Distributional solution theory of linear DAEs

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Contents

Motivation

$$\begin{aligned} E(\cdot)\,\dot{x} &= A(\cdot)\,x + B(\cdot)u \\ y &= C(\cdot)x \end{aligned} \qquad E \text{ singular} \qquad (1)$$

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Equivalence:

$$(1) \quad \overset{x=Tz}{\Longleftrightarrow} \quad \begin{aligned} SET\dot{z} &= (SAT - SET')z + SBu \\ y &= CTz \end{aligned}$$

for invertible matrices S, T

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Assumption

“Type” of transformation matrices S, T equal to “type” of coefficient matrices E, A, B, C.

Assumptions

$$
\begin{aligned}
SET\dot{z} &= (SAT - SET')z + SBu \\
y &= CTz
\end{aligned}
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"Negative" assumptions

- Coefficients time-varying and not necessarily continuous
- Inhomogenity not necessarily continuous
- Initial values not necessarily consistent

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Solution theory under this assumptions.

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Goal

Solution theory under this assumptions. Consequences:

- Distributional solutions
- Distributional coefficients
- Multiplication of distributions!

Contents

Distributions revisited

Definition

- Test functions:

$$\mathcal{C}_0^\infty := \{ \varphi \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}) \mid \operatorname{supp} f \text{ is compact } \}$$

- Distributions:

$$\mathbb{D} := \{ D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous } \}$$

- Distributions with given support $M \subseteq \mathbb{R}$:

$$\mathbb{D}_M := \{ D \in \mathbb{D} \mid \operatorname{supp} D \subseteq M \}$$

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Theorem (Distributions with point support)

$$D \in \mathbb{D}_{\{t\}},\ t \in \mathbb{R} \quad \Rightarrow \quad \exists\, \alpha_0, \ldots, \alpha_n \in \mathbb{R} : D = \sum_{i=0}^n \alpha_i \delta_t^{(i)}$$

Dirac-impulse and its derivatives: $\delta_t^{(i)}(\varphi) = (-1)^i \varphi^{(i)}(t)$

Piecewise smooth distributions

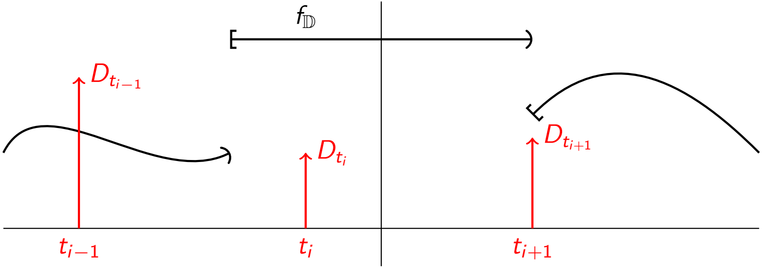
Definition (Piecewise smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$)

$D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \subset \mathbb{D}$ is a piecewise smooth distribution

$:\Longleftrightarrow$

$\exists\, f \in \mathcal{C}^\infty_{\text{pw}} \quad \exists$ **feasible** $T \subseteq \mathbb{R} \quad \exists \left\{ D_t \in \mathbb{D}_{\{t\}} \mid t \in T \right\}$:

$$D = f_{\mathbb{D}} + \sum_{t \in T} D_t$$



Properties of piecewise smooth distributions

Theorem (Properties of $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$)

Let $F = f_{\mathbb{D}} + \sum_{t\in T} F_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$.

