Switched Differential Algebraic Equations: Solution Theory, Lyapunov Functions, and Stability

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Workshop on Hybrid Dynamic Systems, 16:20-16:50, Thursday, July 29, 2010, Waterloo, Ontario, Canada

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Switched DAEs

DAE = Differential algebraic equation

Homogeneous switched nonlinear DAE

$$E_{\sigma(t)}(x(t))\dot{x}(t) = f_{\sigma(t)}(x(t)) \qquad \textbf{(swDAE)}$$

or short $E_\sigma(x)\dot{x} = f_\sigma(x)$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, N\}$
  - piecewise constant
  - locally finite jumps
- subsystems $(E_1, f_1), \ldots, (E_N, f_N)$
  - $E_p : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $f_p : \mathbb{R}^n \to \mathbb{R}^n$ smooth, $p = 1, \ldots, N$
  - linear case: $E_p \in \mathbb{R}^{n\times n}$, $f_p = A_p \in \mathbb{R}^{n\times n}$, $p = 1, \ldots, N$

Questions

Existence and nature of solutions?

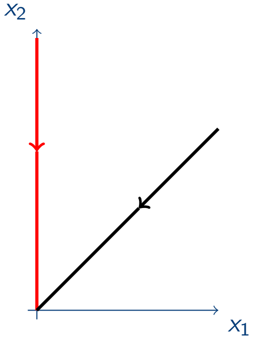
$E_p(x)\dot{x} = f_p(x)$ asymp. stable $\forall p$ $\overset{?}{\Rightarrow}$ $E_\sigma(x)\dot{x} = f_\sigma(x)$ asymp. stable

Example (linear)

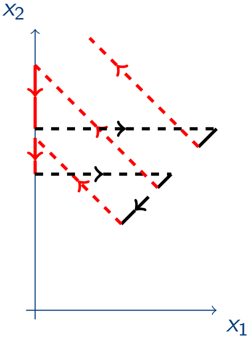
Example (linear, i.e. $E_\sigma \dot{x} = A_\sigma x$):

$$(E_1, A_1) : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x \qquad (E_2, A_2) : \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} x$$

non-switched:



switched:



More (linear) examples in [Liberzon & T., IEEE Proc. CDC 2009]

Observations

Solutions

- Subsystems have constrained dynamics: Consistency spaces
- Switching $\Rightarrow$ Inconsistent initial values
- Inconsistent initial values $\Rightarrow$ Jumps in x

Stability

- Common Lyapunov function not sufficient
- Overall stability depend on jumps

Impulses

- Linear case: switching $\Rightarrow$ Dirac impulses in solution x
- Dirac impulse = infinite peak $\Rightarrow$ Instability
- Nonlinear case: f(Dirac impulse)? Undefined.

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Assumptions on subsystems

Consider non-switched DAE:

$$E(x)\dot{x} = f(x)$$

Definition (Consistency space)

$$\mathfrak{C}_{(E,f)} := \{\ x_0 \in \mathbb{R}^n \mid \exists \text{ (classical) solution } x \text{ with } x(0) = x_0\ \}$$

Time invariance: x solution $\Rightarrow$ $x(t) \in \mathfrak{C}_{(E,f)}\ \forall t$

Assumptions on non-switched DAE

A1 $f(0) = 0$, hence $0 \in \mathfrak{C}_{(E,f)}$

A2 $\mathfrak{C}_{(E,f)}$ is closed manifold (possibly with boundary) in $\mathbb{R}^n$

A3 $\forall x_0 \in \mathfrak{C}_{(E,f)} \exists$ unique solution $x : [0,\infty) \to \mathbb{R}^n$ with $x(0) = x_0$ and $x \in \mathcal{C}^1 \cap \mathcal{C}^\infty_{\text{pw}}$

Linear case

Linear case: **A1**, **A2** trivially fulfilled

Lemma (Linear case and A3)

(E, A) *fulfills* **A3** $\Leftrightarrow$ *matrix pair* (E, A) *is regular, i.e.* $\det(sE - A) \not\equiv 0$

Theorem (Linear switched case: Existence & Uniqueness, [T. 2009])

$$E_\sigma \dot{x} = A_\sigma x$$

with regular matrix pairs (E_p, A_p) *has unique solution for any switching signal and any initial value.*

Impulses in solution

Above solutions might contain impulses!

