]$.

2.2. Consistency projectors

The following result was partly mentioned in [12] and [13], a complete proof is given in [14]. It is crucial for a definition of the so called consistency projectors directly in terms of the original system description; here $B\mathcal{M} := \{\ Bx \in \mathbb{R}^n \mid x \in \mathcal{M}\ \}$ and $B^{-1}\mathcal{M} := \{\ x \in \mathbb{R}^n \mid Bx \in \mathcal{M}\ \}$ for some matrix $B \in \mathbb{R}^{n\times n}$ and some set $\mathcal{M} \subseteq \mathbb{R}^n$.

Theorem 6. *Consider a regular matrix pair* $(E, A) \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$ *and let*

$$\begin{aligned} \mathcal{V}_0 &= \mathbb{R}^n, & \mathcal{V}_{i+1} &= A^{-1}E\mathcal{V}_i,\ i = 0, 1, \ldots, \\ \mathcal{W}_0 &= \{0\}, & \mathcal{W}_{i+1} &= E^{-1}A\mathcal{W}_i,\ i = 0, 1, \ldots. \end{aligned}$$

Then there exists $i^* \in \{0, 1, \ldots, n\}$ *such that*

$$\begin{aligned} \mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \ldots =: \mathcal{V}^* \\ \mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{i^*} = \mathcal{W}_{i^*+1} = \ldots =: \mathcal{W}^* \end{aligned}$$

and $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$. *Furthermore, choose* $V \in \mathbb{R}^{n\times n_1}$, $n_1 \in \mathbb{N}$, *and* $W \in \mathbb{R}^{n\times n-n_1}$ *such that* $\operatorname{im} V = \mathcal{V}^*$ *and* $\operatorname{im} W = \mathcal{V}^*$ *then* $T := [V, W]$ *and* $S^{-1} := [EV, AW]$ *are invertible matrices and*

$$(SET, SAT) = \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right),$$

where $J \in \mathbb{R}^{n_1 \times n_1}$ is some matrix, $N \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ is a nilpotent matrix and I is the identity matrix of appropriate size.

Definition 7. *For a regular matrix pair (E, A), let $T \in \mathbb{R}^{n\times n}$ and $n_1 \in \mathbb{N}$ be given as in Theorem 6. The* consistency projector *for the pair (E, A) is*

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

where $I \in \mathbb{R}^{n_1\times n_1}$ is an identity matrix of size $n_1 \times n_1$.

Note that the consistency projector does not depend on the specific choice of $T = [V, W]$, because for any other choice $\hat{T} = [\hat{V}, \hat{W}]$ with $\operatorname{im} \hat{V} = \mathcal{V}^*$ and $\operatorname{im} \hat{W} = \mathcal{W}^*$ there exists invertible matrices $P \in \mathbb{R}^{n_1\times n_1}$ and $Q \in \mathbb{R}^{n_2\times n_2}$ such that $\hat{V} = VP$ and $\hat{W} = WQ$, hence

$$\begin{aligned} \hat{T} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \hat{T}^{-1} &= [V, W] \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \left([V, W] \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right)^{-1} \\ &= [V, W] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} [V, W]^{-1} = \Pi_{(E,A)} \end{aligned}$$

Theorem 8. *Consider the switched DAE* (1) *with Assumptions S1 and S2. For each $p \in \{1, \dots, N\}$, let $\Pi_p := \Pi_{(E_p,A_p)}$ be the consistency projectors as in Definition 7. Then every ITP solution $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ of* (1) *with initial time t_0 fulfills*

$$\forall t \geq t_0 : \quad x(t+) = \Pi_{\sigma(t)} x(t-)$$

Proof. Let $p = \sigma(t)$ and for the matrix pair (E_p, A_p) choose the matrices S, T, J, N as in Theorem 6. By Assumption S1 there exists $\varepsilon > 0$ such that σ is constant on $[t, t + \varepsilon)$, hence Lemma 4 (together with Remark 5) yields

$$x(t+) = T \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$$

for some $v_0 \in \mathbb{R}^{n_1}$, $n_1 \in \mathbb{N}$. Let $T^{-1}x(t-) = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$, where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_1}$. Then

$$\Pi_{\sigma(t)} x(t-) = T \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