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  $t_0 \in \mathbb{R}: \quad F(t_0-), F(t_0+) \in \mathbb{R}$ *und* $F[t_0] \in \mathbb{D}_{\{t_0\}}$

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  $M \subseteq \mathbb{R}$ *interval* $: \quad F_M \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \cap \mathbb{D}_M$

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- **Restriction to intervals:**
  $M \subseteq \mathbb{R}$ *interval*: $\quad F_M \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \cap \mathbb{D}_M$
- **Associative multiplication (Fuchssteiner multiplication):**
  $G \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}: \quad FG \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ *with*
  - $(FG)' = F'G + FG'$,
  - $(fg)_{\mathbb{D}} = f_{\mathbb{D}} g_{\mathbb{D}} \quad \forall f, g \in \mathcal{C}^\infty_{\mathrm{pw}}$,
  - $\delta_t F = F(t-)$ *and* $F\delta_t = F(t+)$

The Fuchssteiner multiplication

Recall: $F \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \quad \Leftrightarrow \quad F = f_\mathbb{D} + F[\cdot]$, where

- $f \in \mathcal{C}^\infty_{\text{pw}}$ and $F[\cdot] = \sum_{t \in T} F[t]$
- $F[t] = \sum_{i=0}^{n_t} \alpha_i^t \delta_t^{(i)}$

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Definition (Multiplication by Dirac impulses)

For $F \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and $t \in \mathbb{R}$ let

$$\delta_t F := F(t-)\delta_t \quad \text{and} \quad F\delta_t := F(t+)\delta_t$$

and for $n \in \mathbb{N}$

$$\delta_t^{(n+1)} F := \left(\delta_t^{(n)} F\right)' - \delta_t^{(n)} F', \quad F\delta_t^{(n+1)} := \left(F\delta_t^{(n)}\right)' - F'\delta_t^{(n)}.$$

Hence for $F, G \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$:

$$FG = (fg)_{\mathbb{D}} + f_{\mathbb{D}} F[\cdot] + G[\cdot] g_{\mathbb{D}}$$

Distributional DAEs

$$E\,\dot{x} = A\,x + Bu$$
$$y = Cx$$

E, A, B, C matrices with $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$-entries
x, y, u vectors with $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$-entries

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Question: When is a distributional matrix (not) invertible?

Theorem (Invertibility of distributional matrices)

Let $E = (E_{\mathrm{reg}})_{\mathbb{D}} + E[\cdot] \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ *for* $n \in \mathbb{N}$ *and* $E_{\mathrm{reg}} \in \left(\mathcal{C}^\infty_{\mathrm{pw}}\right)^{n\times n}$.

$$E \text{ is invertible} \quad \Leftrightarrow \quad E_{\mathrm{reg}} \text{ is invertible in } \left(\mathcal{C}^\infty_{\mathrm{pw}}\right)^{n\times n}$$

If E *is invertible then*

$$E^{-1} = (E_{\mathrm{reg}})^{-1} - (E_{\mathrm{reg}})^{-1} E[\cdot] (E_{\mathrm{reg}})^{-1}$$

Contents

Examples

Example

$\dot{x} = \delta x$

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$\dot{x} = \delta x$

Solution: $x = (\mathbb{1} + \mathbb{1}_{[0,\infty)})x_0$ for some $x_0 \in \mathbb{R}$

Check: $\dot{x} = x_0\delta = x(0-)\delta = \delta x$

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Example

$\delta\dot{x} = x$

Only solution: $x = 0$

Solutions for pure DAEs

Theorem (Solution of a pure DAE)

For $N \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ with N_{reg} strictly lower triangular and $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ the pure DAE

$$N\dot{x} = x + v$$

has the unique solution

$$x = \sum_{i=0}^{n-1} \left(N \tfrac{\mathrm{d}_{\mathbb{D}}}{\mathrm{d}t}\right)^i (v)$$

Note that for $v = 0$ the only solution is $x = 0$.

Solutions for distributional ODEs

Theorem (Solution of a distributional ODE)

For $A \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$, $t_0 \in \mathbb{R}$ *with* $A[\cdot]_{(-\infty,t_0)} = 0$, *and* $v \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ *all solutions of*

$$\dot{x} = Ax + v$$

are given by

$$x = \Phi_{t_0} x_0 + \Psi_{t_0}(v), \quad x_0 \in \mathbb{R}^n$$

where $\Phi_{t_0} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ *and* $\Psi_{t_0} : (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ *is a linear operator.*

The matrix Φ_{t_0} and the operator Ψ_{t_0} are given by ...

