Edge-wise funnel synchronization

Stephan Trenn*

Fachbereich Mathematik, TU Kaiserslautern, 67663 Kaiserslautern

Recently, it was suggested in [Shim & Trenn 2015] to use the idea of funnel control in the context of synchronization of multi-agent systems. In that approach each agent is able to measure the difference of its own state and the average state of its neighbours and this synchronization error is used in a typical funnel gain feedback law, see e.g. [Ilchmann & Ryan 2008]. Instead of considering one error signal for each node of the coupling graph (corresponding to an agent) it is also possible to consider one error signal for each edge of the graph. In contrast to the node-wise approach this edgewise funnel synchronization approach results (at least in simulations) in a predictable consensus trajectory.

1 Introduction

Consider $N \in \mathbb{N}$ agents with individual scalar dynamics:

$$\dot{x}_i = f_i(t, x_i) + u_i \tag{1}$$

where for $i \in \{1, \ldots, N\} =: V$, $x_i : \mathbb{R} \to \mathbb{R}$ is the state of agent i, $u_i : \mathbb{R} \to \mathbb{R}$ is its input and $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a possibly time-varying nonlinear function which is measurable in t and local Lipschitz in x. The agents are connected via a network which is given by an undirected graph $G = (V, E)$ with $E \subseteq V \times V$. The goal is to design a local feedback law which achieves practical synchronization, i.e.

$$x_1 \approx x_2 \approx \ldots \approx x_n;$$

here "local" means that the input of each agent only depends on the state of itself and its neighbours.

For a graph $G = (V, E)$ let $\mathcal{N}_i := \{\, j \in V \mid (j, i) \in E \,\}$, $d_i := |\mathcal{N}_i|$ and $\mathcal{L}$ be the Laplacian of G. In [4] it was shown that under mild assumptions the classical diffusive coupling

$$u_i = -k \sum_{j \in \mathcal{N}_i} (x_i - x_j) \tag{2}$$

or, equivalently, $\boxed{u = -k\, \mathcal{L}\, x}$ yields practical synchronization, i.e. for all $\varepsilon > 0$ there exists $K > 0$ such that the closed loop (1), (2) with any $k \geq K$ results in solutions satisfying

$$\limsup_{t \to \infty} |x_i(t) - x_j(t)| < \varepsilon \quad \forall i, j \in V.$$

Indeed it can be shown that the consensus trajectory is given by the solution of

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t)) \tag{3}$$

with initial value $s(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$. To illustrate this behavior, consider the following example (taken from [3]) consisting of five agents coupled in a ring topology and with dynamics, $i \in \{1, \ldots, 5\}$:

$$\begin{aligned} f_i(t, x_i) &= (-1 + \delta_i) x_i + 10 \sin t \\ &\quad + 10 m_i^1 \sin(0.1t + \theta_i^1) + 10 m_i^2 \sin(10t + \theta_i^2), \end{aligned}$$

with randomly chosen parameters $\delta_i, m_i^1, m_i^1 \in \mathbb{R}$ and $\theta_i^1, \theta_i^2 \in [0, 2\pi]$. Note that this example with the same parameters will also be used for the simulation in Figure 3. In particular, in all simulations, the second system is unstable because $\delta_2 > 1$. The behavior of the closed loop for some randomly chosen initial values is shown in Figure 1 for two different gain values.

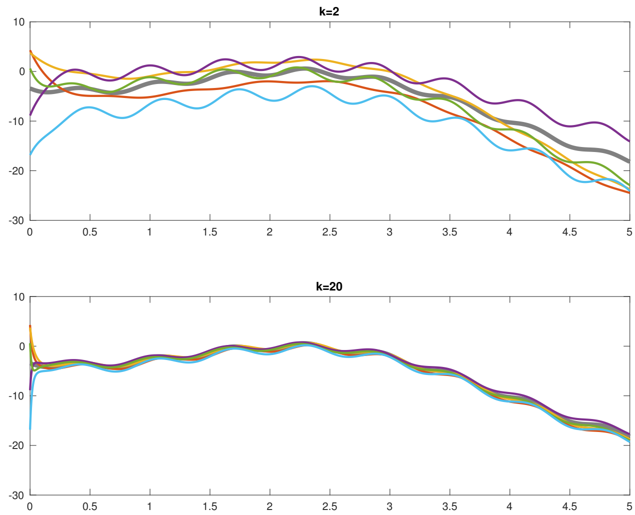

Fig. 1: Closed loop behaviour for diffusive coupling with constant gain $k = 2$ (above) and $k = 20$ (below), the thick grey line is the trajectory of the average dynamics s given by (3).

Funnel control was originally introduced by [2], see also the survey [1], and in its simplest variant was used to achieve reference tracking of nonlinear systems given by $\dot{y} = h(t, y) + u$ for a given reference signal y_{ref}. For the error $e(t) := y(t) - y_{\text{ref}}(t)$ the feedback law has the extremely simple form

$$u(t) = -k(t) e(t), \quad k(t) = \frac{1}{\varphi(t) - |e(t)|}, \tag{4}$$

where $\varphi : [0, \infty) \to [\underline{\varphi}, \overline{\varphi}]$ is the prespecified error bound with $0 < \underline{\varphi} < \overline{\varphi}$, see Figure 2.

