Hybrid Systems with Constraints

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January 16, 2013

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Stability of switched DAEs

Abstract. Differential algebraic equations (DAEs) are used to model dynamical systems with constraints given by algebraic equations. In the presence of sudden structural changes (e.g. switching or faults) this leads to a switched DAE. A special feature of switched DAEs is the presence of induced jumps or even Dirac impulses in the solution. This chapter studies stability of switched DAEs taking into account the presence of these jumps and impulses. For a rigorous mathematical treatment it is first necessary to introduce a suitable solution space - the space of piecewise-smooth distributions. Within this distributional solution space the notion of stability encompasses impulse-freeness which is studied first. Afterwards stability under arbitrary and slow switching is investigated. A generalization to switched DAEs of a classical result concerning stability and commutativity is presented as well as a converse Lyapunov theorem. The theoretical results are illustrated with intuitive examples.

3.1. Introduction

3.1.1. *Systems class: definition and motivation*

The main interest in this chapter are systems described by switched differential algebraic equations (switched DAEs) of the form

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \quad \text{or short} \quad E_\sigma \dot{x} = A_\sigma x \tag{3.1}$$

where $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathtt{P}\}$ denotes the switching signal choosing one of the $\mathtt{P} \in \mathbb{N}$ modes at each time $t \in \mathbb{R}$. Here it is assumed that σ is admissible in the sense that it is piecewise-constant, right-continuous and has locally only finitely many jumps. Furthermore, it is assumed that each mode of the switched DAE (3.1) is given by a regular matrix pair (E_i, A_i), $i \in \{1, 2, \ldots, \mathtt{P}\}$, i.e. $\det(sE_i - A_i)$ is not the zero polynomial.

The main motivation to study this system class is modeling electrical circuits with switches or faults [DOM 10, TRE 12a]. Another motivation might be the analysis of a closed loop composed of a (non-switched) DAE

$$E\dot{x} = Ax + Bu$$

Chapter written by S. TRENN.

together with a switched feedback controller of the form

$$u(t) = F_{\sigma(t)}x(t) \quad \text{or}$$
$$u(t) = F_{\sigma(t)}x(t) + G_{\sigma(t)}\dot{x}(t)$$

resulting in (3.1). Finally, (3.1) might be considered as a piecewise-constant approximation of time-varying DAEs $E(t)\dot{x} = A(t)x$, however, no theory for this kind of approximation is available yet.

Here the main focus is on stability of switched DAEs (3.1), in particular the question: when is it true that asymptotic stability of $E_p\dot{x} = A_p x$ for all $p \in \{1, \ldots, \mathtt{P}\}$ implies asymptotic stability of $E_\sigma\dot{x} = A_\sigma x$ for all switching signals σ?

This chapter is based on the works of the author (in collaboration with several colleagues) which began with the conference article [LIB 09] and the extended journal paper [LIB 12] concerning stability of switched DAEs. Further results also discussed here appeared in [DOM 10] (impulse detection), [LIB 11] (commutativity and stability), [TRE 12b, TRE 12c] (converse Lyapunov Theorem) as well as in the survey article concerning switched DAEs [TRE 12a]. There are only a few papers by other authors who consider switched DAEs (and their stability), e.g. [GEE 96a, GEE 96b, MEN 06a, MEN 06b, RAO 10, WUN 08, ZHA 06]; however, none of these work resolve the dilemma that a switched DAE might exhibit solutions with jumps and impulses (hence distributional solutions) and at the same time the equation (3.1) doesn't make sense for general distributions x. The approach taken here is based on the piecewise-smooth distributional solution framework as introduced by the author in [TRE 09b, TRE 09a]. For an overview on the different solution concepts for DAEs see also the recent survey [TRE 13].

3.1.2. *Examples*

Before making precise the solution concepts for switched DAEs and the stability definition, consider the following set of two examples.

Example 1a:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

Example 1b:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

The solution of the corresponding non-switched DAEs are identical for both examples and are illustrated in the middle of Figure 3.1. A typical trajectory (with jumps

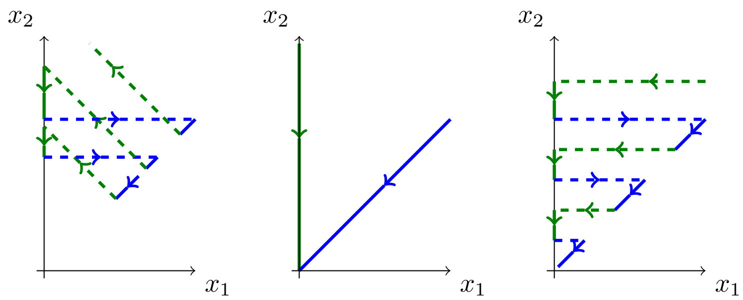

Figure 3.1. *Solution behavior of Examples 1: switched for Example 1a (left), non-switched (middle), switched for Example 1b (right).*

indicated by dashed lines) for Example 1a with sufficiently fast switching is shown in the left of Figure 3.1; clearly, the solution grows unbounded, hence the switched system is unstable. A different behavior results for Example 1b (shown in the right of Figure 3.1): although the non-switched behavior is indistinguishable from the one of Example 1a, the switching now leads to a convergent trajectory (independently of the switching signal). The key observation here is that the behavior of the switched DAE cannot be deduced from the non-switched dynamics alone, the induced jumps play an important role as well. In fact, it is easily seen, that $V(x) = x_1^2 + x_2^2$ is a Lyapunov function for all individual modes in Example 1 in the sense that for all solution of the non-switched system this function (strictly) decreases along solutions. So in contrast to switched ODEs (see e.g. [LIB 03]) the existence of a common Lyapunov function in the usual sense is not sufficient for stability of switched DAEs.

Another phenomena occurring in switched DAEs are impulses (and not only jumps) in the solutions. That this is not only a theoretical possibility shows the following Example 2 based on a simple electrical circuit as given in Figure 3.2.

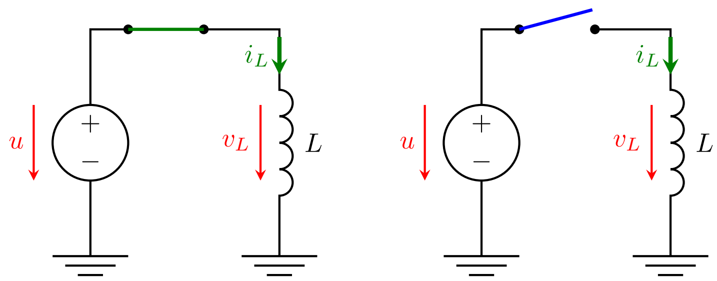

Figure 3.2. *An electrical circuit with a switch leading to Example 2 which exhibits impulsive solutions.*

Under the assumption of a constant input value, i.e. $\dot{u} = 0$, the two modes of the electrical circuit can be written as

Example 2:

switch closed:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} x,$$

switch open:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x,$$

where $x = [u, i_L, v_L]^\top$.

