Solution theory for switched differential-algebraic equations

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Theorem (Unique solution of a pure DAE)

Unique solution of the pure DAE $N\dot{x} = x + f$ is

$$x = -\sum_{i=0}^{d-1} N^i f^{(i)}$$

Regularity and similarity

Definition (Regularity)

(E, A) is called regular $:\Leftrightarrow$

1) Existence: $\forall f \in \mathcal{C}^\infty$ $\exists$ solution x of (DAE) and
2) Uniqueness: $\forall f \in \mathcal{C}^\infty$ $\forall$ solutions x_1, x_2 $\forall t_0 \in \mathbb{R}$:

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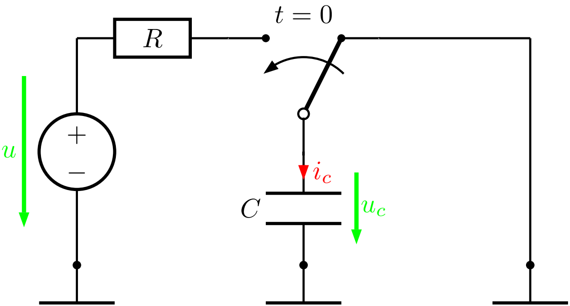
Similarity

For S, T invertible matrices:

$$E\dot{x} = Ax + f \quad \overset{x=Tz}{\Leftrightarrow} \quad SET\dot{z} = SATz + Sf$$

Write: $(E, A) \cong (SET, SAT)$

A simple circuit example



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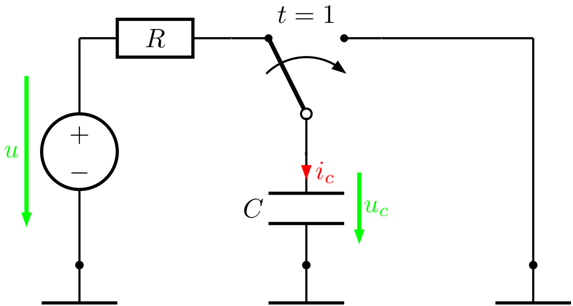

Basic ideas of distributions

Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac impulse δ is "derivative" of step function $\mathbb{1}_{[0,\infty)}$
- Formally defined by Schwartz 1950

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Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

Dirac impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t)$

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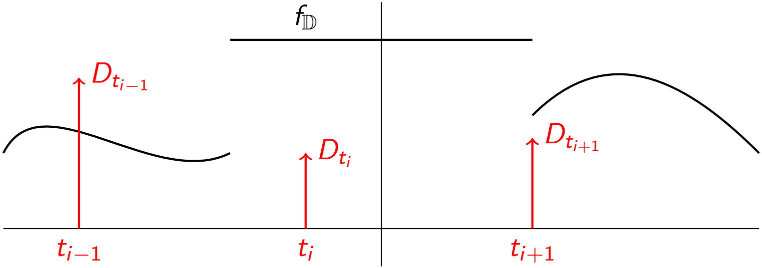
and additionally,

$$(D_{M_1})_{M_2} = 0$$

Piecewise smooth distributions

Definition (Piecewise smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$)

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} = \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Some results for DAE-regularity

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If (E, A) *is DAE-regular, then* $(SET, SAT - SET')$ *is DAE-regular for invertible (time-varying) matrices* S *and* T.

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If (E, A) is DAE-regular, then $(SET, SAT - SET')$ is DAE-regular for invertible (time-varying) matrices S and T.

Theorem (Switching and regularity)

If (E_i, A_i) is DAE-regular for all $i \in \mathbb{Z}$ and $\{\ t_i \in \mathbb{R} \mid i \in \mathbb{Z}\ \}$ is ordered and locally finite, then $\left(\sum_{i\in\mathbb{Z}} E_{i[t_i,t_{i+1})}, \sum_{i\in\mathbb{Z}} A_{i[t_i,t_{i+1})}\right)$ is DAE-regular.

Some questions

- What is the “right” definition for controllability?
- Using impulses for control
- Impulse avoidance with feedback

Summary

- Motivation to study distributional DAEs
  - Jumps in inhomogenity
  - Inconsistent initial values
  - Switching
- Problems with a distributional approach
  - Restriction in general not possible
  - Multiplication with piecewise-smooth coefficients
  - Solution: piecewise-smooth distributions
- Solution theory for distributional DAEs
  - New definition: DAE-regularity
  - Feasible for switched DAEs