On Observability of Switched DAEs

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DAE = Differential algebraic equation

Switched DAEs

Switched linear DAE **(swDAE)**

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned} \quad \text{or short} \quad \begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}$$

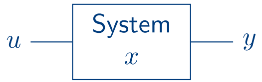
with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathrm{p}\} =: \bar{\mathrm{p}}$
  - piecewise constant
  - locally finite jumps
- matrix tuples $(E_1, A_1, B_1, C_1, D_1), \ldots, (E_\mathrm{p}, A_\mathrm{p}, B_\mathrm{p}, C_\mathrm{p}, D_\mathrm{p})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times r}$, $C_p \in \mathbb{R}^{m\times n}$, $D_p \in \mathbb{R}^{m\times r}$, $p \in \bar{\mathrm{p}}$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$, $p \in \bar{\mathrm{p}}$

Motivation

Electrical circuits, see next talk.

Global Observability of Switched DAEs



Definition (Global observability)

The **(swDAE)** is (globally) observable $:\Leftrightarrow$
$\forall$ solutions $(u_1, x_1, y_1), (u_2, x_2, y_2)$: $(u_1, y_1) \equiv (u_2, y_2) \;\Rightarrow\; x_1 \equiv x_2$

Proposition (0-distinguishability)

The **(swDAE)** *is observable if, and only if,*

$$y \equiv 0 \text{ and } u \equiv 0 \quad \Rightarrow \quad x \equiv 0.$$

Hence consider in the following **(swDAE)** without inputs:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x \\ y &= C_\sigma x \end{aligned}$$

and observability question: $y \equiv 0 \overset{?}{\Rightarrow} x \equiv 0$

Motivating example

System 1:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\dot{x}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}x$$
$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$y=x_3,\ \dot{y}=\dot{x}_3=0,\ x_2=0,\ \dot{x}_1=0$
$\Rightarrow x_1$ unobservable

System 2:

$$\begin{bmatrix}0&0&0\\0&1&0\\1&0&0\end{bmatrix}\dot{x}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}x$$
$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$y=x_3=\dot{x}_1,\ x_1=0,\ \dot{x}_2=0$
$\Rightarrow x_2$ unobservable

$\sigma(\cdot): 1 \to 2$

Jump in x_1 produces impulse in y
$\Rightarrow$ Observability

$\sigma(\cdot): 2 \to 1$

Jump in x_2 no influence in y
$\Rightarrow x_2$ remains unobservable

Question

$$\begin{aligned}E_p\dot{x}&=A_px+B_pu\\ y&=C_px+D_pu\end{aligned}$$ not observable $\overset{?}{\Rightarrow}$ $$\begin{aligned}E_\sigma\dot{x}&=A_\sigma x+B_\sigma u\\ y&=C_\sigma x+D_\sigma u\end{aligned}$$ observable

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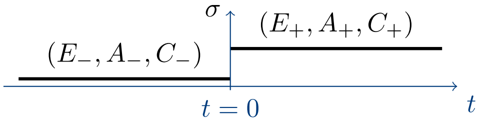
The main result



Theorem (Observability)

The **(swDAE)** *with* **a single switch** *is observable if, and only if,*

$$\{0\} = \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathsf{imp}-}$$

What are these four subspace?

Contents

Solutions of classical DAEs

Consider for now non-switched DAE

$$E\dot{x} = Ax.$$

Theorem (Weierstrass 1868)

(E, A) *regular* $\Leftrightarrow$
$\exists S, T \in \mathbb{R}^{n\times n}$ *invertible:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

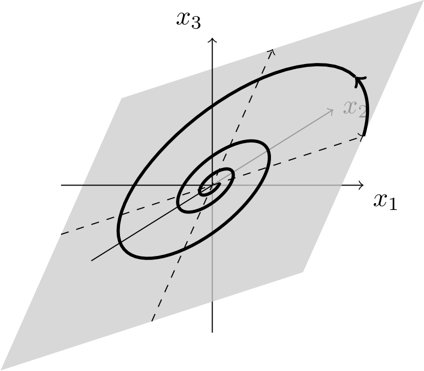
N *nilpotent*

Corollary (for regular (E, A))

x *solves* $E\dot{x} = Ax \Leftrightarrow x(t) = T \begin{pmatrix} e^{Jt}v_0 \\ 0 \end{pmatrix}$

