

Stability of switched DAEs

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Switched DAEs



Switched linear DAE (differential algebraic equation)

(swDAE)

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$$

or short $E_{\sigma}\dot{x} = A_{\sigma}x$

with

- switching signal $\sigma : \mathbb{R} \rightarrow \{1, 2, \dots, P\}$
 - piecewise constant, right-continuous
 - locally finitely many jumps
- matrix pairs $(E_1, A_1), \dots, (E_P, A_P)$
 - $E_p, A_p \in \mathbb{R}^{n \times n}$, $p = 1, \dots, P$
 - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \neq 0$



Motivation and questions

Why switched DAEs $E_\sigma \dot{x} = A_\sigma x$?

- ① modeling of electrical circuits with switches
- ② DAEs $E\dot{x} = Ax + Bu$ with switched feedback

$$u(t) = F_{\sigma(t)}x(t) \quad \text{or}$$

$$u(t) = F_{\sigma(t)}x(t) + G_{\sigma(t)}\dot{x}(t)$$

- ③ approximation of time-varying DAEs $E(t)\dot{x} = A(t)x$ via piecewise-constant DAEs

Question

$$E_p \dot{x} = A_p x \text{ asymp. stable } \forall p \stackrel{?}{\Rightarrow} E_\sigma \dot{x} = A_\sigma x \text{ asymp. stable } \forall \sigma$$



Example 1: jumps and stability

Example 1a:

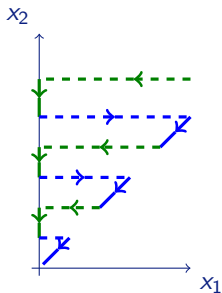
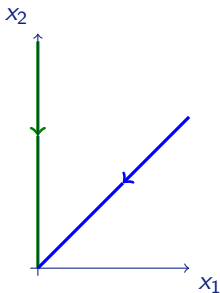
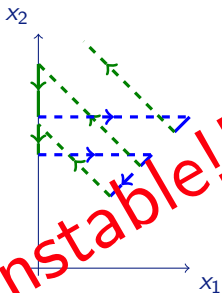
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

Example 1b:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

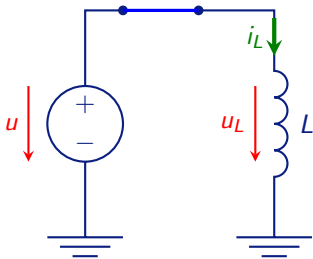


unstable!!!

Remark: $V(x) = x_1^2 + x_2^2$ is Lyapunov function for **all** subsystem



Example 2: impulses in solutions



constant input:

inductivity law:

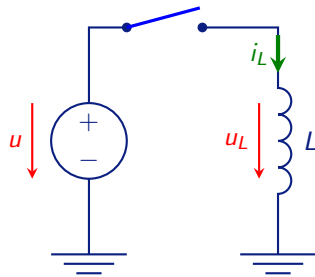
switch dependent:

$$0 = u_L - u$$

$$\dot{u} = 0$$

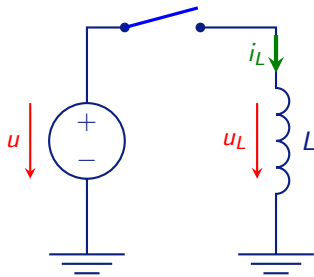
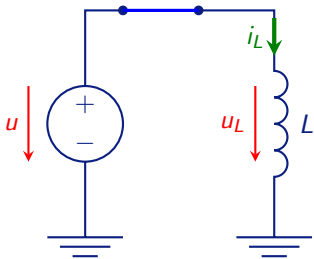
$$L \frac{d}{dt} i_L = u_L$$

$$0 = i_L$$





Example 2: impulses in solutions



$$x = [u, i_L, u_L]^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} x$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x$$

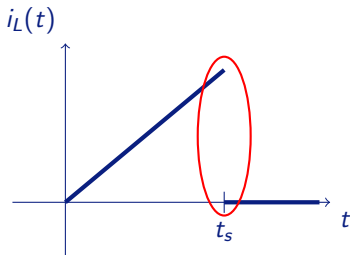
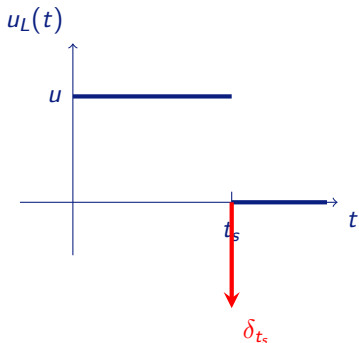


