The bang-bang funnel controller

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49th IEEE Conference on Decision and Control
Wednesday, December 15, 2010, 11:20–11:40, Atlanta, USA
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Feedback loop

\[
\begin{align*}
\dot{x} &= F(x, u) \\
y &= H(x)
\end{align*}
\]

Switching logic

Funnel

Reference signal \(y_{\text{ref}}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}\) absolutely continuous
The funnel

Control objective

Error $e := y - y_{\text{ref}}$ evolves within *funnel*

$$\mathcal{F} = \mathcal{F}(\varphi_-, \varphi_+) := \{ (t, e) \mid \varphi_-(t) \leq e \leq \varphi_+(t) \}$$

where $\varphi_{\pm} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ absolutely continuous

- time-varying strict error bound
- transient behaviour
- practical tracking ($|e(t)| < \lambda$ for $t >> 0$)
The bang-bang funnel controller

Continuous Funnel Controller: Introduced by Ilchmann et al. in 2002

New approach

Achieve control objectives with bang-bang control, i.e. \( u(t) \in \{ U_-, U_+ \} \)
Relative degree one

Definition (Relative degree one)

\[
\begin{align*}
\dot{x} &= F(x, u) \\
y &= H(x)
\end{align*}
\quad \Leftrightarrow \quad
\begin{align*}
\dot{y} &= f(y, z) + g(y, z) u \\
\dot{z} &= h(y, z)
\end{align*}
\]

- Structural assumption
- \(f, g, h\) can be unknown
- feasibility assumption (later) in terms of \(f, g, h\) and funnel

Important property

\[
\begin{align*}
u(t) &<< 0 \quad \Rightarrow \quad \dot{y}(t) < 0 \\
u(t) &>> 0 \quad \Rightarrow \quad \dot{y}(t) > 0
\end{align*}
\]
Switching logic

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Switching logic

\[ u(t) = U_- \quad \text{if} \quad e(t) > \varphi_-(t) \]
\[ u(t) = U_+ \quad \text{if} \quad e(t) \leq \varphi_-(t) \]
\[ e(t) \geq \varphi_+(t) \]
\[ e(t) < \varphi_+(t) \]

Too simple?
⇒ Feasibility assumptions
Feasibility assumptions

\[
\begin{align*}
\dot{y} &= f(y, z) + g(y, z)u, \quad y_0 \in \mathbb{R} \\
\dot{z} &= h(y, z), \quad z_0 \in Z_0 \subseteq \mathbb{R}^{n-1}
\end{align*}
\]

\[
Z_t := \left\{ z(t) \bigg| \begin{array}{l}
z : [0, t] \to \mathbb{R}^{n-1} \text{ solves } \dot{z} = h(y, z) \text{ for some } \\
z^0 \in Z_0 \text{ and for some } y : [0, t] \to \mathbb{R} \\
\text{with } \varphi_-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi_+(\tau) \\
\forall \tau \in [0, t]
\end{array} \right\}.
\]

Feasibility assumption

\[
\forall t \geq 0 \\forall z_t \in Z_t:
\]

\[
U_- < \frac{\varphi_+(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_+(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_+(t), z_t)}
\]

\[
U_+ > \frac{\varphi_-(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_-(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_-(t), z_t)}
\]

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Theorem (Bang-bang funnel controller)

Relative degree one & Funnel & simple switching logic & Feasibility
⇒
Bang-bang funnel controller works:
- existence and uniqueness of global solution
- error remains within funnel for all time
- no zeno behaviour

Necessary knowledge:
- for controller implementation:
  - relative degree (one)
  - signals: error $e(t)$ and funnel boundaries $\varphi_{\pm}(t)$
- to check feasibility:
  - bounds on zero dynamics
  - bounds on $f$ and $g$
  - bounds on $y_{\text{ref}}$ and $\dot{y}_{\text{ref}}$
  - bounds on the funnel
Relative degree two

Definition (Relative degree two)

\[ \dot{x} = F(x, u) \]
\[ y = H(x) \]
\[ \ddot{y} = f(y, \dot{y}, z) + g(y, \dot{y}, z) u \]
\[ \dot{z} = h(y, \dot{y}, z) \]

Important property

\[ u(t) << 0 \implies \ddot{y}(t) << 0 \]
\[ u(t) >> 0 \implies \ddot{y}(t) >> 0 \]
Feedback loop

\[
\begin{align*}
\dot{x} &= F(x, u) \\
y &= H(x)
\end{align*}
\]