Consider now the situation, that the switch is closed at time $t = 0$ and the initial current is zero, i.e. $i_L(0) = 0$. The inductivity law $L\frac{\mathrm{d}}{\mathrm{d}t}i_L = v_L = u$ now yields that the current is given by the formula (see also Figure 3.3):

$$i_L(t) = \frac{u}{L}t$$

as long as the switch is closed. In particular, $u \neq 0$ implies $i(t) \neq 0$ for all $t > 0$. If at time $t = t_s > 0$ the switch is open, then the current i_L must jump to zero because the ideal switch doesn't allow any current when open. However, the inductivity law of the ideal inductor remains valid, hence the voltage over the inductor is the derivative of the current even when the current jumps to zero. There are only two possibility now: Since the current has a jump no classical derivative exists and there is no solution to this switched system unless $i_L(t_s) = 0$ (and hence $u \equiv 0$). The other possibility is to allow for a bigger solution space where the derivative of a jump is well defined: the space of distributions (a.k.a. generalized functions). Adopting the second viewpoint, the (unique) solution for v_L contains now a Dirac impulse of magnitude $-i_L(t_s-)$ at $t = t_s$, because a distributional derivative of a jump yields a Dirac impulse. Although an ideal Dirac impulse does not occur in reality, one might see a spark when opening the switch in a realization of the circuit. This spark can be interpreted as the approximate realization of the Dirac impulse resulting from the mathematical analysis of the circuit model.

To summarize the observations made on the basis of the above examples: 1) Each mode evolves within a certain *consistency space* and switches may lead to inconsistent initial values and jumps. 2) For stability the jumps play an essential role; a common Lyapunov function in the classical sense does not suffice to guarantee stability. 3) Switching can even lead to the presence of Dirac impulses which (interpreted as an infinite peak) makes the switched system unstable.

The remainder of this chapter is organized as follows. In Section 3.2 some basic facts about switched DAEs are collected and recalled. In particular, the solution theory for non-switched DAEs is discussed including the definition of the consistency

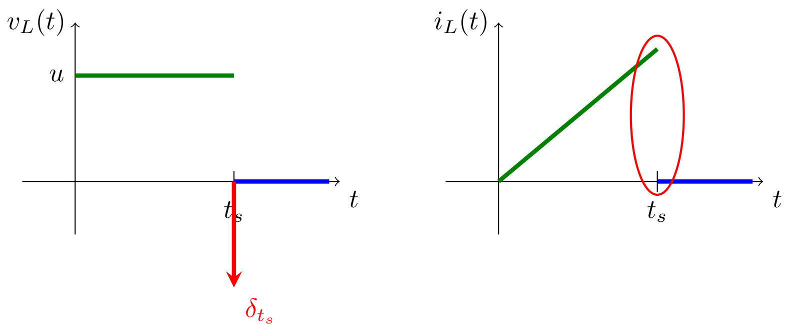

Figure 3.3. *The mathematical solution of Example 2 where the switch initially is closed and is then opened at $t = t_s$.*

projectors and the flow matrices A^{diff} as well as the concept of Lyapunov functions for non-switched DAEs. A brief summary of the classical distribution theory is given and it is highlighted that the space of distributions as a whole is not suitable for a solution space of switched DAEs. The space of piecewise-smooth distributions is introduced as a suitable solution space for switched DAEs. The main focus of this book chapter is on stability results for switched DAE which are presented in Section 3.3. First impulse-freeness as a necessary condition for stability is investigated. Afterwards a sufficient condition for asymptotic stability under arbitrary switching as well as for slow switching is given. A classical result concerning commutativity and stability is generalized to switched DAEs. Finally, a converse Lyapunov theorem is presented for switched DAEs.

3.2. Preliminaries

3.2.1. *Non-switched DAEs: Solutions and consistency projector*

For $E, A \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, consider the non-switched DAE

$$E\dot{x} = Ax \tag{3.2}$$

with regular matrix pair (E, A), i.e. $\det(sE - A)$ is not the zero polynomial. The following result is classical [WEI 68], see also [GAN 59].

THEOREM.– *The matrix pair (E, A) with square matrices E, A is regular if, and only if, there exist invertible matrices S and T such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{3.3}$$

where N is nilpotent.

The decoupling (3.3) is called *Weierstrass canonical form* when it also assumed that J and N are in Jordan canonical form. For the further analysis of DAEs (3.2) this assumption is not needed and is also disadvantageous because aiming for a decoupling with J in Jordan canonical form calls for complex transformation matrices S and T even when E and A are real valued. For that reasons it is more convenient to consider a *quasi-Weierstrass form* (QWF) (3.3) where there is no restriction on J and N apart from the nilpotency of N. In [BER 12] it is shown how to obtain the QWF via the so called Wong-sequences:

$$\mathcal{V}_0 := \mathbb{R}^n \quad , \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \quad i = 0,1,2,\ldots, \quad \mathcal{V}^* := \bigcap_i \mathcal{V}_i,$$

$$\mathcal{W}_0 := \{0\} \quad , \quad \mathcal{W}_{i+1} := A^{-1}(E\mathcal{W}_i), \quad i = 0,1,2,\ldots, \quad \mathcal{W}^* := \bigcup_i \mathcal{W}_i.$$

Choosing full (column) rank matrices V and W such that $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$ one obtains the QWF (3.3) via the transformation matrices $T = [V\,W]$ and $S = [EV\,AW]^{-1}$. Taking into account that the “pure” DAEdifferential algebraic equation!pure

$$N\dot{w} = w$$

for any nilpotent matrix N has only the trivial solution $w = 0$ the following result is immediate.

COROLLARY.– *Consider the DAE* (3.2) *with regular matrix pair* (E, A) *and corresponding QWF* (3.3) *with* $J \in \mathbb{R}^{n_1 \times n_1}$ *obtained by* S *and* $T = [V\,W]$*, where* $V \in \mathbb{R}^{n \times n_1}$*. Then any classical (i.e. differentiable) solution* x *of* (3.2) *is given by*

$$x(t) = Ve^{Jt}v_0, \quad v_0 \in \mathbb{R}^{n_1}.$$

In particular, $\mathfrak{C}_{(E,A)} := \operatorname{im} V = \mathcal{V}^*$ *is the* consistency space *of the DAE* (3.2)*.*

EXAMPLE.– Consider a DAE (3.2) given by the regular matrix pair

$$(E, A) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right).$$

Any solution is given by

$$x(t) = \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} e^{\left[\begin{smallmatrix} -1 & -4\pi \\ \pi & -1 \end{smallmatrix}\right] t} v_0.$$

For $v_0 = (0, 1)^\top$ the corresponding solution and the consistency space is shown in Figure 3.4.

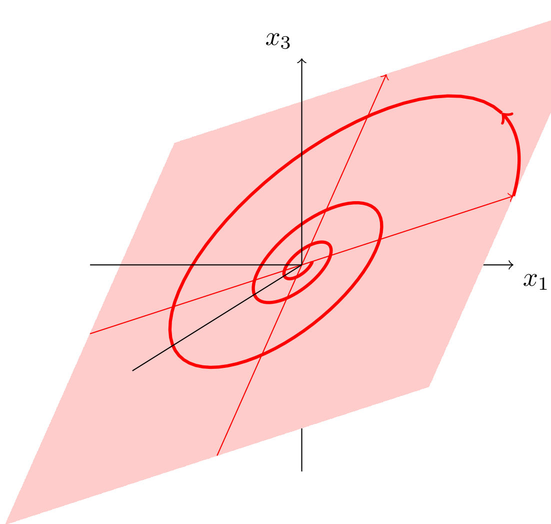

Figure 3.4. *Illustration of a consistency space and a solution of a DAE.*

Assuming that the DAE (3.2) is "switched on" at some time, say $t = 0$, the problem of inconsistent initial values arises. In the QWF-coordinates the DAE (3.2) reads as the following two independent equations

$$\dot{v} = Jv \quad \text{and} \quad N\dot{w} = w$$

and the corresponding initial values v_0 and w_0. For the ODE $\dot{v} = Jv$ any initial value is consistent, in particular $v(0+) = v_0$. For the pure DAE $N\dot{w} = w$ however, only $w_0 = 0$ is consistent and $w(0+) = 0$ whatever the initial value for $w(0-)$ was. Hence any inconsistent initial value $(v(0-), w(0-))^\top = (v_0, w_0)$ jumps to the consistent initial value $(v(0+), w(0+)) = (v_0, 0)$ or in other words:

$$\begin{pmatrix} v(0+) \\ w(0+) \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0-) \\ w(0-) \end{pmatrix}.$$

Translating this jump map back to the original coordinates of (3.2) leads to the following definition.

