Commutativity and asymptotic stability for linear switched DAEs

Stephan Trenn
joint work with
Daniel Liberzon (UIUC) and **Fabian Wirth** (Uni Würzburg)

Technomathematics group, University of Kaiserslautern, Germany

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Content

Switched DAEs

Linear switched DAE (differential algebraic equation)

(swDAE) $\boxed{E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)}$ or short $\boxed{E_\sigma \dot{x} = A_\sigma x}$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathrm{p}\}$
  - piecewise constant, right-continuous
  - locally finitely many jumps (no Zeno behavior)
- matrix pairs $(E_1, A_1), \ldots, (E_\mathrm{p}, A_\mathrm{p})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \ldots, \mathrm{p}$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$

Motivation and question

Why switched DAEs $E_\sigma \dot{x} = A_\sigma x$?

1. modeling of electrical circuits with switches
2. DAEs $E\dot{x} = Ax + Bu$ with switched feedback controller
$$u(t) = F_{\sigma(t)} x(t) \quad \text{or}$$
$$u(t) = F_{\sigma(t)} x(t) + G_{\sigma(t)} \dot{x}(t)$$
3. approximation of time-varying DAEs $E(t)\dot{x}(t) = A(t)x(t)$ via piecewise constant DAEs

Question

$E_p \dot{x} = A_p x$ asymp. stable $\forall p$ $\overset{?}{\Rightarrow}$ $E_\sigma \dot{x} = A_\sigma x$ asymp. stable

Commutativity and stability for switched ODEs

Theorem (Narendra und Balakrishnan 1994)

Consider switched ODE

(swODE) $\dot{x} = A_\sigma x$

with A_p Hurwitz, $p \in \{1, 2, \ldots, \mathrm{p}\}$ and commuting A_p, i.e.

$$[A_p, A_q] := A_p A_q - A_q A_p = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathrm{p}\} \tag{C}$$

$\Rightarrow$ **(swODE)** *asymptotically stable $\forall \sigma$.*

Sketch of proof: Consider switching times $t_0 < t_1 < \ldots < t_k < t$ and $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}(t-t_k)} e^{A_{p_{k-1}}(t_k - t_{k-1})} \cdots e^{A_{p_1}(t_2-t_1)} e^{A_{p_0}(t_1-t_0)} x_0 \\ &\overset{(C)}{=} e^{A_1 \Delta t_1} e^{A_2 \Delta t_2} \cdots e^{A_{\mathrm{p}} \Delta t_{\mathrm{p}}} x_0 \end{aligned}$$

and $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

Generalization to (swDAE)

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Generalization - Questions

- Which matrices have to commute?
- What about the jumps?

Example 1: $(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right), \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$

$[A_1, A_2] = 0$, but instability possible (see next slide)

Example 2: $(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right), \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$

$[A_1, A_2] \neq 0$, but stability for all switching signals (see next slide)

Examples: jumps and stability

Example 1:

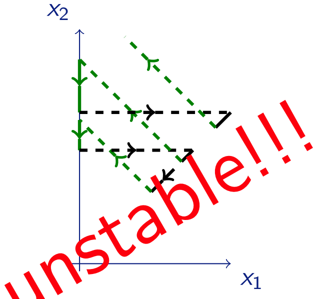
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

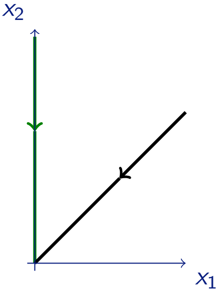
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

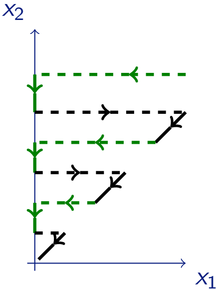
Example 2:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$







Remark: $V(x) = x_1^2 + x_2^2$ is a Lyapunov function for all individuel modes

Observations

Solutions

- modes have restricted dynamics: consistency spaces
- switching $\Rightarrow$ inconsistent initial values
- inconsistent initial values $\Rightarrow$ jumps in x

Stability

- common Lyapunov function not sufficient
- commutativity of A-matrices not sufficient
- stability depends on jumps

Impulses

- switching $\Rightarrow$ Dirac impulses in solution x
- Dirac impulse = infinite peak $\Rightarrow$ instability

Content

1. Introduction
   - Systems class: definition and motivation
   - Examples
2. Nonswitched DAEs
   - Solution theory
   - Consistency projector
   - The matrix A^{diff}
3. Commutativity and stability

Solutions for nonswitched DAE

Consider $E\dot{x} = Ax$

Theorem (Weierstraß 1868)

(E,A) regular $\Leftrightarrow$ $\exists S, T \in \mathbb{R}^{n\times n}$ invertible:

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

N nilpotent, $T = [V, W]$

Corollary (for regular (E,A))

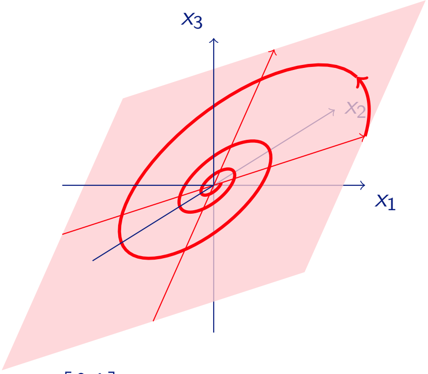
x solves $E\dot{x} = Ax \Leftrightarrow$

$$x(t) = Ve^{Jt}v_0$$

$V \in \mathbb{R}^{n\times n_1}$, $J \in \mathbb{R}^{n_1\times n_1}$, $v_0 \in \mathbb{R}^{n_1}$.