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The switching logic

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Feasibility assumptions

Funnels \( \mathcal{F}(\varphi_+, \varphi_-), \mathcal{F}^d(\varphi^d_+, \varphi^d_-) \)
Safety distances \( \varepsilon^+, \varepsilon^- > 0 \)

Feasibility of funnels

- \( \forall t \geq 0 : \varepsilon_+ < \varphi_+(t) \) and \( \varepsilon_- < \varphi_-(t) \)
- \( \forall t \geq 0 : \varphi^d_+(t) > \dot{\varphi}_-(t) \) and \( \varphi^d_-(t) < \dot{\varphi}_+(t) \)

\[
\ddot{y} = f(y, \dot{y}, z) + g(y, \dot{y}, z)u \\
\dot{z} = h(y, \dot{y}, z)
\]

\( Z_t := \{ z(t) | z \text{ solves } \dot{z} = h(y, \dot{y}, z), z(0) \in Z_0 \} \)

Choose \( \delta_+ > 0 \) such that

\[
\delta_+ > \max\{\varphi^d_-(t), \dot{\varphi}_-(t)\} \quad \text{and} \quad -\delta_- < \min\{\varphi^d_+(t), \dot{\varphi}_+(t)\} \quad \forall t \geq 0
\]
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### Feasibility assumptions

#### Feasibility assumption 1

\[
U_- < \frac{-\delta_+ + \dddot{y}_{\text{ref}}(t) + f(y_t, \dot{y}_t, z_t)}{g(y_t, \dot{y}_t, z_t)},
\]

\[
U_+ > \frac{\delta_- + \dddot{y}_{\text{ref}}(t) + f(y_t, \dot{y}_t, z_t)}{g(y_t, \dot{y}_t, z_t)},
\]

\[\forall t \geq 0, \quad \forall y_t \in [y_{\text{ref}}(t) + \varphi_-(t), y_{\text{ref}}(t) + \varphi_+(t)], \]

\[\forall \dot{y}_t \in [\dot{y}_{\text{ref}}(t) + \dot{\varphi}^d_-(t), \dot{y}_{\text{ref}}(t) + \dot{\varphi}^d_+(t)], \quad \forall z_t \in Z_t\]

#### Feasibility assumption 2

\[
\varepsilon_+ \geq \frac{(||\varphi^d_-|| + ||\min\{\dot{\varphi}_+, 0\}||)^2}{2\delta_-}
\]

\[
\varepsilon_- \geq \frac{(||\varphi^d_+|| + ||\max\{\dot{\varphi}_-, 0\}||)^2}{2\delta_+}
\]
Main result relative degree two

Theorem (Bang-bang funnel controller)

Relative degree two & Funnels & simple switching logic & Feasibility

⇒

Bang-bang funnel controller works:

- existence and uniqueness of global solution
- error and its derivative remain within funnels for all time
- no zeno behaviour
The bang-bang funnel controller

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Model of exothermic chemical reactions

Model from [Ilchmann & T. 2004]:

\[ \dot{y} = br(z_1, z_2, y) - qy + u, \]
\[ \dot{z}_1 = c_1 r(z_1, z_2, y) + d(z_{1}^{\text{in}} - z_1), \]
\[ \dot{z}_2 = c_2 r(z_1, z_2, y) + d(z_{2}^{\text{in}} - z_2), \]

\( b \geq 0, \quad q > 0, \quad c_1 < 0, \quad c_2 \in \mathbb{R}, \quad d > 0, \)
\( z_{1/2}^{\text{in}} \geq 0 \)
\( r : \mathbb{R}_{0+} \times \mathbb{R}_{0+} \times \mathbb{R}_{0+} \rightarrow \mathbb{R}_{0+} \) locally Lipschitz with \( r(0, 0, y) = 0 \forall y > 0 \)
\( y_{\text{ref}} = y^* > 0 \)

Feasibility assumptions from [IT 2004] imply feasibility for bang-bang funnel controller if

\[ \varphi_+(t) \in (0, \bar{y} - y^*], \quad \varphi_-(t) \in (-y^*, 0), \]
\[ \dot{\varphi}_+(t) > -\rho_-, \quad \dot{\varphi}_-(t) < \rho_+, \]
Conclusion

- Introduced new controller design: Bang-bang funnel controller
  - Design only depends on relative degree
  - extremely simple
- Feasibility assumptions
  - $U_+, U_-$ must be large enough
  - in terms of bounds on systems dynamics
  - higher performance $\Rightarrow$ larger values for $U_+, U_-$
- Switching dwell times can be guaranteed
- Higher relative degree (work in progress)
  - Switching logic remains simple (hierarchically)
  - Feasibility assumptions get more complicated
  - Switching frequency increase significantly (exponentially?)