DEFINITION.– *Let $S, T \in \mathbb{R}^{n\times n}$ be invertible such that* (3.3) *holds. Then*

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

with block sizes corresponding to the ones in (3.3) *is called the* consistency projector *of the DAE* (3.2)

Note that the definition of the consistency projector is independent of the choice of T and can be easily calculated with the help of the Wong sequences. Furthermore, the consistency projector is a projector onto $\mathcal{V}^*$ along $\mathcal{W}^*$, in particular

$$\operatorname{im}\Pi_{(E,A)} = \mathfrak{C}_{(E,A)}.$$

The consistency projector now uniquely determines the jump induced by an inconsistent value (for details on a suitable distributional solution concept see Section 3.2.4):

THEOREM.– *Consider a DAE* (3.2) *with regular matrix pair* (E, A) *and corresponding consistency projector* $\Pi_{(E,A)}$. *Assume the DAE is "switched on" at* $t = 0$ *and* $x(0-)$ *is given. Then there exists a unique solution* x *of the corresponding inconsistent initial value problem and*

$$x(0+) = \Pi_{(E,A)}x(0-).$$

Finally, the so called *differential projector* [TAN 10] is defined which also plays an import role in the solution representation of the DAE (3.2).

DEFINITION.– *Consider the DAE* (3.2) *with regular matrix pair* (E, A) *and choose (e.g. by using the Wong sequences) invertible matrices* S, T *such that* (3.3) *holds. The* differential projector *is defined as*

$$\Pi^{\text{diff}}_{(E,A)} := T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

where the block sizes correspond to the block sizes in the QWF (3.3)*. Furthermore, define the* flow matrix *corresponding to* (3.2) *by*

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)}A = T\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

The relevance of the flow matrix for (3.2) is as follows [TAN 10]:

THEOREM.– *Consider the DAE* (3.2) *with regular matrix pair and corresponding matrix* A^{diff} *and consistency space* $\mathfrak{C}_{(E,A)}$. *Then the following equivalence holds:*

$$x \textit{ solves } (3.2) \quad \Leftrightarrow \quad x \textit{ solves } \boxed{\dot{x} = A^{\text{diff}}x} \;\wedge\; x(0) \in \mathfrak{C}_{(E,A)}$$

In particular, if the DAE (3.2) *is "switched on" at* $t = 0$ *and* $x(0-) = x_0 \in \mathbb{R}^n$ *is arbitrary, then the unique solution* x *on* $(0, \infty)$ *is given by:*

$$x(t) = e^{A^{\text{diff}}t}\Pi_{(E,A)}x_0. \tag{3.4}$$

Note that the differential projector (and not only A^{diff}) plays some role in the explicit solution formula for the inhomogeneous DAE $E\dot{x} = Ax + f$, see [TRE 12a].

3.2.2. *Lyapunov functions for non-switched DAEs*

DEFINITION.– *Consider a DAE* (3.2) *with regular matrix pair* (E, A) *and corresponding consistency space* $\mathfrak{C}_{(E,A)}$*. Assume that there exist matrices* $P = P^\top > 0$ *(i.e. positive definite) and* $Q = Q^\top > 0$ *on* $\mathfrak{C}_{(E,A)}$ *which solve the* generalize Lyapunov equation

$$A^\top PE + E^\top PA = -Q. \tag{3.5}$$

Then $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top PEx$ *is called* Lyapunov function *for* (3.2).

Note that any Lyapunov function V is monotonically decreasing along (nonzero) solutions of (3.2):

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V\big(x(t)\big) &= \big(Ex(t)\big)^\top PE\dot{x}(t) + \big(E\dot{x}(t)\big)^\top PEx(t)\\ &= x(t)^\top E^\top PAx(t) + x(t)^\top A^\top PEx(t)\\ &= -x(t)^\top Qx(t) < 0.\end{aligned}$$

For a more general definition of a Lyapunov function for (also nonlinear) DAEs, the reader is referred to [LIB 12].

For ODEs it is well known that asymptotic stability is equivalent with the existence of a Lyapunov function; a similar statement also holds for DAEs [OWE 85].

THEOREM.– *The DAE* (3.2) *is asymptotically stable if, and only if, there exists a Lyapunov function for* (3.2).

3.2.3. *Classical distribution theory*

Example 2 from the Introduction showed that switched DAEs of the form (3.1) can lead to distributional solutions. Therefore, it is necessary to recapitulate some basic facts concerning distributions as formally introduced by Schwartz [SCH 51].

The space of distributions $\mathbb{D}$ is defined as the set of all linear and continuous operators $D : \mathcal{C}_0^\infty \to \mathbb{R}$, where $\mathcal{C}_0^\infty$ denotes the set of smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with bounded support (i.e. $\varphi(t) = 0$ for all sufficiently large $|t|$) equipped with a suitable topology. The distributional derivate of a distribution $D \in \mathbb{D}$ is given by $D'(\varphi) = D(\varphi')$. Functions [1] can be imbedded into the space of distributions via the injective homomorphism $f \mapsto f_\mathbb{D}$ given by:

$$f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_\mathbb{R} f(t)\varphi(t)\,\mathrm{d}s\,t.$$

1. to be precise: locally integrable functions which are identified with each other if they only differ on a set of measure zero

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function then the distributional derivative coincides with the classical derivative, i.e.

$$(f')_{\mathbb{D}} = (f_{\mathbb{D}})'.$$

The famous Dirac impulse (a.k.a. Dirac Delta, Delta function) at $t \in \mathbb{R}$ is given by

$$\delta_t(\varphi) = \varphi(t), \quad \delta := \delta_0.$$

It is easily seen that the Dirac impulse is the (distributional) derivative of the Heaviside step function and, in general, any (distributional) derivative of a jump will lead to a Dirac impulse. Distributions can be multiplied with smooth functions via

$$(\alpha D)(\varphi) := D(\alpha\varphi) \quad \text{for } D \in \mathbb{D}, \varphi \in \mathcal{C}_0^\infty, \alpha \in \mathcal{C}^\infty.$$

This multiplication corresponds to the pointwise multiplication of functions.