so it remains to show that $x_1 = v_0$. Let $T^{-1}x = \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ then $v(t-) = x_1$ and $v(t+) = v_0$ and, since x is an ITP solution of (1),

$$E_p \dot{x}_{[t,t+\varepsilon)} = A_p x_{[t,t+\varepsilon)},$$

multiplying from the left with S and substituting x by $T\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ yield

$$\dot{v}_{[t,t+\varepsilon)} = J v_{[t,t+\varepsilon)}$$

It remains to show that $v(t-) = v(t+)$. Restricting the last differential equation to the point t, i.e. considering the impulsive part of it, gives $\dot{v}[t] = Jv[t]$ and since $v[t]$ is a distribution with point support there exists $a_0, a_1, \ldots, a_N \in \mathbb{R}^{n_1}$, $K \in \mathbb{N}$, such that

$$v[t] = a_0\delta_t + a_1\delta_t' + \ldots + a_K\delta^{(n)},$$

hence, invoking [7, Prop. 11],

$$\big(v(t+) - v(t-)\big)\delta_t + \sum_{k=0}^{K} a_k\delta_t^{(k+1)} = \sum_{k=0}^{K} a_k\delta_t^{(k)},$$

or

$$0 = \sum_{k=0}^{K+1} b_k\delta_t^{(k)},$$

where $b_{N+1} = a_N$, $b_k = a_{k-1} - a_k$, $k = N, \ldots, 1$, and $b_0 = v(t+) - v(t-) - a_0$. Since $\delta_t, \delta_t', \ldots, \delta_t^{(N+1)}$ are linearly independent it follows that $0 = b_{N+1} = \ldots = b_0$. Hence $0 = a_N = \ldots a_0 = 0$ and finally $v(t+) - v(t-) = 0$ which completes the proof. qed

Combining Lemma 4 and Remark 5 with Theorem 8 immediately gives

Corollary 9. *Consider the switched DAE* (1) *with assumptions S1 and S2 and let* $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ *be an ITP solution of* (1) *with initial time* $t_0 \in \mathbb{R}$. *Then*

$$\forall t > t_0: \quad \sigma(t-) = \sigma(t+) \;\Rightarrow\; x(t+) = x(t-),$$

i.e. jumps in the solutions can only occur at switching times or at the initial time t_0.

3. Main results

In general, a solution of (1) will have jumps and impulses. In the following, sufficient conditions will be given which ensure that every solution of (1) under arbitrary switching is impulse free or, additionally, has no jumps.

Assumptions. *For the switched DAE* (1) *and* $p = 1, \ldots, N$*, let* $\Pi_p := \Pi_{(E_p, A_p)}$ *be the consistency projectors as in Definition 7.*

- ***A1*** $\forall p \in \{1, \ldots, N\} : \ E_p(I - \Pi_p) = 0.$
- ***A2*** $\forall p, q \in \{1, \ldots, N\} : E_p(I - \Pi_p)\Pi_q = 0.$
- ***A3*** $\forall p, q \in \{1, \ldots, N\} : \ (I - \Pi_p)\Pi_q = 0.$

Since the consistency projectors Π_p can easily be calculated by a finite sequence of subspaces (see Theorem 6 and Definition 7) only depending on the original matrix pairs (E_p, A_p), the Assumptions A1-A3 can be checked directly in terms of the original data. The following theorems state the properties of the solutions if one of the Assumptions A1-A3 is fulfilled.

Theorem 10 (A1)**.** *Consider the switched DAE* (1) *satisfying Assumptions S1, S2 and* ***A1****. Then, for every impulse free initial trajectory and any initial time, the unique ITP solution* $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ *is impulse free, i.e.* $x[t] = 0$ *for all* $t \in \mathbb{R}$ *or, in other words, the distributional solution is actually a piecewise-smooth function.*