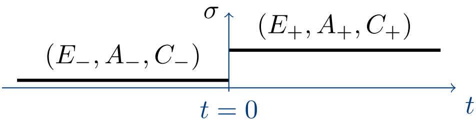
Consistency space: $\mathfrak{C}_{(E,A)} := T \begin{pmatrix} * \\ 0 \end{pmatrix}$

$$(E, A) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right)$$



$$T = \begin{bmatrix} 0 & 4 & * \\ 1 & 0 & * \\ 1 & 1 & * \end{bmatrix}, \quad J = \begin{bmatrix} -1 & -4\pi \\ \pi & -1 \end{bmatrix}$$

The subspace $\mathfrak{C}_-$



Property of solution

x solves $E_\sigma \dot{x} = A_\sigma x$, then

- $x \equiv 0 \quad \Leftrightarrow \quad x(0-) = 0$
- $x(0-) \in \mathfrak{C}_- := \mathfrak{C}_{(E_-, A_-)}$

Reminder:

$(S_- E_- T_-, S_- A_- T_-) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_- \end{smallmatrix}\right], \left[\begin{smallmatrix} J_- & 0 \\ 0 & I \end{smallmatrix}\right] \right)$ and $\mathfrak{C}_{(E_-, A_-)} = T_- \begin{pmatrix} * \\ 0 \end{pmatrix}$

The differential projector

Let $S, T \in \mathbb{R}^{n\times n}$ be invertible with $(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right)$.

Definition (Differential "projector")

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S \quad \text{and} \quad \boxed{A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A}$$

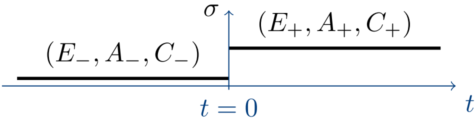
Following Implication holds:

$$x \text{ solves } E\dot{x} = Ax \quad \Rightarrow \quad \dot{x} = A^{\text{diff}} x$$

Hence, with $y = Cx$,

$$y \equiv 0 \quad \Rightarrow \quad x(0) \in \ker[C / CA^{\text{diff}} / C(A^{\text{diff}})^2 / \cdots / C(A^{\text{diff}})^{n-1}]$$

The spaces O_+ and O_-



Hence

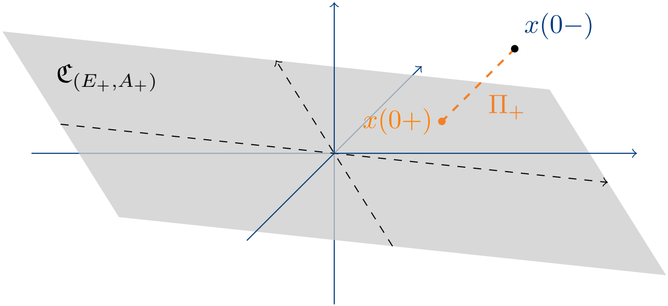
$$y_{(-\infty,0)} \equiv 0 \quad \Rightarrow \quad x(0-) \in \ker \underbrace{[C_-/C_-A_-^{\text{diff}}/C_-(A_-^{\text{diff}})^2/\cdots/C_-(A_-^{\text{diff}})^{n-1}]}_{:=O_-}$$

and

$$y_{(0,\infty)} \equiv 0 \quad \Rightarrow \quad x(0+) \in \ker \underbrace{[C_+/C_+A_+^{\text{diff}}/C_+(A_+^{\text{diff}})^2/\cdots/C_+(A_+^{\text{diff}})^{n-1}]}_{:=O_+}$$