When considering distributional solutions of the switched DAE (3.1) one has to consider $x \in \mathbb{D}^n$ and it is necessary to define the products $E_\sigma \dot{x}$ as well as $A_\sigma x$. However, this is *not possible* in general because $t \mapsto E_{\sigma(t)}$ and $t \mapsto A_{\sigma(t)}$ are not smooth functions of t. One possible solution to this problem could be the following observation:

$$\left.\begin{array}{r} E_\sigma \dot{x} = A_\sigma x \\ i \in \mathbb{Z} : \ \sigma\big|_{[t_i, t_{i+1})} \equiv p_i \end{array}\right\} \quad \Leftrightarrow \quad \forall i \in \mathbb{Z} : \ (E_{p_i}\dot{x})_{[t_i,t_{i+1})} = (A_{p_i}x)_{[t_i,t_{i+1})}$$

where $D_I \in \mathbb{D}$ denotes a restriction of the distribution $D \in \mathbb{D}$ to the interval $I \subseteq \mathbb{R}$ which generalizes the restriction of functions f_I given by

$$f_I(t) := \begin{cases} f(t), & t \in I, \\ 0, & t \notin I. \end{cases}$$

Hence one arrives at a new question: How to define a *restriction of distributions* to intervals? The following example shows (for details see [TRE 09a, Thm. 2.2.2]) that this definition is in general *not possible*!

EXAMPLE.– Consider the following (well defined!) distribution (see also Figure 3.5):

$$D := \sum_{i \in \mathbb{N}} d_i \, \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}.$$

A restriction of D to the interval $(0, \infty)$ should give:

$$D_{(0,\infty)} = \sum_{k \in \mathbb{N}} d_{2k} \, \delta_{d_{2k}}.$$

Choose any $\varphi \in \mathcal{C}_0^\infty$ with $\varphi_{[0,1]} \equiv 1$, then

$$D_{(0,\infty)}(\varphi) = \sum_{k\in\mathbb{N}} d_{2k} = \sum_{k\in\mathbb{N}} \frac{1}{2k+1} = \infty,$$

hence $D_{(0,\infty)}$ is not a well defined distribution.

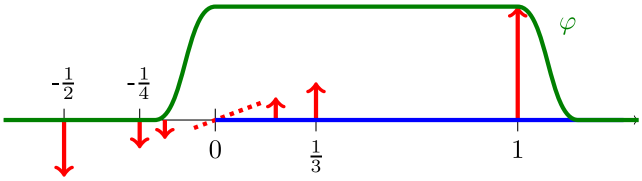

Figure 3.5. *Illustration of a "bad" distribution (red arrows) for which a restriction to the interval $(0,\infty)$ is not possible, because $D_{(0,\infty)}(\varphi)$ is not well defined.*

Hence one arrives at the following dilemma: Switched DAEs have distributional solutions so it is necessary to read (3.1) as an equation of distributions which in turn makes it necessary to define the product of a piecewise-constant function with a distribution or, equivalently, define a distributional restriction. But a distributional restriction cannot be defined in general. Another problem not mentioned so far is that for distributions it is not possible to speak of an initial value because distributions are not functions of time, e.g. $D(0)$ doesn't makes sense for general distributions $D \in \mathbb{D}$.

The underlying problem for this dilemma is the fact that the space of distributions is just *too big*; it contains distributions which are troublesome and lead to the above mentioned difficulties. For example there exist continuous functions which are nowhere differentiable (e.g. the Weierstrass function), but its distributional derivatives exist and it is hard to handle this kind of distributions. However, these "pathological" distributions are not really of interest when studying switched DAEs (3.1); the only important thing is that Dirac impulses (and its derivatives) should be handled. This leads to the definition of a suitable smaller space of distribution as defined in the next subsection.

3.2.4. *Piecewise-smooth distributions and solvability of* (3.1)

The following definition first appears in [TRE 08] (with more details in [TRE 09a]). For an overview on other approaches concerning distributional solution see the recent survey article [TRE 13].

DEFINITION.– *The space of* piecewise-smooth distributions *is defined as (see also Figure 3.6)*

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, T \subseteq \mathbb{R} \textit{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\},$$

where $\mathcal{C}^\infty_{\mathrm{pw}}$ *denotes the space of piecewise-smooth functions, i.e.* $f \in \mathcal{C}^\infty_{\mathrm{pw}}$ *if, and only if, there exists smooth functions* $f_i \in \mathcal{C}^\infty$, $i \in \mathbb{Z}$ *and a strictly ordered unbounded set* $\{ t_i \in \mathbb{R} \mid i \in \mathbb{Z} \}$ *such that* $f = \sum_{i\in\mathbb{Z}} (f_i)_{[t_i,t_{i+1})}$.

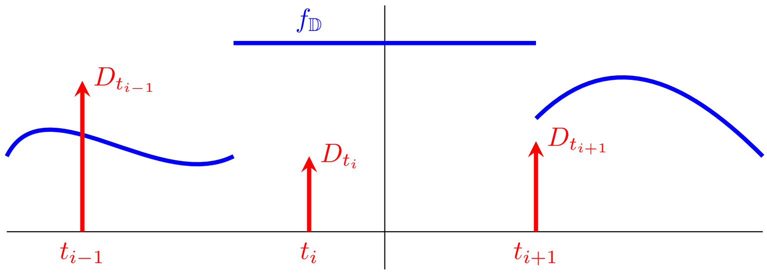

Figure 3.6. *Illustration of a piecewise-smooth distribution.*

Piecewise-smooth distributions have, among others, the following properties which are relevant here:

1. Closed under differentiation: $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \;\Rightarrow\; D' \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$.
2. Restriction to any interval $I \subseteq \mathbb{R}$ well defined:

$$D_I := (f_I)_{\mathbb{D}} + \sum_{t\in T\cap I} D_t, \quad \text{where } D = f_{\mathbb{D}} + \sum_{t\in T} D_t.$$

3. Multiplication with piecewise-smooth functions well-defined [2]:

$$\alpha D = \sum_{i\in\mathbb{Z}} \alpha_i D_{[t_i,t_{i+1})}, \quad \text{where } \alpha = \sum_{i\in Z} (\alpha_i)_{[t_i,t_{i+1})}.$$

4. Left and right evaluation possible:

$$D(t+) := f(t+),\; D(t-) := f(t-), \quad \text{where } D = f_{\mathbb{D}} + \sum_{t\in T} D_t.$$

2. This multiplication is in fact a special case of the Fuchssteiner multiplication [FUC 68, FUC 84] defined for piecewise-smooth distributions, for details see [TRE 09a, Sec. 2.4]

5. Impulse evaluation possible:

$$D[t] := \begin{cases} D_t, & \text{if } t \in T \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } D = f_{\mathbb{D}} + \sum_{t \in T} D_t.$$

With these properties it is now possible to speak of distributional solutions of the switched DAE (3.1):

$$x \text{ solves (3.1)} \quad :\Leftrightarrow \quad x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \text{ and (3.1) holds in } \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$$

and the following important result holds [TRE 12a, Cor. 6.5.2].

THEOREM.– *Consider the switched DAE* (3.1) *with regular matrix pairs* (E_p, A_p), $p \in \{1, 2, \ldots, \mathtt{P}\}$. *Then for all admissible switching signals* σ *there* exists *a solution* $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ *and* x *is* uniquely *determined on* $[0, \infty)$ *by the value* $x(0-)$. *In particular, all jumps and impulses induced by the switches are unique.*

3.3. Stability results

This section starts with a definition of asymptotic stability of switched DAEs (3.1) which takes into account that solutions are now distributions.

DEFINITION.– *The switched DAE* (3.1) *is called asymptotically stable if, and only if, for all switching signals* σ *and all solutions* x *of* (3.2) *it holds that* x *is impulse free (i.e.* $x[t] = 0$ *for all* $t \in \mathbb{R}$*) and* $x(t\pm) \to 0$ *for* $t \to \infty$

Note that impulse-freeness does not exclude the presence of jumps. The property of impulse-freeness and convergence to zero can be handled independently and a characterization of impulse-freeness is presented first [TRE 09a].