* e-mail trenn@mathematik.uni-kl.de

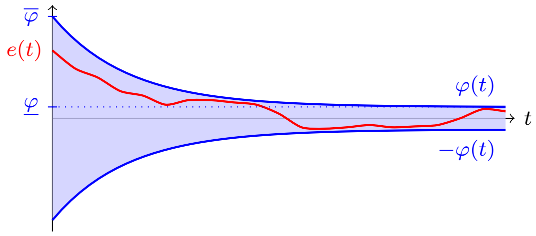

Fig. 2: Illustration of funnel control.

2 Funnel synchronization

The basic idea of funnel synchronization is the replacement of the constant gain k in (2) by the funnel gain $k(t)$ as in (4) with the error for agent i given by

$$e_i = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} (x_i - x_j) = x_i - \overline{x}_i,$$

where $\overline{x}_i$ denotes the average of the neighbours of agent i. The corresponding feedback law is then

$$u_i(t) = -k_i(t)\, d_i\, e(t), \quad k_i(t) = \frac{1}{\varphi(t) - |e_i(t)|}.$$

This approach was studied in [5] and it was (numerically) observed that synchronization occurs, but not to the trajectory given by (3). This problem can be avoided by using a weakly centralized approach by applying the same gain (defined by the maximum of all individual funnel gains) for all agents, i.e.

$$u_i(t) = k_{\max}(t) d_i e_i(t), \quad k_{\max}(t) = \max_i k_i(t).$$

Although no general proof is available it seems that a key fact is that the weakly centralized approach preserves the Laplacian feedback form because

$$u(t) = -k_{\max}(t)\mathcal{L}x(t),$$

where $k_{\max}(t)\mathcal{L}$ still is a (time-varying, weighted) Laplacian matrix, in contrast to the node-wise funnel feedback which takes the form

$$u(t) = -K(t)\mathcal{L}x(t) = -\begin{bmatrix} k_1(t) & & & \\ & k_2(t) & & \\ & & \ddots & \\ & & & k_N(t) \end{bmatrix} \mathcal{L}x(t),$$

where $K(t)\mathcal{L}$ is not symmetric in general. Motivated by this observation, another approach is the consideration of edgewise gains, i.e. the following generalization of diffusive coupling:

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} k_{ij}(x_i(t) - x_j(t)),$$

with $k_{ij} = k_{ji} > 0$. This results in an overall feedback law

$$u = -\mathcal{L}_K x(t)$$

where $(\mathcal{L}_K)_{ij} := -k_{ij}$ if $(i,j) \in E$, $(\mathcal{L}_K)_{ij} := 0$ if $i \neq j$ and $(i,j) \notin E$ and $(\mathcal{L}_K)_{ij} := \sum_{\ell \in \mathcal{N}_i} k_{i\ell}$ if $i = j$ is again a (weighted) Laplacian matrix. In particular, $(1, 1, \ldots, 1)^\top$ is in the left kernel of $\mathcal{L}_K$ and therefore the proof idea of [4, Sec. II.A] goes through without significant changes and one obtains the following result:

Theorem 2.1 *Consider N agents given by* (1) *satisfying the boundedness assumption from [4, Ass. 1] coupled with an undirected connected graph. If the average dynamics* (3) *remain bounded for each initial value then for all $\varepsilon > 0$ there exists $K > 0$ such that for all $k_{ij} \geq K$ the closed loop satisfies:*

$$\limsup_{t \to \infty} |x_i(t) - s(t)| < \varepsilon.$$

Note that the proof idea cannot be used in case the gain is time-varying, because then the used coordinate transformation becomes time-varying and the derivative of the transformation occurs as an additional term. Nevertheless, the novel approach is now to replace the constant gain k_{ij} by the funnel feedback law:

$$k_{ij}(t) = \frac{1}{\varphi(t) - |e_{ij}(t)|}, \quad e_{ij} := x_i - x_j.$$

The overall feedback then takes the form

$$u(t) = -\mathcal{L}_K(t)x(t),$$

with $\mathcal{L}_K(t)$ defined as $\mathcal{L}_K$ with k_{ij} replaced by $k_{ij}(t)$. In particular, $\mathcal{L}_K(t)$ is a (time-varying, weighted) Laplacian matrix and [5, Lem. 2] can directly be applied to guarantee boundedness of all solutions of the closed loop. However, as of now no proof is available yet that all errors stay away from the funnel boundary, but simulations look promising, see Figure 3. In that simulation the funnel boundary was chosen to be

$$\varphi(t) = (\overline{\varphi} - \underline{\varphi})e^{-\lambda t} + \underline{\varphi}$$

with $\overline{\varphi} = 20$, $\underline{\varphi} = 1$, $\lambda = 1$. With this choice the initial funnel size was large enough so that all initial (edgewise) errors where inside the funnel.

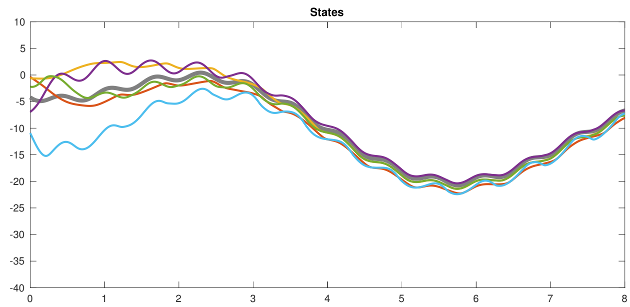

Fig. 3: Simulation results for edgewise funnel synchronization; the grey thick line is the solution of (3)

References

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