Proof. Let $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ be the ITP solution to some given initial trajectory and initial time $t_0 \in \mathbb{R}$ and let $t_0 < t_1 < t_2 < \ldots$ be the switching times of the switching signal σ after the initial time t_0. Lemma 4 already shows that $x_{(t_i, t_{i+1})}$ is impulse free for all $i \in \mathbb{N}$, hence it remains to show that $x[t_i] = 0$ for all $i \in \mathbb{N}$. Therefore, consider a fixed $i \geq 0$ and let $p = \sigma(t_i)$. For the matrix pair (E_p, A_p), choose matrices S, T, J, N as in Theorem 6, i.e. $(SE_pT, SA_pT) = (\left[\begin{smallmatrix} I & \\ & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & \\ & I \end{smallmatrix}\right])$ and let $T^{-1}x = \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$. Then $x[t_i] = 0$ if and only if $v[t_i] = 0$ and $w[t_i] = 0$, where v and w fulfill

$$\begin{aligned} \dot{v}[t_i] &= Jv[t_i], \\ N\dot{w}[t_i] &= w[t_i]. \end{aligned}$$

In the proof of Theorem 8 it was already shown that $\dot{v}[t_i] = Jv[t_i]$ implies $v[t_i] = 0$. Hence it remains to show that $N\dot{w}[t_i] = w[t_i]$ together with Assumption A1 implies $w[t_i] = 0$. First observe that $N\dot{w}[t_i] = w[t_i]$ implies, invoking [7, Prop. 11],

$$N(w[t_i])' = w[t_i] - N\big(w(t_i+) - w(t_i-)\big)\delta_{t_i},$$

taking the derivative of the equations and multiplying it from the left with N yields

$$\begin{aligned} N^2(w[t_i])'' &= N(w[t_i])' - N^2\big(w(t_i+) - w(t_i-)\big)\delta'_{t_i} \\ &= w[t_i] - N\big(w(t_i+) - w(t_i-)\big)\delta_{t_i} - N^2\big(w(t_i+) - w(t_i-)\big)\delta'_{t_i}. \end{aligned}$$

Repeating this process yields, since N is nilpotent,

$$0 = N^{n_1}(w[t_i])^{(n_1)} = w[t_i] - \sum_{k=0}^{n_1-1} N^{k+1}\big(w(t_i+) - w(t_i-)\big)\delta_{t_i}^{(k)}$$

or

$$w[t_i] = \sum_{k=0}^{n_1-1} N^{k+1}\big(w(t_i+) - w(t_i-)\big)\delta_{t_i}^{(k)}.$$

Assumption A1 and Theorem 8 yield

$$\begin{aligned} 0 \overset{\text{A1}}{=} E_p(I - \Pi_p)x(t_i-) \overset{\text{Thm. 8}}{=} & E_p\big((x(t_i-) - x(t_i+)\big) \\ &= S \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{pmatrix} v(t_i-) - v(t_i+) \\ w(t_i-) - w(t_i+) \end{pmatrix} \end{aligned}$$

and, in particular,

$$0 = N\big(w(t_i-) - w(t_i+)\big), \tag{4}$$

hence $w[t_i] = 0$. qed

Remark 11. *Lemma 4 together with* (4) *reveals that Assumption A1 is equivalent to the condition that all matrix pairs* (E_p, A_p), $p = 1, \ldots, N$, *have index one or less [5, Def. 2.9].*

Theorem 12 (A2)**.** *Consider the switched DAE* (1) *satisfying Assumptions S1, S2 and* ***A2****. Then every consistent solution* $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ *of* (1) *is impulse free, i.e.* $x[t] = 0$ *for all* $t \in \mathbb{R}$.

Proof. Let $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ be some consistent solution of (1) for some switching signal $\sigma \in \mathcal{S}$. Using the same notation as in the proof of Theorem 10 the proof can be repeated identically up to where Assumption A1 was used. Let $q = \sigma(t_i-)$ and choose for the matrix pair (E_q, A_q) the matrices S_q, T_q, J_q, N_q and $n_{1,q} \in \mathbb{N}$ as in Theorem 6 then Lemma 4 applied to the interval $(t_i - \varepsilon, t_i)$ for sufficiently small $\varepsilon > 0$ yields that there exists some $v_q \in \mathbb{R}^{n_{1,q}}$ such that

$$x(t_i-) = T_q \begin{bmatrix} v_q \\ 0 \end{bmatrix} = T_q \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_q^{-1} T_q \begin{bmatrix} v_q \\ 0 \end{bmatrix} = \Pi_q x(t_i-).$$