THEOREM.– *Consider a switched DAE* (3.1) *with regular matrix pairs* (E_p, A_p), $p \in \{1, 2, \ldots, \mathtt{P}\}$, *and corresponding consistency projector* Π_p. *Then all solutions* $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ *of* (3.1) *are impulse free for all admissible switching signals* σ *if, and only if, the following* impulse-freeness condition *holds*

$$\forall p, q \in \{1, 2, \ldots, \mathtt{P}\}: \quad E_q(I - \Pi_q)\Pi_p = 0 \tag{3.6}$$

EXAMPLE.– Consider Example 2 from the Introduction again. With the help of the Wong-sequences the matrices V_c, W_c corresponding to the closed switch and V_o, W_o corresponding to the open switch can be calculated easily:

$$V_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad V_o = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad W_o = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence the corresponding consistency projectors are

$$\Pi_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Pi_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since

$$E_o(I - \Pi_o)\Pi_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

the impulse-freeness condition (3.6) is not fulfilled and hence Dirac impulses can occur, which is in agreement of the ad-hoc analysis of Example 2 in the Introduction.

3.3.1. *Stability under arbitrary switching*

In this subsection the question shall be answered when a switched DAE (3.1) is asymptotically stable for all admissible switching signals. One necessary condition is by definition impulse-freeness of all solutions, hence (3.6) has to hold; furthermore, each individual mode $E_p\dot{x} = A_p x$ has to be asymptotically stable (which is one instance of the switched DAE (3.1) where the switching signal is constant), hence according to Theorem 3.2.2 there exist Lyapunov-functions V_p, $p \in \{1, 2, \ldots, \mathtt{P}\}$ for each mode of the switched DAE. For switched ODEs a sufficient condition for stability of the switched system would be $V_1 = V_2 = \ldots V_{\mathtt{P}}$, however Example 1a showed that this is not the case for switched DAEs anymore because the jumps have to be incorporated as well. This leads to the following result [LIB 09].

THEOREM.– *Consider a switched DAE* (3.1) *with regular matrix pairs* (E_p, A_p), $p \in \{1, 2, \ldots, \mathtt{P}\}$, *and corresponding consistency projectors* Π_p *and consistency spaces* $\mathfrak{C}_{E_p,A_p} = \operatorname{im}\Pi_p$. *Assume that the impulse-freeness condition* (3.6) *holds and that each mode has a Lyapunov function* V_p *according to Theorem 3.2.2. Then the following* Lyapunov-jump condition

$$\forall p, q = 1, \ldots, \mathtt{P}\ \forall x \in \mathfrak{C}_{(E_p,A_p)}: \quad V_q(\Pi_q x) \leq V_p(x) \tag{3.7}$$

ensures asymptotic stability of the switched DAE (3.1) *for all admissible switching signals.*

Note that the Lyapunov-jump condition implies that V_p and V_q must coincide on the intersection $\mathfrak{C}_{(E_p,A_p)} \cap \mathfrak{C}_{(E_q,A_q)}$ because for all x in that intersection it holds that $\Pi_q x = x = \Pi_p x$ and hence:

$$V_q(x) = V_q(\Pi_q x) \leq V_p(x) = V_p(\Pi_p x) \leq V_q(x).$$

In particular, for switched ODEs the condition (3.7) simplifies to the well known common-Lyapunov-function condition, because then all consistency space are $\mathbb{R}^n$.

Furthermore, if (3.7) holds it is possible to define a “common” Lyapunov function for (3.1) as follow:

$$V(x) := \begin{cases} V_p(x), & x \in \mathfrak{C}_{(E_p,A_p)}, \\ \text{arbitrary}, & \text{otherwise} \end{cases}$$

So Theorem 3.3.1 can also be read as: The switched DAE is asymptotically stable if there exists a common Lyapunov function V such that (3.7) holds. An interesting question is whether the converse also holds; this question is answered positively in Subsection 3.3.3 under certain commutativity assumptions and in Subsection 3.3.4 for the case of exponential stability.

EXAMPLE.– Consider again the Examples 1a and 1b from the Introduction. The corresponding consistency projectors are given by

Example 1a	Example 1b
$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$,	$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

It is easily seen that the impulse-freeness condition (3.6) holds for both examples. Consider now two Lyapunov functions V_1 and V_2 for Example 1a, then the Lyapunov-jump condition (3.7) would imply for all $\xi \in \mathbb{R}$

$$V_1((\xi,\xi)^\top) \geq V_2(\Pi_2(\xi,\xi)^\top) = V_2((0,2\xi)^\top) \geq V_1(\Pi_1(0,2\xi)^\top) = V_1((2\xi,2\xi)^\top).$$

Hence for all $x \in \mathfrak{C}_{(E_1,A_1)}$ it follows that $V_1(x) \leq V_1(2x)$ and V_1 cannot be a Lyapunov function for $E_1\dot{x} = A_1x$. Hence in agreement of the observed instability of Example 1a, it is not possible to find Lyapunov functions V_1 and V_2 fulfilling (3.7). On the other hand consider Example 1b where one can choose $P_1 = I = P_2$ and $Q_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & -2 \end{smallmatrix}\right] = Q_2$ as solutions of the corresponding generalized Lyapunov equation (3.5) leading to the Lyapunov function $V_1(x) = V_2(x) = x_2^2$. Now the Lyapunov-jump condition (3.7) is satisfied with equality:

$$x \in \mathfrak{C}_{(E_1,A_1)} \Leftrightarrow x = \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \text{ hence } V_2(\Pi_2 x) = V_2((0,\xi)^\top) = \xi^2 = V_1(x),$$

$$x \in \mathfrak{C}_{(E_2,A_2)} \Leftrightarrow x = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \text{ hence } V_1(\Pi_1 x) = V_1((\xi,\xi)^\top) = \xi^2 = V_2(x).$$

The above results also hold for nonlinear switched DAEs in quasi-linear form:

$$E_\sigma(x)\dot{x} = f_\sigma(x),$$

the impulse freeness condition (3.6) is then replaced by the condition:

$$\forall p,q \in \{1,\dots,\mathtt{P}\}\ \forall x_0^- \in \mathfrak{C}_p\ \exists \text{ unique } x_0^+ \in \mathfrak{C}_q :\ x_0^+ - x_0^- \in \ker E_q(x_0^+)$$

where $\mathfrak{C}_p$ is the consistency manifold of $E_p(x)\dot{x} = f_p(x)$; for details see [LIB 12].

3.3.2. *Slow switching*

For switched ODEs it is well known that fast switching might be the reason for an unstable behavior and that under a "slow switching" assumption asymptotic stability holds. Consider the set of switching signals with dwell time $\tau_d > 0$:

$$\Sigma^{\tau_d} := \left\{ \sigma : \mathbb{R} \to \{1, \dots, N\} \;\middle|\; \begin{array}{l} \forall \text{ switching times } t_i \in \mathbb{R}, i \in \mathbb{Z} : \\ t_{i+1} - t_i \geq \tau_d \end{array} \right\}.$$

The following result [LIB 09] shows now that for any impulse-free switched DAE with asymptotically stable modes a dwell time exists such that the switched DAE is asymptotically stable.

