

Switched differential algebraic equations: Jumps and impulses

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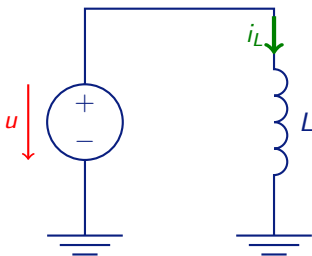


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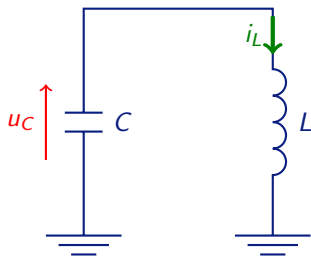


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Standard modeling of circuits



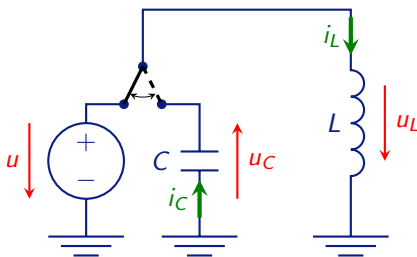
$$\frac{d}{dt} i_L = \frac{1}{L} u$$



$$\begin{aligned} \frac{d}{dt} i_L &= -\frac{1}{L} u_C \\ \frac{d}{dt} u_C &= \frac{1}{C} i_L \end{aligned}$$

General form: $\dot{x} = Ax + Bu$

Switched ODE?



$$\text{Mode 1: } \frac{d}{dt} i_L = \frac{1}{L} u$$

$$\text{Mode 2: } \begin{aligned} \frac{d}{dt} i_L &= -\frac{1}{L} u_C \\ \frac{d}{dt} u_C &= \frac{1}{C} i_L \end{aligned}$$

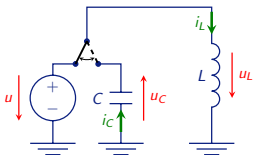
No switched ODE

Not possible to write as

$$\dot{x}(t) = A_{\sigma(t)} x + B_{\sigma(t)} u$$



Include algebraic equations in description



With $x := (i_L, u_L, i_C, u_C)$ write each mode as:

$$E_p \dot{x} = A_p x + B_p u$$

Algebraic equations $\Rightarrow E_p$ singular

Mode 1: $L \frac{d}{dt} i_L = u_L, C \frac{d}{dt} u_C = i_C, 0 = u_L - u, 0 = i_C$

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} u$$

Mode 2: $L \frac{d}{dt} i_L = u_L, C \frac{d}{dt} u_C = i_C, 0 = i_L - i_C, 0 = u_L + u_C$

Switched DAEs



DAE = Differential algebraic equation

Switched DAE

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \quad (\text{swDAE})$$

$$\text{or short } E_{\sigma}\dot{x} = A_{\sigma}x + B_{\sigma}u$$

with

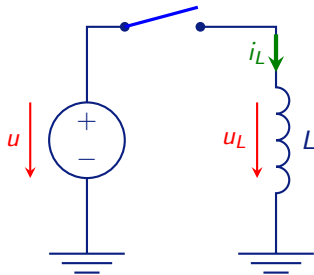
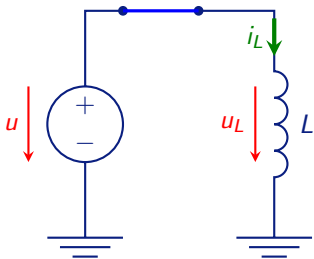
- switching signal $\sigma : \mathbb{R} \rightarrow \{1, 2, \dots, p\}$
 - piecewise constant
 - locally finitely many jumps
- modes $(E_1, A_1, B_1), \dots, (E_p, A_p, B_p)$
 - $E_p, A_p \in \mathbb{R}^{n \times n}$, $p = 1, \dots, p$
 - $B_p : \mathbb{R}^{n \times m}$, $p = 1, \dots, p$
- input $u : \mathbb{R} \rightarrow \mathbb{R}^m$

Question

Existence and nature of solutions?