Question: $x(0+) \in \ker O_+ \quad \Rightarrow \quad x(0-) \in \ ?$

Consistency projector and O_+^-



Assume $(S_+E_+T_+, S_+A_+T_+) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_+ \end{smallmatrix}\right], \left[\begin{smallmatrix} J_+ & 0 \\ 0 & I \end{smallmatrix}\right]\right)$:

Consistency projector

$x(0+) = \Pi_+ x(0-)$ where

$$\Pi_+ := T_+ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_+^{-1}$$

$$x(0+) \in \ker O_+$$

$$\Rightarrow \quad x(0-) \in \Pi_+^{-1} \ker O_+ = \ker \underbrace{O_+ \Pi_+}_{=: O_+^-}$$

Main result revisited

Reminder of main result

The **(swDAE)** with **a single switch** is observable if, and only if,

$$\{0\} = \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathsf{imp}-}.$$

So far:

$$y_{(-\infty,0)} = 0 \ \wedge\ y_{(0,\infty)} = 0 \quad \Rightarrow \quad x(0-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^-$$

where

$$O_- = [C_- \ / \ C_- A_-^{\mathsf{diff}} \ / \ C_-(A_-^{\mathsf{diff}})^2 \ / \cdots / \ C_-(A_-^{\mathsf{diff}})^{n-1}]$$

and

$$O_+^- = [C_+ \ / \ C_+ A_+^{\mathsf{diff}} \ / \ C_+(A_+^{\mathsf{diff}})^2 \ / \cdots / \ C_+(A_+^{\mathsf{diff}})^{n-1}]\Pi_+$$

The impulsive effect

Assume $(S_+E_+T_+, S_+A_+T_+) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_+ \end{smallmatrix}\right], \left[\begin{smallmatrix} J_+ & 0 \\ 0 & I \end{smallmatrix}\right]\right)$:

Definition (Impulse "projector")

$$\Pi_+^{\text{imp}} := T_+ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S_+ \quad \text{and} \quad \boxed{E_+^{\text{imp}} := \Pi_+^{\text{imp}} E_+}$$

Impulsive part of solution:

$$x[0] = \sum_{i=0}^{n-1} (E_+^{\text{imp}})^{i+1} (x(0+) - x(0-)) \delta_0^{(i)}$$

← Dirac impulses

Conclusion:

$$y[0] = 0 \quad \Rightarrow \quad C_+ x[0] = 0 \quad \Rightarrow \quad \boxed{x(0+) - x(0-) \in \ker O_+^{\text{imp}}}$$

where

$$O_+^{\text{imp}} := [C_+ E_+^{\text{imp}} \,/\, C_+ (E_+^{\text{imp}})^2 \,/\, \cdots \,/\, C_+ (E_+^{\text{imp}})^{n_2 - 1}]$$

The unobservable space

$x(0+) = \Pi_+ x(0-)$ and $x(0+) - x(0-) \in \ker O_+^{\mathsf{imp}}$ gives

$$x(0-) \in (\Pi_+ - I)^{-1} \ker O_+^{\mathsf{imp}} = \ker \underbrace{O_+^{\mathsf{imp}}(\Pi_+ - I)}_{=:O_+^{\mathsf{imp}-}}$$

Altogether:

$$y \equiv 0 \quad \Rightarrow \quad x(0-) \in \mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathsf{imp}-}$$

Theorem (Unobservable subspace)

$$y \equiv 0 \quad \Leftrightarrow \quad x(0-) \in \mathcal{M}$$

Corollary: **(swDAE)** observable $\Leftrightarrow$ $\mathcal{M} = \{0\}$

Example revisited

System 1:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\dot{x}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}x$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$\sigma(\cdot):1\to 2$ gives

$$\begin{aligned}\mathfrak{C}_- &= \operatorname{span}\{e_1,e_3\},\\ \ker O_- &= \operatorname{span}\{e_1,e_2\}\\ \ker O_+^- &= \operatorname{span}\{e_1,e_2,e_3\},\\ \ker O_+^{\mathsf{imp}-} &= \operatorname{span}\{e_2\}\end{aligned}$$