THEOREM.– *Consider the switched DAE* (3.1) *for which* (3.6) *is satisfied and each mode is asymptotically stable. Then there exists a dwell time* $\tau_d > 0$ *such that* (3.1) *is asymptotically stable for all* $\sigma \in \Sigma^{\tau_d}$.

Note that Examples 1a and 1b both satisfy (3.6) and all modes are asymptotically stable, hence both examples are asymptotically stable for slow switching; in fact a dwell time $\tau_d > \ln \sqrt{2}$ will make the switched DAE of Example 1a asymptotically stable and, as Example 1b is asymptotically stable for any switching signal, any dwell time $\tau_d > 0$ will make Example 1b asymptotically stable.

The above result also holds when considering an averaged dwell time condition [HES 99], for details see[LIB 12].

3.3.3. *Commutativity and stability*

It is well known [NAR 94] that for switched ODEs the following result holds:

THEOREM.– *Consider the switched ODE*

$$\dot{x} = A_\sigma x \tag{3.8}$$

with A_p *Hurwitz,* $p \in \{1, 2, \dots, \mathtt{P}\}$ *and commuting* A_p*, i.e.*

$$[A_p, A_q] := A_p A_q - A_q A_p = 0. \quad \forall p, q \in \{1, 2, \dots, \mathtt{P}\} \tag{3.9}$$

Then (3.8) *is asymptotically stable for all switching signals* σ.

The proof idea is in fact rather simple: Consider the switching times $t_0 < t_1 < \ldots < t_k < t$ of a switching signal σ and let $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}(t-t_k)} e^{A_{p_{k-1}}(t_k - t_{k-1})} \cdots e^{A_{p_1}(t_2-t_1)} e^{A_{p_0}(t_1-t_0)} x_0 \\ &\overset{(3.9)}{=} e^{A_1 \Delta t_1} e^{A_2 \Delta t_2} \cdots e^{A_{\mathtt{P}} \Delta t_{\mathtt{P}}} x_0 \end{aligned}$$

and $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

The obvious question is now: How to generalize this result to switched DAEs (3.1)? A straightforward generalization is however not obvious because it is not clear which matrices have to commute and how the jumps have to be incorporated. In fact, Example 1a is not asymptotically stable for fast switching but the A-matrices do commute! Hence simply assuming that the A-matrices commute does not work for switched DAEs. It turns out [LIB 11] that in fact the matrices A^{diff} as defined in Definition 3.2.1 are the ones of interest.

THEOREM.– *Consider the switched DAE* (3.1) *with regular matrix pairs satisfying* (3.6) *and with corresponding matrices A_p^{diff}, $p \in \{1, 2, \ldots, \mathtt{P}\}$ as in Definition 3.2.1. Assume the impulse-freeness condition* (3.6) *is satisfied and each mode of the switched DAE is asymptotically stable. If all A^{diff}-matrices commute, i.e.*

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathtt{P}\} \tag{3.10}$$

then (3.1) *is asymptotically stable for all admissible switching signals. Furthermore, if* (3.10) *holds then there exists a common* quadratic *Lyapunov function satisfying* (3.7).

It is interesting to note that there is no explicit conditions on the jumps via the consistency projectors. However, it can be shown that (3.10) implies

$$[\Pi_p, A_q^{\text{diff}}] = 0 \quad \wedge \quad [\Pi_p, \Pi_q] = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathtt{P}\}.$$

Invoking (3.4) it is now easy to see that for given switching times $t_0 < t_1 < \ldots < t_k < t$ and $p_i := \sigma(t_i+)$, the solution of (3.1) is given by

$$x(t) = e^{A_{p_k}^{\text{diff}}(t-t_k)}\Pi_{p_k} e^{A_{p_{k-1}}^{\text{diff}}(t_k-t_{k-1})}\Pi_{p_{k-1}} \cdots e^{A_{p_1}^{\text{diff}}(t_2-t_1)}\Pi_{p_1} e^{A_{p_0}^{\text{diff}}(t_1-t_0)}\Pi_{p_0} x_0$$

and invoking the above commutativity properties allows for the following rearrangement

$$x(t) = e^{A_1^{\text{diff}}\Delta t_1}\Pi_1\, e^{A_2^{\text{diff}}\Delta t_2}\Pi_2 \cdots e^{A_{\mathtt{P}}^{\text{diff}}\Delta t_{\mathtt{P}}}\Pi_{\mathtt{P}} x_0$$

where $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

The construction of the common quadratic Lyapunov function is as follows (for details see [LIB 11]): First choose a coordinate transformation T which simultaniously block-diagonalizes all A^{diff}-matrices, i.e. for all $p \in \{1, 2, \ldots, \mathtt{P}\}$

$$T^{-1} A_p^{\text{diff}} T = \operatorname{diag}(A_{p1}, A_{p2}, \ldots, A_{pl}) \quad \text{for some } l \in \mathbb{N},$$

where each A_{pk} is either the zero matrix or Hurwitz. This is possible due to the commutativity condition (3.10). Then, again due to (3.10) it is possible to find for each k a symmetric positive definite matrix P_k and a scalar $\alpha_k > 0$ such that:

$$A_{pk}^\top P_k + P_k A_{pk} < -\alpha_k P_k \quad \forall p \in \mathfrak{P}_k := \{\, p \in \{1, 2, \ldots, \mathtt{P}\} \mid A_{pk} \neq 0 \,\}.$$

If $\mathfrak{P}_k = \emptyset$ for some $k \in \{1, 2, \ldots, l\}$ then set $P_k = I$. A common quadratic Lyapunov function is now given by

$$V(x) = x^\top T^{-T} \operatorname{diag}(P_1, P_2, \ldots, P_l) T^{-1} x.$$

3.3.4. *Lyapunov exponent and converse Lyapunov theorem*

The goal of this chapter is to study the maximal exponential growth rate of the switched DAE (3.1) and then, using a characterization of a finite growth rate, establish the existence of a common Lyapunov function when (3.1) is (uniformly) exponentially stable for all admissible switching functions. For this consider again the explicit solution formula (under the assumption that (3.6) holds) already used in the previous subsection:

$$x(t) = \underbrace{e^{A^{\text{diff}}_{p_k}(t\text{-}t_k)}\Pi_{p_k} e^{A^{\text{diff}}_{p_{k-1}}(t_k\text{-}t_{k-1})}\Pi_{p_{k-1}} \cdots e^{A^{\text{diff}}_{p_1}(t_2\text{-}t_1)}\Pi_{p_1} e^{A^{\text{diff}}_{p_0}(t_1\text{-}t_0)}\Pi_{p_0}}_{=:\,\Phi^\sigma(t,t_0)} x(t_0-),$$

where $\Phi^\sigma(t, t_0)$ is the evolution matrix of (3.1) from t_0 to t corresponding to the switching signal σ. When considering all admissible switching signals σ then the set of evolution matrices only depend on the time difference $t - t_0$ and this leads to the following definition.