Assumption **A4**

Consider switched nonlinear DAE

$$E_{\sigma(t)}(x(t))\dot{x}(t) = f_{\sigma(t)}(x(t)) \qquad \textbf{(swDAE)}$$

with consistency spaces $\mathfrak{C}_p := \mathfrak{C}_{(E_p, f_p)}$

Assumption A4

A4 $\forall p, q \in \{1, \ldots, N\}\ \forall x_0^- \in \mathfrak{C}_p\ \exists$ unique $x_0^+ \in \mathfrak{C}_q :\ x_0^+ - x_0^- \in \ker E_q(x_0^+)$

Motivation:

- $x_0^+ - x_0^-$ jump at switching time
- Dirac impulse in $\dot{x}$ in direction $x_0^+ - x_0^-$
- **A4** $\Rightarrow$ no Dirac impulse in product $E_\sigma(x)\dot{x}$
- **A4** $\Rightarrow$ unique jump with above property

Existence & Uniqueness of solutions

Definition (Solution)

$x \in \mathcal{C}^\infty_{\text{pw}}$ is called *solution* of **swDAE** $:\Leftrightarrow$

$$E_\sigma(x)_{\mathbb{D}}(x_{\mathbb{D}})' = f_\sigma(x)_{\mathbb{D}}$$

within the space of piecewise-smooth distributions [T. 2009]

Theorem (Existence & uniqueness of solutions)

(swDAE) + **A1**-**A4** *has unique solution for all (consistent) initial values*

Remark (Consistency projectors)

A4 induces unique map $\Pi_q : \bigcup_p \mathfrak{C}_p \to \mathfrak{C}_q$ such that

$$x(t+) = \Pi_q(x(t-))$$

for all solutions of **(swDAE)** with $\sigma(t+) = q$.

A4 for the linear case

Lemma (Linear consistency projector)

Choose invertible $S_p, T_p \in \mathbb{R}^{n\times n}$ such that

$$(S_p E_p T_p, S_p A_p T_p) = \left(\begin{bmatrix} I & 0 \\ 0 & N_p \end{bmatrix}, \begin{bmatrix} J_p & 0 \\ 0 & I \end{bmatrix} \right)$$

with N_p nilpotent, then

$$\Pi_p = T_p \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1}$$

Theorem (Linear equivalent of A4)

A4 $\Leftrightarrow$ $\forall p, q : E_q(\Pi_q - I)\Pi_p = 0$

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Reminder: Lyapunov function for ODEs

$V : \mathbb{R}^n \to \mathbb{R}^n$ is called *Lyapunov function* for $\dot{x} = f(x)$ $\quad :\Leftrightarrow$

- V is positiv definite and radially unbounded
- $\dot{V}(x) := \nabla V(x) f(x) < 0$ for all $x \neq 0$

$\dot{V}(x) < 0 \quad \Leftrightarrow \quad V$ decreases along solutions

No reference to solutions

Definition of Lyapunov function does not refer to any solutions.

Definition (Lyapunov function for non-switched DAE)

$V : \mathfrak{C}_{(E,f)} \to \mathbb{R}^n$ is called *Lyapunov function* for $E(x)\dot{x} = f(x)$ $\quad :\Leftrightarrow$

L1 V is positive definite and $V^{-1}[0, V(x)]$ is compact

L2 $\exists F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}\ \forall x \in \mathfrak{C}_{(E,f)}\ \forall z \in T_x\mathfrak{C}_{(E,f)}$:
$\nabla V(x) z = F(x, E(x)z)$

L3 $\dot{V}(x) := F(x, f(x)) < 0\ \forall x \in \mathfrak{C}_{(E,f)} \setminus \{0\}$

Lyapunov function and asymptotic stability

Decreasing along solutions

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \nabla V(x(t))\dot{x}(t) \overset{\textbf{L2}}{=} F(x(t), E(x(t))\dot{x}(t)) = F(x(t), f(x(t))) \overset{\textbf{L3}}{<} 0$$

Theorem (Lyapunov's Direct Method for DAEs)

Consider DAE $E(x)\dot{x} = f(x)$ with **A1**-**A3**.

$$\exists \text{ Lyapunov function } V \quad \Rightarrow \quad \text{DAE is asymptotically stable}$$

Theorem (Linear case, [Owens & Debeljkovic 1985])