Impulse example



inductivity law:

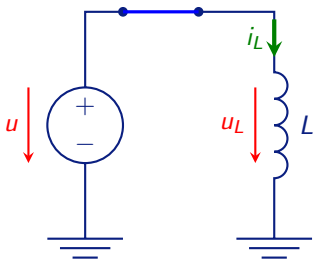
$$L \frac{d}{dt} i_L = u_L$$

switch dependent:

$$0 = u_L - u$$

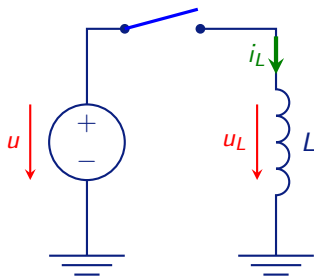
$$0 = i$$

Impulse example



$$x = [i_L, u_L]^T$$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$



$$x = [i_L, u_L]^T$$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

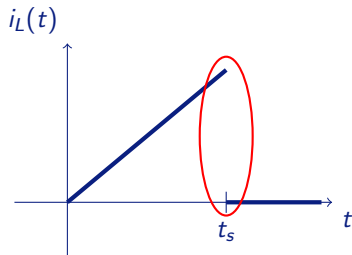
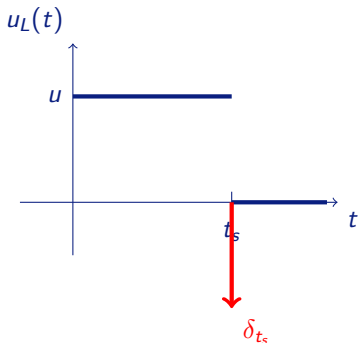


Solution of example

$$L \frac{d}{dt} i_L = u_L, \quad 0 = u_L - u \text{ or } 0 = i_L$$

Assume: u constant, $i_L(0) = 0$

$$\text{switch at } t_s > 0: \sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$$



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Distribution theorie - basic ideas



Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ_0 is “derivative” of Heaviside step function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

- 1 Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
- 2 Axiomatic: Space of all “derivatives” of continuous functions (J. Sebastião e Silva 1954)

Distributions - formal



Definition (Test functions)

$$\mathcal{C}_0^\infty := \{ \varphi : \mathbb{R} \rightarrow \mathbb{R} \mid \varphi \text{ is smooth with compact support} \}$$

Definition (Distributions)

$$\mathbb{D} := \{ D : \mathcal{C}_0^\infty \rightarrow \mathbb{R} \mid D \text{ is linear and continuous} \}$$

Definition (Regular distributions)

$$f \in L_{1,\text{loc}}(\mathbb{R} \rightarrow \mathbb{R}): \quad f_{\mathbb{D}} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \quad \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)dt \in \mathbb{D}$$

Definition (Derivative)

$$D'(\varphi) := -D(\varphi')$$

Dirac Impulse at $t_0 \in \mathbb{R}$

$$\delta_{t_0} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$$

Multiplication with functions



Definition (Multiplication with smooth functions)

$$\alpha \in \mathcal{C}^\infty : (\alpha D)(\varphi) := D(\alpha\varphi)$$

$$\text{(swDAE)} \quad E_\sigma \dot{x} = A_\sigma x + B_\sigma u$$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma, B_\sigma \notin \mathcal{C}^\infty$

Observation:

$$\begin{aligned} E_\sigma \dot{x} = A_\sigma x + B_\sigma u \\ i \in \mathbb{Z} : \sigma_{[t_i, t_{i+1})} \equiv p_i \end{aligned} \quad \Leftrightarrow \quad \forall i \in \mathbb{Z} : (E_{p_i} \dot{x})_{[t_i, t_{i+1})} = (A_{p_i} x + B_{p_i} u)_{[t_i, t_{i+1})}$$