DEFINITION.– *Consider the switched DAE* (3.1) *with regular matrix pairs and let*

$$\mathcal{M} := \left\{ \, (A^{\text{diff}}_p, \Pi_p) \;\middle|\; \textit{corresponding to } (E_p, A_p), p = 1, \ldots, \mathtt{P} \, \right\}$$

be the set of A^{diff}-matrices and consistency projectors Π_p corresponding to the matrix pairs (E_p, A_p), $p \in \{1, 2, \ldots, \mathtt{P}\}$. The set of all evolution matrices with time span $t > 0$ is then

$$\mathcal{S}_t := \left\{ \prod_{i=0}^{k} e^{A^{\text{diff}}_i \tau_i} \Pi_i \;\middle|\; \begin{array}{l} (A^{\text{diff}}_i, \Pi_i) \in \mathcal{M}, \ \sum_{i=0}^{k} \tau_i = \Delta t, \\[2ex] \tau_i > 0, i = 1, 2, \ldots, k-1, \ \tau_k \geq 0 \end{array} \right\}.$$

Furthermore, denote the set of all evolution operators as $\mathcal{S} := \bigcup_{t>0} \mathcal{S}_t$.

An interesting property of the sets $\mathcal{S}_t$ is their semigroup property [TRE 12c]:

$$\mathcal{S}_{s+t} = \mathcal{S}_s \mathcal{S}_t := \left\{ \, \Phi_s \Phi_t \;\middle|\; \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t \, \right\}.$$

Note that this semigroup property relies on the fact that

$$[A^{\text{diff}}, \Pi] = 0 \quad \forall (A^{\text{diff}}, \Pi) \in \mathcal{M}.$$

The exponential growth bound for the switched DAE (3.1) is now defined as follows.

DEFINITION.– *For* $t > 0$ *the* exponential growth bound *of* (3.1) *is*

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\}.$$

This definition implies that for all solutions x of $E_\sigma \dot{x} = A_\sigma x$ satisfying (3.6) and all $t > 0$:

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\| \, \|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t)\, t} \|x(0-)\|.$$

A major difference to switched ODEs without jumps is the possibility that $\lambda_t(\mathcal{S}_t) = \pm\infty$! In fact, when all jumps are given by zero consistency projections then $\lambda_t(\mathcal{S}_t) = -\infty$. On the other hand $\lambda_t(\mathcal{S}_t) = \infty$ is also possible as Example 1a from the Introduction shows:

EXAMPLE.– Consider again Example 1a. For a high switching frequency the dynamics are dominated by the jumps, i.e.

$$\Phi^\sigma(t,0) \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

where $k \in \mathbb{N}$ is half the number of switches of σ in the interval $[0, t]$. Therefore, $\Phi^\sigma(t,0)$ is not bounded uniformly in σ and $\lambda_t(\mathcal{S}_t) = \infty$. The effect of increasing the switching frequency is illustrated in Figure 3.7.

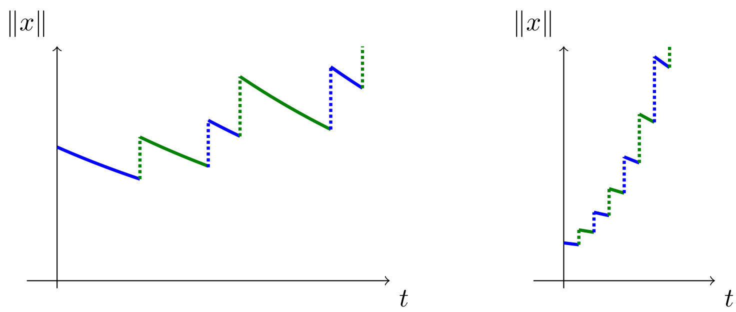

Figure 3.7. *The norm of a typical solution x of the switched DAE of Example 1a where the switching frequency is higher in the right picture than in the left picture. Apparently, the growth rate is much higher for faster switching.*

There is however a simple characterization for boundedness of $\mathcal{S}_t$ [TRE 12c]:

THEOREM.– *Consider the switched DAE* (3.1) *with the corresponding sets* $\mathcal{S}_t$, $t > 0$, *of evolution matrices and denote with* $\mathcal{M}_\Pi := \left\{ \Pi \;\middle|\; (A^{\text{diff}}, \Pi) \in \mathcal{M} \text{ for some } A^{\text{diff}} \right\}$ *the set of all consistency projectors. Then*

$$\mathcal{S}_t \text{ is bounded} \quad \Leftrightarrow \quad \mathcal{M}_\Pi \text{ is product bounded},$$

where a set is called product bounded if, and only if, all finite products are uniformly bounded.

If the set of evolution operators $\mathcal{S}_t$ is bounded it makes sense [TRE 12c] to consider the long-term growth bound or the (upper) Lyapunov exponent of $\mathcal{S}$.

THEOREM.– *Consider the switched DAE* (3.1) *with regular matrix pairs and associated evolution operator sets* $\mathcal{S}_t$ *and* $\mathcal{S}$. *Assume that the set of consistency projectors is product bounded and contains at least one nonzero projector. Then the* (upper) Lyapunov exponent

$$\lambda(\mathcal{S}) := \lim_{t\to\infty} \lambda_t(\mathcal{S}_t) = \lim_{t\to\infty} \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t}$$

of (3.1) *is well defined and finite.*

The above results does not rely on the assumption that the switched DAE (3.1) fulfills the impulse-freeness condition (3.6) because it is only concerned with the set $\mathcal{S}_t$ defined as products of exponentials and consistency projectors. In fact, even when impulses occur in the solutions of (3.1) it still holds that $x(t+) = \Phi_t x(0-)$ for some $\Phi_t \in \mathcal{S}_t$.

For formulating a converse Lyapunov theorem the following notion of exponential stability is needed.

DEFINITION.– *Consider the switched DAE* (3.1) *with regular matrix pairs satisfying* (3.6). *Then* (3.1) *is called* uniformly exponentially stable *if, and only if, there exists* $M \geq 1$ *and* $\mu > 0$ *such that for all admissible switching signals and all solutions it holds that*

$$\|x(t+)\| \leq M e^{-\mu t} \|x(0-)\|, \quad \forall t \geq 0.$$

The following converse Lyapunov theorem was first reported in [TRE 12b] and the details are presented in [TRE 12c].

THEOREM.– *Consider the switched DAE* (3.1) *with regular matrix pairs satisfying* (3.6). *If* (3.1) *is uniformly exponentially stable then there exists a common Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *for* (3.1) *in the sense that it decreases exponentially along solutions and* (3.7) *holds.*

The basic idea of the proof is as follows: Exponential stability with constants M and μ implies that the Lyapunov exponent fulfills

$$\lambda(\mathcal{S}) \leq -\mu < 0.$$

Choose any $\varepsilon \in (0, \mu)$ and define the Lyapunov function candidate

$$V(x) := \sup_{t>0} \sup_{\Phi_t \in \mathcal{S}_t} e^{-(\lambda(\mathcal{S})+\varepsilon)t} \|\Phi_t x\|.$$

It is then easily seen that

$$V(\Phi_t x) \leq e^{(\lambda(\mathcal{S})+\varepsilon)t} V(x),$$

in particular,

$$V(\Pi_p x) \leq V(x)$$

for all consistency projectors Π_p, $p \in \{1, 2, \ldots, \mathtt{P}\}$. Hence V is indeed a common Lyapunov function for (3.1).

Note that the above constructed Lyapunov function V is in general not smooth. The smoothing techniques from [LIN 96] or [CAI 07] might not work here because they might result in a Lyapunov function candidate violating (3.7).