Therefore, Assumption A2 implies

$$0 = E_p(I - \Pi_p)\Pi_q x(t_i-) = E_p(I - \Pi_p)x(t_i-)$$

and the claim follows as in the proof of Theorem 10. qed

Remark 13. *In general Assumption A2 is independent of the index of the matrix pairs* (E_p, A_p)*,* $p = 1, \ldots, N$*. However, if the index of one matrix pair* (E_q, A_q)*,* $q \in \{1, \ldots, N\}$*, is zero, i.e.* E_q *is invertible, then the consistency projector* Π_q *is the identity matrix and Assumption A2 is equivalent to Assumption A1.*

Theorem 14 (A3)**.** *Consider the switched DAE* (1) *satisfying Assumptions S1, S2 and **A3**. Then every consistent solution* $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ *of* (1) *is impulse free and has no jumps, i.e.* $x[t] = 0$ *and* $x(t-) = x(t+)$ *for all* $t \in \mathbb{R}$ *or in other words, the distribution* x *is actually a absolutely continuous function.*

Proof. Since Assumption A3 implies Assumption A2, Theorem 12 already shows that all solutions of (1) are impulse free, hence it remains to show that all solutions have no jumps, i.e. every solution $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ fulfills $x(t-) = x(t+)$ for all $t \in \mathbb{R}$. Let $\sigma \in \mathcal{S}$ be the switching signal of (1), $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ an arbitrary solution of (1), $t \in \mathbb{R}$, $q := \sigma(t-)$ and $p := \sigma(t+)$. If $p = q$ then Corollary 9 already shows that $x(t-) = x(t+)$, hence it remains to consider $p \neq q$. Identically as in the proof of Theorem 12 it follows that $\Pi_q x(t-) = x(t-)$, hence Assumption A3 together with Theorem 8 yield

$$0 = (I - \Pi_p)\Pi_q x(t-) = (I - \Pi_p)x(t-) = x(t-) - x(t+).$$

qed

4. Example

As an example consider a simple circuit with a switch as depicted in Figure 1. The switch in the circuit can be in three different position: 1) Left position, the capacitor is connected with the voltage source and is charged, 2) middle position, the switch is between the two connections, 3) right position, the capacitor is short-circuited. For the analysis the input source is assumed to be constant and can therefore be included as state variable described by the simple equation $\dot{u} = 0$. Furthermore, $C\dot{u}_C = i_C$ and $u_R = Ri_R$ hold

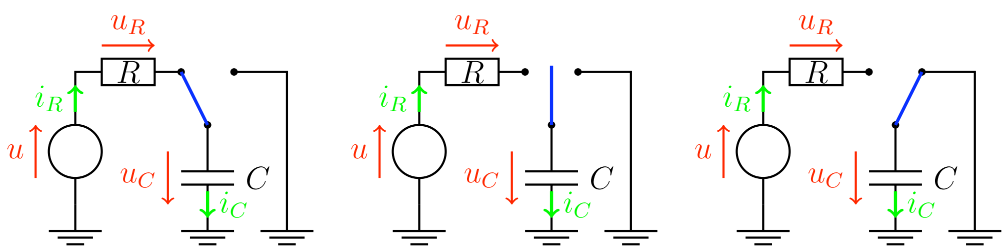

Figure 1: Example circuit with switch in three different positions.

independently of the switch. If the switch is in the left position, then the two equations $i_C = i_R$ and $u_c + u_R + u = 0$ hold, if the switch is in the middle position, then $i_R = 0 = i_C$ and if the switch is in the right position then $i_R = 0$ and $u_C = 0$. Writing $x = (u, u_C, u_R, i_C, i_R)$ the corresponding switched system (1) is described by some switching signal $\sigma : \mathbb{R} \to \{1, 2, 3\}$ satisfying Assumption S1 and the matrix pairs $(E_1, A_1), (E_2, A_2), (E_3, A_3)$ given by

$$E_1 = E_2 = E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -R \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -R \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -R \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Constructing the subspaces as in Theorem 6 yields the consistency projectors

