Stabilization of switched DAEs via fast switching

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Switched differential algebraic equations (switched DAEs) can model dynamical systems with state constraints together with sudden structural changes (switches). These switches may lead to induced jumps and can destabilize the system even in the case that each mode is stable. However, the opposite effect is also possible; in particular, the question of finding a stabilizing switching signal is of interest. Two approaches are presented how to stabilize a switched DAE via fast switching.

1 Introduction

In this note the stabilization of switched differential-algebraic equations (switched DAEs)

$$E_\sigma \dot{x} = A_\sigma x \tag{1}$$

via fast switching is investigated. In particular, the Mironchenko-Wirth-Wulff (MWW) approach [5, 6] is compared with our recently proposed approach via averaging [9]. It is well known, that switched DAEs of the form (1) exhibit jumps in the state-variable x as well as possible Dirac-impulses (derivatives of jumps), c.f. [12]. The latter impulsive behavior is important, but does not influence the jumps and the continuous flow of the solutions. Because of that, only the impulse-free part of the solution will be considered in the following. The effect of fast switching on the impulsive part of the solution is not yet fully understood, some first (surprising) results are discussed in [11].

A sufficient criterium for stabilizability via fast switching for switched ordinary differential equations (switched ODEs) of the form

$$\dot{x} = A_\sigma x \tag{2}$$

is the existence of a Hurwitz convex combination $\sum_i d_i A_i$, $d_i \in [0, 1]$ with $\sum_i d_i = 1$. In that case the classical averaging results yield that the solution of (2) can be approximated arbitrarily well via fast switching by the averaged non-switched ODE

$$\dot{x} = A_{av} x \text{ with } A_{av} := \sum_i d_i A_i. \tag{3}$$

In this note, the generalization of this result to switched DAEs is discussed.

2 DAE preliminaries

A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is called regular, iff $\det(Es - A) \in \mathbb{R}[s]$ is not the zero polynomial. It is a well known fact, that (E, A) is regular if, and only if, there exist invertible $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{4}$$

where $N \in \mathbb{R}^{n_N \times n_N}$ is nilpotent. Following [1] the decoupling (4) is called quasi-Weierstrass form (QWF), which can easily obtained via the so-called Wong sequences.

Definition 2.1 (Consistency projector and flow matrix) For a regular matrix pair (E, A) with QWF (4), the consistency projector and the flow matrix are

$$\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad A^{\text{diff}} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

where block sizes correspond to the QWF.

The relevance of these two matrices is highlighted in the following statement, which was first utilized in [10] in the context of observability characterization for switched DAEs.

Lemma 2.2 *All solutions of $E\dot{x} = Ax$ (restricted on $[0, \infty)$) satisfy*

$$x(t) = e^{A^{\text{diff}} t} \Pi x(0^-), \quad t > 0,$$

where $x(0^-)$ is the (possibly inconsistent) initial value just before the DAE gets activated.

3 The MWW-approach

The key ingredient of the MWW-approach is the "jump & flow approximation matrix"

$$A^\varepsilon := T \begin{bmatrix} J & 0 \\ 0 & -\frac{1}{\varepsilon} I \end{bmatrix} T^{-1} = A^{\text{diff}} - \frac{1}{\varepsilon}(I - \Pi),$$

where $\varepsilon > 0$, and the corresponding approximation result [6, Lem 4]:

$$\left(e^{A^{\text{diff}} t} \Pi - e^{A^\varepsilon t} \right) x_0 = -e^{-\frac{t}{\varepsilon}} (I - \Pi) x_0,$$

in particular,

$$\forall \varepsilon > 0: \quad x_\varepsilon(t) \to x(t) \text{ as } t \to \infty,$$
$$\forall t > 0: \quad x_\varepsilon(t) \to x(t) \text{ as } \varepsilon \to 0,$$

where $x_\varepsilon(\cdot)$ denotes the solution of the ODE

$$\dot{x}_\varepsilon = A^\varepsilon x_\varepsilon, \quad x_\varepsilon(0) = x_0$$

and $x(\cdot)$ denotes the (impulse-free part of the) solution of $E\dot{x} = Ax$ restricted to $[0, \infty)$ with (possibly inconsistent) initial condition $x(0^-) = x_0$.