$\exists V$ *for* $E\dot{x} = Ax \quad \Leftrightarrow \quad \exists P, Q :$ $E^\top PA + A^\top PE = -Q$
where $P = P^\top$ *pos. def. and* $Q = Q^\top$ *pos. def. on* $\mathfrak{C}_{(E,A)}$

$$V(x) = (Ex)^\top PEx \quad \Rightarrow \quad \nabla V(x)z = (Ex)^\top PEz + (Ez)^\top PEx =: F(x, Ez)$$

$$\dot{V}(x) = F(x, Ax) = (Ex)^\top PAx + (Ax)^\top PEx = -x^\top Qx < 0$$

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Reminder **A4** and stability of subsystems

$$E_{\sigma(t)}(x(t))\dot{x}(t) = f_{\sigma(t)}(x(t)) \qquad \textbf{(swDAE)}$$

with consistency spaces $\mathfrak{C}_p := \mathfrak{C}_{(E_p,f_p)}$

Assumption A4

A4 $\forall p,q \in \{1,\ldots,N\}\ \forall x_0^- \in \mathfrak{C}_p\ \exists$ unique $x_0^+ \in \mathfrak{C}_q :\ x_0^+ - x_0^- \in \ker E_q(x_0^+)$

Induced *consistency projectors*:

$$\Pi_q : \bigcup_p \mathfrak{C}_p \to \mathfrak{C}_q, \quad x_0^- \mapsto x_0^+$$

Assumption: Subsystem have Lyapunov functions

$\exists V_p : \mathfrak{C}_p \to \mathbb{R}_{\geq 0}$ Lyapunov function for $E_p(x)\dot{x} = f_p(x)$

Stability results

Theorem (Stability under arbitrary switching)

Consider **(swDAE)** *with* **A1-A4** *with induced consistency projectors* Π_p *and Lyapunov functions* V_p. *If*

$$\forall p, q\ \forall x \in \mathfrak{C}_p: \quad V_q(\Pi_q(x)) \leq V_p(x)$$

then **(swDAE)** *is asymptotically stable for all* σ.

Theorem (Stability under average dwell time switching)

$$\left.\begin{array}{ll} \exists \lambda > 0\ \forall p\ \forall x \in \mathfrak{C}_p: & \dot{V}_p(x) \leq \lambda V_p(x) \text{ and} \\ \exists \mu \geq 1\ \forall p, q\ \forall x \in \mathfrak{C}_p: & V_q(\Pi_q(x)) \leq \mu V_p(x) \end{array}\right\} \Rightarrow$$

(swDAE) *is asymptotically stable for all* σ *with average dwell time*

$$\tau_a > \frac{\ln \mu}{\lambda}$$

Average dwell time for linear case

Consider $E_\sigma \dot{x} = A_\sigma x$ with (E_p, A_p) regular and stable, i.e. exists Lyapunov functions

$$V_p(x) = (E_p x)^\top P_p E_p x, \text{ where } P_p \text{ solves } E_p^\top P_p A_p + A_p^\top P_p E_p = -Q_p$$

and choose minimal O_p such that $\operatorname{im} O_p = \operatorname{im} \Pi_p$.

Theorem (Linear always stable under average dwell time)

$E_\sigma \dot{x} = A_\sigma x$ is asymptotically stable for all σ with average dwell time

$$\tau_a > \frac{\ln \mu}{\lambda}$$

where

$$\lambda := \max_p \lambda_p, \quad \mu := \max_{p,q} \mu_{p,q},$$

$$\lambda_p := \frac{\lambda_{\min}(O_p^\top Q_p O_p)}{\lambda_{\max}(O_p^\top E_p^\top P_p E_p O_p)}, \quad \mu_{p,q} := \frac{\lambda_{\max}(O_p^\top \Pi_q^\top E_q^\top P_q E_q \Pi_q O_p)}{\lambda_{\min}(O_p^\top E_p^\top P_p E_p O_p)}$$

Summary

- Solution theory for switched DAEs
  - Consistency spaces $\Rightarrow$ inconsistent initial values
  - Jumps in solutions $\Rightarrow$ **A4** & consistency projectors
  - Existence & Uniqueness of solutions
- Lyapunov functions for DAEs
- Stability results
  - Arbitrary switching
  - Average dwell time switching
  - Linear case: Explicit lower bound for average dwell time
- Open questions:
  - State dependent switching (e.g. models of Diodes)
  - Converse Lyapunov theorems