New question: **Restriction of distributions**



Desired properties of distributional restriction

Distributional restriction:

$$\{ M \subseteq \mathbb{R} \mid M \text{ interval} \} \times \mathbb{D} \rightarrow \mathbb{D}, \quad (M, D) \mapsto D_M$$

and for each interval $M \subseteq \mathbb{R}$

- ① $D \mapsto D_M$ is a projection (linear and idempotent)
- ② $\forall f \in L_{1,loc} : (f_{\mathbb{D}})_M = (f_M)_{\mathbb{D}}$
- ③ $\forall \varphi \in C_0^\infty : \left[\begin{array}{ll} \text{supp } \varphi \subseteq M & \Rightarrow D_M(\varphi) = D(\varphi) \\ \text{supp } \varphi \cap M = \emptyset & \Rightarrow D_M(\varphi) = 0 \end{array} \right]$
- ④ $(M_i)_{i \in \mathbb{N}}$ pairwise disjoint, $M = \bigcup_{i \in \mathbb{N}} M_i$:

$$D_M = \sum_{i \in \mathbb{N}} D_{M_i}, \quad D_{M_1 \dot{\cup} M_2} = D_{M_1} + D_{M_2}, \quad (D_{M_1})_{M_2} = 0$$

Theorem

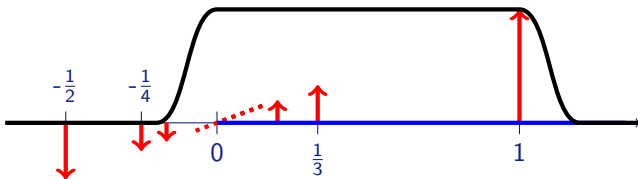
Such a distributional restriction does not exist.



Proof of non-existence of restriction

Consider the following distribution(!):

$$D := \sum_{i \in \mathbb{N}} d_i \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}$$



Restriction should give

$$D_{(0,\infty)} = \sum_{k \in \mathbb{N}} d_{2k} \delta_{d_{2k}}$$

Choose $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi_{[0,1]} \equiv 1$:

$$D_{(0,\infty)}(\varphi) = \sum_{k \in \mathbb{N}} d_{2k} = \sum_{k \in \mathbb{N}} \frac{1}{2k+1} = \infty$$

Dilemma



Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

Distributions

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

Space of distributions **too big**.

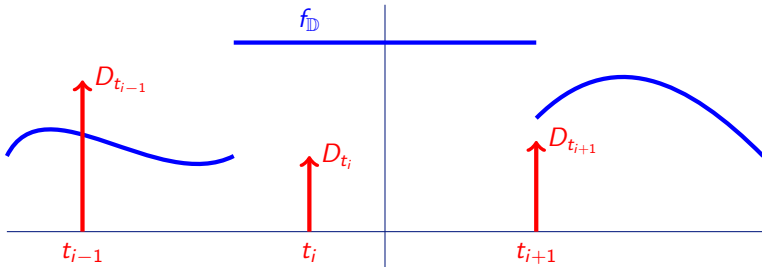


Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions \mathbb{D}_{pwC^∞})

$$\mathbb{D}_{pwC^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t \in T} D_t \mid \begin{array}{l} f \in C_{pw}^\infty, \\ T \subseteq \mathbb{R} \text{ locally finite,} \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$





Properties of \mathbb{D}_{pwC^∞}

- C_{pw}^∞ “ \subseteq ” \mathbb{D}_{pwC^∞}
- $D \in \mathbb{D}_{pwC^\infty} \Rightarrow D' \in \mathbb{D}_{pwC^\infty}$
- Restriction $\mathbb{D}_{pwC^\infty} \rightarrow \mathbb{D}_{pwC^\infty}$, $D \mapsto D_M$ for all intervals $M \subseteq \mathbb{R}$ well defined
- Multiplication with C_{pw}^∞ -functions well defined
- Left and right sided evaluation at $t \in \mathbb{R}$: $D(t-), D(t+)$
- Impulse at $t \in \mathbb{R}$: $D[t]$

(swDAE) $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with input $u \in (\mathbb{D}_{pwC^\infty})^m$

Application to (swDAE)

x solves (swDAE) $:\Leftrightarrow x \in (\mathbb{D}_{pwC^\infty})^n$ and (swDAE) holds in \mathbb{D}_{pwC^∞}



Relevant questions

Consider $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with **regular matrix pairs** (E_p, A_p) .