As a consequence, for sufficiently small $\varepsilon > 0$, the (impulse-free part of the) solution of the switched DAE (1) is approximated well by the solution of the switched ODE

$$\dot{x} = A^\varepsilon_\sigma x, \quad x(0) = x_0. \tag{5}$$

Now the classical averaging result can be used (provided there exists a Hurwitz convex combination $\sum_i d_i A^\varepsilon_i$) to stabilize

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(5). As highlighted already in [6], this approach does not always work, some additional assumptions need to be satisfied. Note that although the following approximation holds:

$$\forall t > 0: \quad x_{\sigma,\varepsilon}(t) \to x_\sigma(t^-) \text{ as } \varepsilon \to 0,$$
$$\forall t > 0: \quad x_{\sigma,\varepsilon}(t) \to x_{av}(t) \text{ as } p \to 0$$

where x_σ is the solution of (1) with initial condition $x_\sigma(0^-) = x_0$ and periodic switching signal σ with period $p > 0$, $x_{\sigma,\varepsilon}$ is the solution of (5) and x_{av} is the solution of the corresponding averaged ODE (3), it is *not true* in general that x_σ approximates $x_{\sigma,\varepsilon}$ as $p \to 0$, in fact

$$x_\sigma(t^-) - x_{\sigma,\varepsilon}(t) \to \infty \text{ as } p \to 0 \text{ is possible.}$$

As shown in [5, 6] this blow up can be prevented, if the family of switched ODEs (5) can be stabilized uniformly via a single switching signal for all sufficiently small $\varepsilon > 0$. However, in general this will not be satisfied because the averaging approach will result in switching signals depending on ε. In the special situation that the flow matrices commute, then it can be shown [6, Lem. 8], that such a uniform stabilization is possible.

The underlying problem of the MWW-approach is that the destabilizing effect of the consistency projectors is not directly taken into account; furthermore, the nonexistence of an averaged model for the switched DAE (i.e. non-convergence of the solution trajectory to some trajectory of a non-switched system for increasingly fast switching) is not considered. On the other hand, the MWW-approach does not necessarily need the latter assumption to work, because the main result in [5,6] is not restricted to stabilization via the averaging approach (e.g. in [5] the stabilization capability of the projectors itself is investigated).

4 A direct averaging approach

The MWW-approach can be seen as an indirect averaging approach for stabilization via fast switching, because the switched DAE is first approximated by a switched ODE and the latter is then approximated by a non-switched ODE via fast switching. An alternative approach is to directly consider the averaged model for the switched DAE (without introducing the approximation parameter $\varepsilon > 0$), i.e. utilizing our averaging results [2–4, 7–9] for the problem of stabilization via fast switching. Assume the switching signal $\sigma : [0,\infty) \to \{1, 2, \dots, q\}$ is periodic with period $p > 0$ and duty cycles $d_i \in (0, 1)$, $\sum_i d_i = 1$. Without restriction of generality it can be assumed that σ is monotonically increasing on each periodicity interval. Then the following averaging results holds:

Theorem 4.1 ([9]) *Consider the regular switched DAE* (1) *with consistency projectors* Π_i *and flow matrices* A_i^{diff} *and let*

$$\Pi_\cap := \Pi_q \Pi_{q-1} \cdots \Pi_2 \Pi_1.$$

If the consistency projectors satisfy

$$\begin{aligned} \forall i: &\quad \operatorname{im}\Pi_i \supseteq \operatorname{im}\Pi_\cap \\ \forall i: &\quad \ker\Pi_i \subseteq \ker\Pi_\cap \end{aligned} \qquad \text{(PA)}$$

then on each compact interval in $(0, \infty)$

$$\|x_\sigma - x_{av}\|_\infty \le C\|x_\sigma(0^-)\|p,$$

where x_{av} *is the solution of*

$$\dot{x}_{av} = \Pi_\cap A_{av}^{\text{diff}} \Pi_\cap x_{av}, \quad x_{av}(0) = \Pi_\cap x_\sigma(0^-) \quad (6)$$

with $A_{av}^{\text{diff}} := \sum_i d_i A_i^{\text{diff}}$. *In particular, exponential stability of* (6) *implies exponential stability of* (1) *for sufficiently fast switching.*

Note that condition (PA) is weaker than assuming commutativity of the consistency projector which in turn is a weaker than assuming commutativity of the flow matrices A^{diff}. The latter assumption was used in [5, 6] to show uniform stabilizability of (5) and hence applicability of the averaging approach. With the direct averaging approach the class of switched DAEs for which the averaging approach can be utilized for stabilization is therefore significantly enlarged.

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