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1/R & -1/R & 0 & 0 & 0 \\ -1/R & -1/R & 0 & 0 & 0 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Pi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For Assumption A1 the condition $E_p(I - \Pi_p) = 0$, $p = 1, 2, 3$, must be checked:

$$E_1(I - \Pi_1) = 0, \quad E_2(I - \Pi_2) = 0, \quad E_3(I - \Pi_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence Assumption A1 is not fulfilled and impulses in the solution cannot be excluded. But it is still possible that impulses cannot occur if only consistent solutions are considered, for this, Assumption A2 must be verified, i.e. $E_p(I-\Pi_p)\Pi_q = 0$ for $p,q = 1,2,3$ must be checked (note that for $p = q$ the condition is always fulfilled because $\Pi_p^2 = \Pi_p$):

$$E_1(I-\Pi_1)\Pi_2 = 0,\ E_1(I-\Pi_1)\Pi_3 = 0,\ E_2(I-\Pi_2)\Pi_1 = 0,\ E_2(I-\Pi_2)\Pi_3 = 0,$$

$$E_3(I-\Pi_3)\Pi_1 = \begin{pmatrix} 0&0&0&0&0\\ 0&C&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad E_3(I-\Pi_3)\Pi_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&C&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}.$$

Therefore, impulses cannot be excluded. However, if the switch is not allowed to move into the right position, i.e. $\sigma : \mathbb{R} \to \{1,2\}$, then Assumptions A1 and A2 are fulfilled and no impulses can occur. In this case one can also check condition A3:

$$(I-\Pi_1)\Pi_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 1&1&0&0&0\\ 1/R&1/R&0&0&0\\ 1/R&1/R&0&0&0 \end{pmatrix}, \quad (I-\Pi_2)\Pi_1 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ -1&-1&0&0&0\\ -1/R&-1/R&0&0&0\\ -1/R&-1/R&0&0&0 \end{pmatrix}.$$

So jumps cannot be excluded.

5. Conclusions

For switched differential algebraic equations a suitable concept of solutions was introduced. Solutions are so called piecewise-smooth distributions which, roughly speaking, consists of a sum of piecewise-smooth functions and impulses. The main result of this paper are sufficient conditions which ensure that the solutions are impulse free and/or have no jumps. These conditions are based on so called consistency projectors which can be calculated easily in terms of the original data. Furthermore, the conditions are independent of the specific switching signal and guarantee impulse/jump freeness under arbitrary switching. However if more is known about the switching signal it might be possible in future research to refine the results. Furthermore, throughout this paper it was assumed that all matrix pairs are regular, it is also of interest whether similar results can be obtained for non-regular matrix pairs.

6. Acknowledgements

The author thanks Professor Achim Ilchmann and Professor Daniel Liberzon for helpful comments on the manuscript of this paper.

References

[1] S. L. Campbell, Singular Systems of Differential Equations I, Pitman, New York, 1980.

[2] S. L. Campbell, Singular Systems of Differential Equations II, Pitman, New York, 1982.

[3] E. Griepentrog, R. März, Differential-algebraic equations and their numerical treatment, No. 88 in Teubner-Texte zur Mathematik, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1986.

[4] L. Dai, Singular Control Systems, No. 118 in Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1989.

[5] P. Kunkel, V. Mehrmann, Differential-Algebraic Equations. Analysis and Numerical Solution, EMS Publishing House, Zürich, Switzerland, 2006.

[6] D. Liberzon, Switching in Systems and Control, Systems and Control: Foundations and Applications, Birkhäuser, Boston, 2003.

[7] S. Trenn, Regularity of distributional differential algebraic equations, submitted for publication (Institut for Mathematics, TU Ilmenau, Preprint Number 08-24), preprint available online.

[8] L. Schwartz, Théorie des Distributions, Hermann, Paris, 1957, 1959.

[9] L. Jantscher, Distributionen, De Gruyter Lehrbuch, Walter de Gruyter, Berlin, New York, 1971.

[10] K. Weierstraß, Zur Theorie der bilinearen und quadratischen Formen, Monatsh. Akademie. Wiss. (1868) 310–338.

[11] W. Walter, Einführung in die Theorie der Distributionen, 3rd Edition, BI Wissenschaftsverlag, Mannheim Leipzig Wien Zürich, 1994.

[12] F. L. Lewis, A survey of linear singular systems, IEEE Proc. Circuits, Systems and Signal Processing 5 (1) (1986) 3–36.

[13] D. H. Owens, D. L. Debeljkovic, Consistency and Liapunov stability of linear descriptor systems: A geometric analysis, IMA J. Math. Control & Information (1985) 139–151.

[14] T. Berger, A. Ilchmann, S. Trenn, A generalized jordan canonical form for linear matrix pencils, in preparation.