- Existence of solutions?
- Uniqueness of solutions?
- Inconsistent initial value problems?
- Jumps and impulses in solutions?
- Conditions for impulse free solutions?

Theorem (Existence and uniqueness)

$\forall x^0 \in (\mathbb{D}_{\text{pwC}^\infty})^n \quad \forall t_0 \in \mathbb{R} \quad \forall u \in (\mathbb{D}_{\text{pwC}^\infty})^m \quad \exists! x \in (\mathbb{D}_{\text{pwC}^\infty})^n:$

$$\begin{aligned}x_{(-\infty, t_0)} &= x^0_{(-\infty, t_0)} \\(E_\sigma \dot{x})_{[t_0, \infty)} &= (A_\sigma x + B_\sigma u)_{[t_0, \infty)}\end{aligned}$$

Remark: x is called *consistent solution* $:\Leftrightarrow E_\sigma \dot{x} = A_\sigma x + B_\sigma u$

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Regularity: Definition and characterization



Definition (Regularity)

(E, A) regular $:\Leftrightarrow \det(sE - A) \neq 0$

Theorem (Characterizations of regularity)

The following statements are equivalent:

- (E, A) is regular.
- $\exists S, T \in \mathbb{R}^{n \times n}$ invertible which yield *quasi-Weierstrass form*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad (\text{QWF})$$

where N is a nilpotent matrix.

- \forall smooth $f \exists$ *classical solution* x of $E\dot{x} = Ax + f$ which is *uniquely given* by $x(t_0)$ for any $t_0 \in \mathbb{R}$.



Wong sequences and the quasi-Weierstrass form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad (\text{QWF})$$

Theorem ([Armentano '86], [Berger, Ilchmann, T. '10])

For regular (E, A) define the *Wong sequences*

$$\begin{aligned} \mathcal{V}^{i+1} &:= A^{-1}(E\mathcal{V}^i), & \mathcal{V}^0 &:= \mathbb{R}^n, \\ \mathcal{W}^{i+1} &:= E^{-1}(A\mathcal{W}^i), & \mathcal{W}^0 &:= \{0\}. \end{aligned}$$

Then $\mathcal{V}^i \xrightarrow{\text{finite}} \mathcal{V}^*$ and $\mathcal{W}^i \xrightarrow{\text{finite}} \mathcal{W}^*$. Choose V, W such that $\text{im } V = \mathcal{V}^*$ and $\text{im } W = \mathcal{W}^*$ then

$$T := [V, W], \quad S := [EV, AW]^{-1}$$

yield (QWF).

Matlab Code for calculating quasi-Weierstrass form



Calculating a basis of the pre-image $A^{-1}(\text{im } S)$:

```
function V=getPreImage(A,S)
[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
end;
```

Calculating V with $\text{im } V = \mathcal{V}^*$:

```
function V = getVspace(E,A)
[m,n]=size(E);
if (m==n) & [m,n]==size(A)
    V=eye(n,n);
    oldsize=n+1; newsize=n; finished=0;
    while (newsize~=oldsize);
        EV=colspace(E*V);
        V=getPreImage(A,EV);
        oldsize=newsize; newsize=rank(V);
    end;
end;
```

Calculating W with $\text{im } W = \mathcal{W}^*$ analogously.

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Consistency projector



$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \quad (\text{QWF})$$

Definition (Consistency projector)

Let (E, A) be regular with (QWF), **consistency projector**:

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Theorem

x solves $E_\sigma \dot{x} = A_\sigma x \Rightarrow \forall t \in \mathbb{R} :$

$$x(t+) = \Pi_{(E_q, A_q)} x(t-), \quad q := \sigma(t+)$$



Differential projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad (\text{QWF})$$

Definition (Differential projector)

Let (E, A) be regular with (QWF), **differential projector**:

$$\Pi_{(E,A)}^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

$$A^{\text{diff}} := \Pi_{(E,A)}^{\text{diff}} A$$

Theorem

x solves $E_\sigma \dot{x} = A_\sigma x \Rightarrow \forall t \in \mathbb{R} :$

$$\dot{x}(t+) = A_{\sigma(t+)}^{\text{diff}} x(t+), \quad \dot{x}(t-) = A_{\sigma(t-)}^{\text{diff}} x(t-)$$