Consistency space: $\mathfrak{C}_{(E,A)} := \operatorname{im} V$

$$(E,A) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right)$$



$$V = \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} -1 & -4\pi \\ \pi & -1 \end{bmatrix}$$

Consistency projectors

Observation

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{pmatrix} \dot v \\ \dot w \end{pmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

Consistent initial values: $\begin{pmatrix} v_0 \\ 0 \end{pmatrix} \in \mathbb{R}^n$

arbitrary initial value $\mathbb{R}^n \ni \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \overset{\Pi}{\mapsto} \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$ consistent initial value

Definition (Consistency projector for regular (E, A))

Let $S, T \in \mathbb{R}^{n\times n}$ invertible with $(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] \right)$:

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Remark: $\Pi_{(E,A)}$ can be calculated easily and directly from (E, A)

The matrix A^{diff}

Let (E,A) be regular with $(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right)$, N nilpotent

consistency projector: $\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Definition (Differential "projector")

$$\Pi^{\text{diff}}_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

Theorem (Differential dynamic of DAE)

x solves $E\dot{x} = Ax$ $\quad \Rightarrow \quad \dot{x} = \Pi^{\text{diff}}_{(E,A)} A x$

$$\boxed{A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A} = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Content

Stability result

Consider again switched DAE: $E_\sigma \dot{x} = A_\sigma x$

Impulse freeness condition

(IFC): $\forall p, q \in \{1, \ldots, N\}: \quad E_p(I - \Pi_p)\Pi_q = 0$

Theorem (T. 2009)

(IFC) $\Rightarrow$ *All solutions of* $E_\sigma \dot{x} = A_\sigma x$ *are impulse free*

Theorem (Main result)

(IFC) $\wedge$ (E_p, A_p) *asymp. stable* $\forall p \wedge$

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathrm{p}\}$$

$\Rightarrow$ **(swDAE)** *asymptotically stable* $\forall \sigma$

Interesting: no additional condition on jumps!

Sketch of proof

From

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \dots, \mathrm{p}\} \tag{C}$$

follows also

$$[\Pi_p, A_p^{\text{diff}}] = 0 \quad \wedge \quad [\Pi_p, \Pi_q] = 0 \quad \wedge \quad [A_p^{\text{diff}}, \Pi_q] = 0.$$

Consider switching times $t_0 < t_1 < \dots < t_k < t$ and $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}^{\text{diff}}(t-t_k)} \Pi_{p_k} e^{A_{p_{k-1}}^{\text{diff}}(t_k - t_{k-1})} \Pi_{p_{k-1}} \cdots e^{A_{p_1}^{\text{diff}}(t_2 - t_1)} \Pi_{p_1} e^{A_{p_0}^{\text{diff}}(t_1 - t_0)} \Pi_{p_0} x_0 \\ &\overset{(K)}{=} e^{A_1^{\text{diff}} \Delta t_1} \Pi_1 \; e^{A_2^{\text{diff}} \Delta t_2} \Pi_2 \cdots e^{A_{\mathrm{p}}^{\text{diff}} \Delta t_{\mathrm{p}}} \Pi_{\mathrm{p}} x_0 \end{aligned}$$

and $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

Quadratic Lyapunov function

Theorem (Existence of common quadratic Lyapunov function)

(IFC) $\wedge$ (E_p, A_p) *asymp. stable* $\forall p$ $\wedge$ $[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0$ $\forall p, q$
$\Rightarrow$ $\exists$ *common quadratic Lyapunov function with*

$$V(\Pi_p x) \leq V(x) \quad \forall x\ \forall p$$

Key observation for proof: $[A_1^{\text{diff}}, A_2^{\text{diff}}] = 0 \quad \Rightarrow \quad \exists T$ invertierbar:

$$T A_1^{\text{diff}} T^{-1} = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad T A_2^{\text{diff}} T^{-1} = \begin{bmatrix} A_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with A_{ij} Hurwitz und $[A_{11}, A_{21}] = 0$

Common quadratic Lyapunov function: Construction

$$TA_1^{\text{diff}}T^{-1} = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad TA_2^{\text{diff}}T^{-1} = \begin{bmatrix} A_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with A_{ij} Hurwitz und $[A_{11}, A_{21}] = 0 \quad \Rightarrow \quad \exists P_1,\ P_2,\ P_3$ s.p.d.:

$$\begin{gathered} A_{11}^\top P_1 + P_1 A_{11} < 0 \quad \wedge \quad A_{21}^\top P_1 + P_1 A_{21} < 0 \\ A_{12}^\top P_2 + P_2 A_{12} < 0 \\ A_{22}^\top P_3 + P_3 A_{22} < 0 \end{gathered}$$

$\Rightarrow$

$$P = T^{-\top} \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & P_2 & 0 & 0 \\ 0 & 0 & P_3 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} T^{-1}$$

gives sought quadratic Lyapunov function $V(x) = x^\top P x$.

Summary

We considered switched DAEs: $\boxed{E_\sigma \dot{x} = A_\sigma x}$

- Solution theory
  - no classical solutions: jumps and impulses
  - impulse freeness condition
  - jumps still permitted
- Commutativity and stability
  - commutativity of A-matrices not sufficient
  - but commutativity of A^{diff}-matrices sufficient
  - also takes care of jumps
  - commutativity $\Rightarrow$ quadratic Lyapunov function
- Next step: Converse Lyapunov theorem for general case

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