$$\Rightarrow \quad \mathcal{M}=\{0\}$$

System 2:

$$\begin{bmatrix}0&0&0\\0&1&0\\1&0&0\end{bmatrix}\dot{x}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}x$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$\sigma(\cdot):2\to 1$ gives

$$\begin{aligned}\mathfrak{C}_- &= \operatorname{span}\{e_2\},\\ \ker O_- &= \operatorname{span}\{e_1,e_2\}\\ \ker O_+^- &= \operatorname{span}\{e_1,e_2\},\\ \ker O_+^{\mathsf{imp}-} &= \operatorname{span}\{e_1,e_2,e_3\}\end{aligned}$$

$$\Rightarrow \quad \mathcal{M}=\operatorname{span}\{e_2\}$$

Conclusions

- We have studied observability of switched DAE

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}$$

- Full characterization of observability for single switch case, based on intersection of four subspaces:
  - Consistency: $x(0-) \in \mathfrak{C}_-$
  - Left unobservability: $y^{(i)}(0-) = 0 \;\Leftrightarrow\; x(0-) \in \ker O_-$
  - Jump unobservability: $y^{(i)}(0+) = 0 \;\Leftrightarrow\; x(0-) \in \ker O_+^-$
  - Impulse unobervability: $y[0] = 0 \;\Leftrightarrow\; x(0-) \in \ker O_+^{\text{imp}-}$
- Understanding of single switch case fundamental for general switching signal (future work)

Forward Observability

Definition (Forward observability)

The **(swDAE)** is forward observable $:\Leftrightarrow \forall\ (u_1, x_1, y_1), (u_2, x_2, y_2)$:
$(u_1, y_1) = (u_2, y_2) \ \Rightarrow\ \exists T \geq 0 :\ x_{1(T,\infty)} = x_{2(T,\infty)}$

- in general, weaker than global observability
- presumably more useful for observer design

Theorem (Forward Observability for single switching)

The **(swDAE)** *with* **single switching** *is forward observable if, and only if,*

$$\Pi_+(\mathcal{M}) = \{0\}.$$

Example revisited:
$\sigma(\cdot) : 2 \to 1$, **(swDAE)** globally unobservable but

$\Pi_+(\mathcal{M}) = \{0\}$ hence **(swDAE)** is forward observable

General Class of Switching Signals

Let $\mathcal{M}_k := \mathfrak{C}_k \cap \ker O_k \cap \ker O_{k+1}^{-} \cap \ker O_{k+1}^{\mathsf{imp}-}$,

$$\mathcal{N}_m^m := \mathcal{M}_m$$
$$\mathcal{N}_{k-1}^m := \mathcal{M}_{k-1} \cap \Pi_k^{-1}(\exp(-A_k^{\mathsf{diff}}\tau_k)\mathcal{N}_k^m); \qquad 1 \leq k \leq m$$

Theorem (Global Observability)

(swDAE) *is globally observable if, and only if,* $\exists\, m \in \mathbb{N}$ *such that,*

$$\mathcal{N}_0^m = \{0\}$$

Similarly, let $\mathcal{P}_k := \Pi_{k+1}(\mathfrak{C}_k \cap \ker O_k \cap \ker O_{k+1}^{\mathsf{imp}-}) \cap \ker O_{k+1}$,

$$\mathcal{Q}_0^0 = \mathcal{P}_0$$
$$\mathcal{Q}_0^k = \mathcal{P}_k \cap \Pi_{k+1}(\exp(A_{k+1}\tau_{k+1})\mathcal{Q}_0^{k-1}), \qquad k \geq 1$$

Theorem (Forward Observability)

(swDAE) *is forward observable if, and only if,* $\exists\, m \in \mathbb{N}$ *such that,*

$$\mathcal{Q}_0^m = \{0\}$$