Impulse projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad (\text{QWF})$$

Definition (Impulse projector)

Let (E, A) be regular with (QWF), **impulse projector**:

$$\Pi_{(E,A)}^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$$

$$E^{\text{imp}} := \Pi_{(E,A)}^{\text{imp}} E$$

Theorem

x solves $E_{\sigma} \dot{x} = A_{\sigma} x \Rightarrow \forall t \in \mathbb{R} :$

$$x[t] = \sum_{i=0}^{n-2} (E_{\sigma(t+)}^{\text{imp}})^{i+1} (x(t+) - x(t-)) \delta_t^{(i)}$$

Impulse freeness



Consider $E_\sigma \dot{x} = A_\sigma x$

Theorem (Impulse freeness)

$$\forall p, q \in \{1, \dots, p\} : E_q (\Pi_{(E_q, A_q)} - I) \Pi_{(E_p, A_p)} = 0 \Rightarrow x[t] = 0 \quad \forall t$$

Proof:

$$x[t] = \sum_{i=0}^{n-2} (\Pi_{\sigma(t+)}^{\text{imp}} E_{\sigma(t+)})^{i+1} (x(t+) - x(t-)) \delta_t^{(i)}$$

$$x(t+) = \Pi_{(E_{\sigma(t+)}, A_{\sigma(t+)})} x(t-)$$

$$x(t-) \in \text{im } \Pi_{(E_{\sigma(t-)}, A_{\sigma(t-)})}$$

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Solution formula, inhomogeneous non-switched case



Consider $E\dot{x} = Ax + f$

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \quad (\text{QWF})$$

$$\begin{aligned} \Pi_{(E,A)} &:= T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, & \Pi_{(E,A)}^{\text{diff}} &:= T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, & \Pi_{(E,A)}^{\text{imp}} &:= T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S, \\ A^{\text{diff}} &:= \Pi_{(E,A)}^{\text{diff}} A, & E^{\text{imp}} &:= \Pi_{(E,A)}^{\text{imp}} E \end{aligned}$$

Theorem (Explicit solution formula, non-switched)

x solves $E\dot{x} = Ax + f \iff \exists c \in \mathbb{R}^n \forall t \in \mathbb{R} :$

$$x(t) = e^{A^{\text{diff}}t} \Pi_{(E,A)} c + \int_0^t e^{A^{\text{diff}}(t-s)} \Pi_{(E,A)}^{\text{diff}} f(s) ds - \sum_{i=0}^{n-1} (E^{\text{imp}})^i \Pi_{(E,A)}^{\text{imp}} f^{(i)}(t)$$



Jumps and impulses for switched DAE

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \quad (\text{swDAE})$$

$$B_q^{\text{imp}} := \Pi_{(E_q, A_q)}^{\text{imp}} B_q, \quad q \in \{1, \dots, p\}, \quad u[\cdot] = 0$$

Theorem (Jumps and impulses)

x solves (swDAE) $\Rightarrow \forall t \in \mathbb{R} :$

$$x(t+) = \Pi_{(E_q, A_q)} x(t-) - \sum_{i=0}^{n-1} (E_q^{\text{imp}})^i B_q^{\text{imp}} u^{(i)}(t+),$$

$$x[t] = - \sum_{i=0}^{n-1} (E_q^{\text{imp}})^{i+1} (I - \Pi_{(E_q, A_q)}) x(t-) \delta_t^{(i)} \quad q := \sigma(t+)$$

$$- \sum_{i=0}^{n-1} (E_q^{\text{imp}})^{i+1} \sum_{j=0}^i B_q^{\text{imp}} u^{(i-j)}(t+) \delta_t^{(j)}$$



Conclusions

- DAEs natural for modeling electrical circuits
- Switches induce jumps and impulses \Rightarrow Distributional solutions
 - General distributions not suitable
 - Smaller space: Piecewise-smooth distributions
- Regularity \Leftrightarrow Existence & uniqueness of solutions
- Unique consistency jumps
- Condition for impulse-freeness
- Explicit solution formulas