3.4. Conclusion

This chapter has studied stability related questions for switched DAEs. It turns out that most classical stability results for switched ODEs can be generalized for switched DAEs; however, these generalizations are not always straightforward due the possible presence of jumps and impulses in the solutions of switched DAEs. The analysis presented here always assumed regularity of matrix pairs defining the modes of the switched DAE; this assumption guarantees existence and uniqueness of solutions, it would be interesting to investigate switched DAEs without this assumption. Another avenue of further research concerns the commutativity results which might be generalized to certain Lie-algebraic conditions. Another question is concerned how exactly switched DAEs fit into the general hybrid-systems framework as proposed in [GOE 09]; one apparent incompatibility is the possible presence of Dirac impulses in solutions of switched DAEs.

3.5. Acknowledgements

Many thanks to Carmen Pedicini and Daniel Liberzon for providing valuable feedback on the manuscript of this chapter.

3.6. Bibliography

[BER 12] BERGER T., ILCHMANN A., TRENN S., "The quasi-Weierstraß form for regular matrix pencils", *Lin. Alg. Appl.*, vol. 436, num. 10, p. 4052–4069, 2012.

[CAI 07] CAI C., TEEL A. R., GOEBEL R., "Smooth Lyapunov functions for hybrid systems. I. Existence is equivalent to robustness", *IEEE Trans. Autom. Control*, vol. 52, num. 7, p. 1264–1277, 2007.

[DOM 10] DOMÍNGUEZ-GARCÍA A. D., TRENN S., "Detection of Impulsive Effects in Switched DAEs with Applications to Power Electronics Reliability Analysis", *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA*, p. 5662–5667, 2010.

[FUC 68] FUCHSSTEINER B., "Eine assoziative Algebra über einen Unterraum der Distributionen", *Mathematische Annalen*, vol. 178, p. 302–314, 1968.

[FUC 84] FUCHSSTEINER B., "Algebraic foundation of some distribution algebras", *Studia Mathematica*, vol. 76, p. 439–453, 1984.

[GAN 59] GANTMACHER F. R., *The Theory of Matrices (Vol. I & II)*, Chelsea, New York, 1959.

[GEE 96a] GEERTS A. H. W. T., SCHUMACHER J. M. H., "Impulsive-Smooth Behavior in Multimode Systems. Part I: State-Space and Polynomial Representations", *Automatica*, vol. 32, num. 5, p. 747–758, 1996.

[GEE 96b] GEERTS A. H. W. T., SCHUMACHER J. M. H., "Impulsive-Smooth Behavior in Multimode Systems. Part II: Minimality and Equivalence", *Automatica*, vol. 32, num. 6, p. 819–832, 1996.

[GOE 09] GOEBEL R., SANFELICE R. G., TEEL A. R., "Hybrid dynamical systems", *IEEE Control Systems Magazine*, vol. 29, num. 2, p. 28–93, 2009.

[HES 99] HESPANHA J. P., MORSE A. S., "Stability of switched systems with average dwell-time", *Proc. 38th IEEE Conf. Decis. Control*, p. 2655–2660, 1999.

[LIB 03] LIBERZON D., *Switching in Systems and Control*, Systems and Control: Foundations and Applications, Birkhäuser, Boston, 2003.

[LIB 09] LIBERZON D., TRENN S., "On stability of linear switched differential algebraic equations", *Proc. IEEE 48th Conf. on Decision and Control*, p. 2156–2161, December 2009.

[LIB 11] LIBERZON D., TRENN S., WIRTH F. R., "Commutativity and asymptotic stability for linear switched DAEs", *Proc. 50th IEEE Conf. Decis. Control and European Control Conference ECC 2011, Orlando, USA*, p. 417–422, 2011.

[LIB 12] LIBERZON D., TRENN S., "Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability", *Automatica*, vol. 48, num. 5, p. 954–963, May 2012.

[LIN 96] LIN Y., SONTAG E. D., WANG Y., "A smooth converse Lyapunov theorem for robust stability", *SIAM J. Control Optim.*, vol. 34, num. 1, p. 124–160, 1996.

[MEN 06a] MENG B., "Observability Conditions of Switched Linear Singular Systems", *Proceedings of the 25th Chinese Control Conference*, Harbin, Heilongjiang, China, p. 1032–1037, 2006.

[MEN 06b] MENG B., ZHANG J.-F., "Reachability Conditions for Switched Linear Singular Systems", *IEEE Trans. Autom. Control*, vol. 51, num. 3, p. 482–488, 2006.

[NAR 94] NARENDRA K. S., BALAKRISHNAN J., "A common Lyapunov function for stable LTI systems with commuting A-matrices", *IEEE Trans. Autom. Control*, vol. 29, num. 12, p. 2469–2471, December 1994.

[OWE 85] OWENS D. H., DEBELJKOVIC D. L., "Consistency and Liapunov Stability of Linear Descriptor Systems: A Geometric Analysis", *IMA J. Math. Control & Information*, vol. 2, p. 139–151, 1985.

[RAO 10] RAOUF J., MICHALSKA H. H., "Exponential Stabilization of Singular Systems by Controlled Switching", *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA*, IEEE Control Systems Society, IEEE, p. 414–419, December 2010.

[SCH 51] SCHWARTZ L., *Théorie des Distributions I,II*, Num. IX,XPublications de l'institut de mathématique de l'Universite de Strasbourg, Hermann, Paris, 1950, 1951.

[TAN 10] TANWANI A., TRENN S., "On observability of switched differential-algebraic equations", *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA*, p. 5656–5661, 2010.

[TRE 08] TRENN S., "Distributional solution theory for linear DAEs", *PAMM - Proc. Appl. Math. Mech.*, vol. 8, Weinheim, GAMM Annual Meeting 2008, Bremen, Wiley-VCH Verlag GmbH, p. 10077–10080, 2008, published online: 25 Feb 2009.

[TRE 09a] TRENN S., Distributional differential algebraic equations, PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany, 2009.

[TRE 09b] TRENN S., "Regularity of distributional differential algebraic equations", *Math. Control Signals Syst.*, vol. 21, num. 3, p. 229–264, 2009.

[TRE 12a] TRENN S., "Switched differential algebraic equations", VASCA F., IANNELLI L., Eds., *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, Chapter 6, p. 189–216, Springer-Verlag, London, 2012.

[TRE 12b] TRENN S., WIRTH F., "A converse Lyapunov theorem for switched DAEs", *PAMM (Special Issue: 83rd Annual Meeting of the GAMM)*, vol. 12, Weinheim, Wiley-VCH Verlag GmbH, p. 789–792, 2012.

[TRE 12c] TRENN S., WIRTH F., "Linear switched DAEs: Lyapunov exponent, converse Lyapunov Theorem, and Barabanov norm", *Proc. 51st IEEE Conf. Decis. Control, Maui, Hawaii*, p. 2666–2671, 2012.

[TRE 13] TRENN S., "Solution concepts for linear DAEs: a survey", ILCHMANN A., REIS T., Eds., *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, Springer-Verlag, 2013, to appear.

[WEI 68] WEIERSTRASS K., "Zur Theorie der bilinearen und quadratischen Formen", *Berl. Monatsb.*, p. 310–338, 1868.

[WUN 08] WUNDERLICH L., Analysis and Numerical Solution of Structured and Switched Differential-Algebraic Systems, PhD thesis, Fakultät II Mathematik und Naturwissenschaften, Technische Universität Berlin, Berlin, Germany, 2008.

[ZHA 06] ZHAI G., KOU R., IMAE J., KOBAYASHI T., "Stability analysis and design for switched descriptor systems", *Proceedings of the 2006 IEEE International Symposium on Intelligent Control*, p. 482–487, 2006.

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