

# A converse Lyapunov theorem for switched DAEs

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# Switched DAEs

## Linear switched DAE (differential algebraic equation)

(swDAE)

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$$

or short

$$E_{\sigma}\dot{x} = A_{\sigma}x$$

with

- switching signal  $\sigma : \mathbb{R} \rightarrow \{1, 2, \dots, p\}$ 
  - piecewise constant, right-continuous
  - locally finitely many jumps (no Zeno behavior)
- matrix pairs  $(E_1, A_1), \dots, (E_p, A_p)$ 
  - $E_p, A_p \in \mathbb{R}^{n \times n}$ ,  $p = 1, \dots, p$
  - $(E_p, A_p)$  regular, i.e.  $\det(E_p s - A_p) \neq 0$
  - impulse-free solutions (but jumps are allowed!)

## Question

$E_{\sigma}\dot{x} = A_{\sigma}x$  asymp. stable  $\forall \sigma \stackrel{?}{\Rightarrow}$  common Lyapunov function



# Lyapunov norms



More general approach:

## Definition (Lyapunov norm)

$\| \cdot \|$  is a  $\lambda$ -Lyapunov norm,  $\lambda \in \mathbb{R}$ ,

$:\Leftrightarrow \forall \sigma : \boxed{\|x(t)\| \leq e^{\lambda t} \|x(0-)\|} \quad \forall \text{ solutions } x \text{ of } E_\sigma \dot{x} = A_\sigma x$

In particular:  $\lambda < 0 \quad \Rightarrow \quad V = \| \cdot \|$  defines Lyapunov function

## New question

Find Lyapunov norm for  $E_\sigma \dot{x} = A_\sigma x$  (stable or unstable)

# Solution formula



## Theorem ( $A^{\text{diff}}$ and $\Pi_{(E,A)}$ , Tanwani & T. 2010)

Let  $(E, A)$  be regular and consider

$$E\dot{x} = Ax \quad \text{on } [0, \infty)$$

$\Rightarrow \exists$  unique *consistency projector*  $\Pi_{(E,A)}$  and unique *flow matrix*  $A^{\text{diff}}$ :

$$x(0) = \Pi_{(E,A)}x(0-)$$

$$\dot{x} = A^{\text{diff}}x \quad \text{on } (0, \infty)$$

Furthermore,  $A^{\text{diff}}\Pi_{(E,A)} = \Pi_{(E,A)}A^{\text{diff}}$ .

## Corollary (Solution formula for switched DAE)

Any solution of the switched DAE  $E_\sigma\dot{x} = A_\sigma x$  has the form

$$x(t) = e^{A_k^{\text{diff}}(t-t_k)} \Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})} \Pi_{k-1} \dots e^{A_1^{\text{diff}}(t_2-t_1)} \Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)} \Pi_0 x(t_0-)$$



# Evolution operator

$$x(t) = \underbrace{e^{A_k^{\text{diff}}(t-t_k)} \Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})} \Pi_{k-1} \cdots e^{A_1^{\text{diff}}(t_2-t_1)} \Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)} \Pi_0}_{=: \Phi^\sigma(t, t_0)} x(t_0)$$

Let  $\mathcal{M} := \{ (A_p^{\text{diff}}, \Pi_p) \mid \text{corresponding to } (E_p, A_p), p = 1, \dots, p \}$ .

**Definition (Set of all evolutions with fixed time span  $\Delta t > 0$ )**

$$\begin{aligned} \mathcal{S}_{\Delta t} &:= \bigcup_{\sigma} \{ \Phi^\sigma(t_0 + \Delta t, t_0) \mid t_0 \in \mathbb{R} \} \\ &= \left\{ \prod_{i=0}^k e^{A_i^{\text{diff}} \tau_i} \Pi_i \mid (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M}, \sum_{i=0}^k \tau_i = \Delta t, \tau_i > 0 \right\} \end{aligned}$$

Note that  $\forall t_0 \in \mathbb{R} \forall \Delta t > 0$ :

$$x \text{ solves } E_\sigma \dot{x} = A_\sigma x \quad \Leftrightarrow \quad \exists \Phi_{\Delta t} \in \mathcal{S}_{\Delta t} : \quad x(t_0 + \Delta t) = \Phi_{\Delta t} x(t_0)$$

# Semi group property



## Lemma (Semi group)

The set

$$\mathcal{S} := \bigcup_{\Delta t > 0} \mathcal{S}_{\Delta t}$$

is a semi group with

$$\mathcal{S}_{s+t} = \mathcal{S}_s \mathcal{S}_t := \{ \Phi_s \Phi_t \mid \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t \}$$

Need **commutativity** to show “ $\subseteq$ ”:

$$e^{A^{\text{diff}} \tau} \Pi = e^{A^{\text{diff}}(\tau-\tau')} \overset{\curvearrowright}{e^{A^{\text{diff}} \tau'}} \Pi \Pi = e^{A^{\text{diff}}(\tau-\tau')} \Pi e^{A^{\text{diff}} \tau'} \Pi$$

for any  $(A^{\text{diff}}, \Pi) \in \mathcal{M}$  and  $0 < \tau' < \tau$



# Exponential growth bound

## Definition (Exponential growth bound)

For  $t > 0$  the *exponential growth bound* of  $E_\sigma \dot{x} = A_\sigma x$  is

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\}$$

Definition implies for all solutions  $x$  of  $E_\sigma \dot{x} = A_\sigma x$ :

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\| \|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t)t} \|x(0-)\|$$

## Difference to switched ODEs without jumps

$\lambda_t(\mathcal{S}_t) = \pm\infty$  is possible!