The matrix Φ_{t_0}

$\dot{x} = Ax + v$ with solution $x = \Phi_{t_0} x_0 + \Psi_{t_0}(v)$

The matrix Φ_{t_0}

Let $\phi(\cdot, t_0) \in (\mathcal{C} \cap \mathcal{C}^\infty_{\text{pw}})^{n\times n}$ be the transfer matrix of the standard (time-varying) ODE $\dot{x} = A_{\text{reg}} x$.

$$\Phi^0_{t_0} := \phi(\cdot, t_0)_{\mathbb{D}},$$

and for $n \in \mathbb{N}$

$$\Phi^{n+1}_{t_0} := \phi(\cdot, t_0)_{\mathbb{D}} \left(I + \int_{t_0} \phi(\cdot, t_0)^{-1} A[\cdot] \Phi^n_{t_0} \right).$$

Finally

$$\Phi_{t_0} := \lim_{n\to\infty} \Phi^n_{t_0}.$$

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Finally

$$\Phi_{t_0} := \lim_{n\to\infty} \Phi^n_{t_0}.$$

Note: $\forall\, t > t_0 \,\exists\, n \in \mathbb{N}: \quad {\Phi^{n+1}_{t_0}}_{(-\infty,t)} = {\Phi^n_{t_0}}_{(-\infty,t)}$

The operator Ψ_{t_0}

$\dot{x} = Ax + v$ with solution $x = \Phi_{t_0} x_0 + \Psi_{t_0}(v)$

The operator Ψ_{t_0}

Let

$$\Psi_{t_0}^0 := v \mapsto \phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)^{-1} v,$$

and for $n \in \mathbb{N}$

$$\Psi_{t_0}^{n+1} := v \mapsto \phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)^{-1} \left(v + A[\cdot]\Psi_{t_0}^n(v) \right).$$

Finally

$$\Psi_{t_0} := v \mapsto \lim_{n \to \infty} \Psi_{t_0}^n(v)$$

Open question: Can Ψ_{t_0} be written as a (generalized) convolution operator?

Summary and outlook

- Definition of piecewise smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$
  - Suitable for discontinuous coefficients and discontinuous inputs
  - Inconsistent initial values (not in this talk)
  - Discontinuous coordinate transformations

Summary and outlook

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  - Discontinuous coordinate transformations
- Solution theory for distributional DAEs
  - Even for "ODEs" new solution theory
  - Still many open questions, e.g. "regularity"
  - Interesting special case: Piecewise *constant* coefficients

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- Solution theory for distributional DAEs
  - Even for "ODEs" new solution theory
  - Still many open questions, e.g. "regularity"
  - Interesting special case: Piecewise *constant* coefficients
- Further topics
  - Controllability and observability
  - Numerical issues
  - Behavioural approach

Inconsistent initial values

$$E\dot{x} = Ax + v \tag{2}$$

Definition (Inconsistent initial value problem)

Given an initial time $t_0 \in \mathbb{R}$ and some past "trajectory" $x_0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \cap \mathbb{D}_{(-\infty,t_0)})^n$,

$$x \text{ solves the IIVP (2)}, x_{(-\infty,t_0)} = x_0$$
$$\Longleftrightarrow$$
$$(E\dot{x})_{[t_0,\infty)} = (Ax+v)_{[t_0,\infty)} \text{ and } x_{(-\infty,t_0)} = x_0.$$

Theorem (Reformulation of an IIVP)

$$x \text{ solves the IIVP (2)}, \; x_{(-\infty,t_0)} = x_0 \quad \Leftrightarrow \quad \tilde{E}\dot{x} = \tilde{A}x + \tilde{v},$$

where $\tilde{E} = E_{[t_0,\infty)}$, $\tilde{A} = I_{(-\infty,t_0)} + A_{[t_0,\infty)}$, $\tilde{v} = -x_0 + v_{[t_0,\infty)}$.