All jumps are trivial, i.e.  $\Pi_p = 0 \Rightarrow \lambda_t(\mathcal{S}_t) = -\infty$

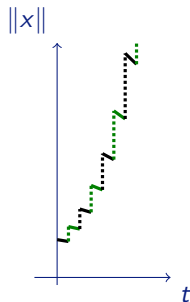
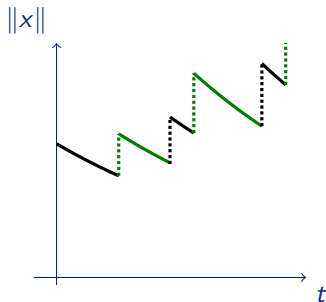
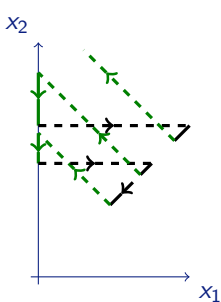




# Infinite exponential growth bound

Example:

$$(E_1, A_1) = \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right) \quad (E_2, A_2) = \left( \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



For small dwell times:  $\Phi_t \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$



# Existence of exponential growth rate

## Theorem (Boundedness of $\mathcal{S}_t$ )

$\mathcal{S}_t$  is bounded  $\Leftrightarrow$  the set of consistency projectors is product bounded

Reminder:

$$\mathcal{S}_t := \left\{ \prod_{i=0}^k e^{A_i^{\text{diff}} \tau_i} \Pi_i \mid (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M}, \sum_{i=0}^k \tau_i = \Delta t, \tau_i > 0 \right\}$$

## Theorem (Exponential growth rate well defined)

Let the consistency projectors be product bounded and not all be trivial, then the (upper) Lyapunov exponent

$$\lambda(\mathcal{S}) := \lim_{t \rightarrow \infty} \lambda_t(\mathcal{S}_t) = \lim_{t \rightarrow \infty} \sup_{\Phi_t \in \mathcal{S}_t} \frac{\|\Phi_t\|}{t}$$

of  $E_\sigma \dot{x} = A_\sigma x$  is well defined and finite.



# Barabanov norm

## Definition (Barabanov norm)

$\|\cdot\|$  is called **Barabanov norm** for  $E_\sigma \dot{x} = A_\sigma x$ , iff

- 1  $\|x(t)\| = \|\Phi_t x(0-)\| \leq e^{\lambda t} \|x(0-)\|$ ,  $\Phi_t \in \mathcal{S}_t$
- 2  $\forall x^0 \in \mathbb{R}^n \exists \bar{\Phi}_t \in \bar{\mathcal{S}}_t: \|\bar{\Phi}_t x^0\| = e^{\lambda t} \|x^0\|$

In particular, every Barabanov norm is also a  $\lambda$ -Lyapunov norm, hence if  $\lambda < 0$  we have a Lyapunov function

## Theorem (Existence of Barabanov norm)

Assume  $\mathcal{S}$  is *irreducible*, i.e.  $\mathcal{S}\mathcal{M} \subseteq \mathcal{M}$  implies  $\mathcal{M} = \emptyset$  or  $\mathcal{M} = \mathbb{R}^n$ .

Then the following are *equivalent*:

- 1 The consistency projectors are *product bounded*
- 2 The *Lyapunov exponent*  $\lambda(\mathcal{S})$  is *bounded*
- 3 There *exists a Barabanov norm* with  $\lambda = \lambda(\mathcal{S})$



# Construction of Barabanov norm

Construction of Barabanov norm similar as in (Wirth 2002, LAA):

$$\mathcal{S}_\infty := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} e^{-\lambda(S)t} \mathcal{S}_t}$$

is a compact nontrivial semigroup, the **limit semigroup**.

$$\|x\| := \max \{ \|Sx\| \mid S \in \mathcal{S}_\infty \}$$

is the sought Barabanov norm.



# The reducible case

## Theorem (Lyapunov norm)

For each  $\varepsilon > 0$

$$\|x\|_\varepsilon := \sup_{t>0} \sup_{\Phi_t \in \mathcal{S}_t} e^{-(\lambda(S)+\varepsilon)t} \|\Phi_t x\|$$

defines a Lyapunov norm for  $E_\sigma \dot{x} = A_\sigma x$ .

## Corollary (Converse Lyapunov Theorem)

$E_\sigma \dot{x} = A_\sigma x$  is uniformly exp. stable  $\Rightarrow V = \|\cdot\|_\varepsilon$  is Lyapunov function

In particular:  $V(\Pi x) \leq V(x)$  for all consistency projectors  $\Pi$

## Non-smooth Lyapunov function

$\|\cdot\|_\varepsilon$  in general **non-smooth**. Smoothification as in Yin, Sontag & Wang 1996 **might violate jump condition!**



# Conclusions

- Studied switched DAEs  $E_\sigma \dot{x} = A_\sigma x$
- Key observation:

$$x(t) = e^{A_k^{\text{diff}}(t-t_k)} \Pi_k \dots e^{A_1^{\text{diff}}(t_2-t_1)} \Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)} \Pi_0 x(t_0-)$$

- Flow set

$$\mathcal{S}_t := \left\{ \prod_{i=0}^k e^{A_i^{\text{diff}} \tau_i} \Pi_i \mid (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M}, \sum_{i=0}^k \tau_i = \Delta t, \tau_i > 0 \right\}$$

- **Product boundedness** of consistency projectors necessary and sufficient for boundedness of  $\mathcal{S}_t$
- Construction of **Barabanov norm** in irreducible case
- Construction of **Lyapunov norm** in reducible case