Solution of example

$$L \frac{d}{dt} i_L = u_L, \quad 0 = u_L - u \quad \text{or} \quad 0 = i_L$$

$$u \text{ constant, } i_L(0) = 0$$

$$\text{switch at } t_s > 0: \sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$$



Observations



Solutions

- modes have constrained dynamics: **consistency spaces**
- switches \Rightarrow **inconsistent initial values**
- inconsistent initial values \Rightarrow **jumps in x**

Stability

- common Lyapunov function **not sufficient**
- stability depends on **jumps**

Impulses

- switching \Rightarrow **Dirac impulse** in solution x
- Dirac impulse = infinite peak \Rightarrow **instability**

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Solutions for unswitched DAEs

Consider $E\dot{x} = Ax$.

Theorem (Weierstrass 1868)

(E, A) regular \Leftrightarrow

$\exists S, T \in \mathbb{R}^{n \times n}$ invertible:

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

N nilpotent, $T = [V, W]$

Corollary (for regular (E, A))

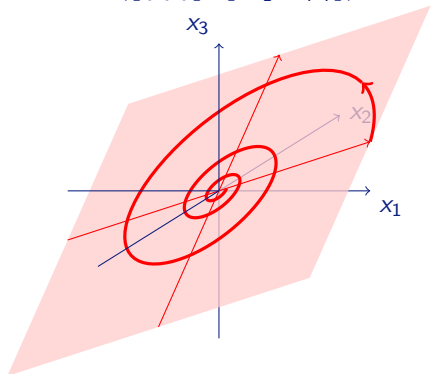
x solves $E\dot{x} = Ax \Leftrightarrow$

$$x(t) = Ve^{Jt}v_0$$

$V \in \mathbb{R}^{n \times n_1}$, $J \in \mathbb{R}^{n_1 \times n_1}$, $v_0 \in \mathbb{R}^{n_1}$.

Consistency space: $\mathfrak{C}_{(E,A)} := \text{im } V$

$$(E, A) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right)$$



$$V = \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} -1 & -4\pi \\ \pi & -1 \end{bmatrix}$$

Consistency projector



Observation

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

Consistent initial value: $\begin{pmatrix} v_0 \\ 0 \end{pmatrix}$, because $N\dot{w} = w \Leftrightarrow w \equiv 0$

arbitrary initial value $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \xrightarrow{\Pi} \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$ consistent initial value

Definition (Consistency projector for regular (E, A))

Let $S, T \in \mathbb{R}^{n \times n}$ be invertible with $(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$:

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Remark: $\Pi_{(E,A)}$ can be calculated **easily** and **directly** from (E, A) (via the Wong sequences)



Lyapunov functions for regular (E, A)

Definition (Lyapunov function for $E\dot{x} = Ax$)

$Q = \overline{Q}^\top > 0$ on $\mathfrak{C}_{(E,A)}$ and $P = \overline{P}^\top > 0$ solutions of

$$A^\top P E + E^\top P A = -Q \quad (\text{generalize Lyapunov equation})$$

Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top P Ex$

V monotonically decreasing along solutions:

$$\begin{aligned} \frac{d}{dt} V(x(t)) &= (Ex(t))^\top P E \dot{x}(t) + (E \dot{x}(t))^\top P Ex \\ &= x(t)^\top E^\top P A x(t) + x(t)^\top A^\top P Ex(t) \\ &= -x(t)^\top Q x(t) < 0 \end{aligned}$$

Theorem (Owens & Debeljkovic 1985)

$E\dot{x} = Ax$ asymptotically stable $\Leftrightarrow \exists$ Lyapunov function

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Distribution theory - basics



Distributions - overview

- generalized functions
- arbitrarily often differentiable
- Dirac impulse δ_0 is “derivative” of unit jump $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

- 1 functional analytical: dual of the space test functions (L. Schwartz 1950)
- 2 axiomatic: space of all “derivatives” of continuous functions (J. Sebastião e Silva 1954)



Dilemma

$$\text{(swDAE)} \quad E_\sigma \dot{x} = A_\sigma x$$

Problem

Multiplication of **non smooth coefficients** E_σ, A_σ with general distribution x **not defined!**

switched DAEs

- example: distributional solutions
- multiplication with non-smooth coefficients

distributions

- multiplication with non-smooth coefficients not well-defined
- *initial value problems cannot be formulated*

Underlying problem

Space of distributions **too big.**

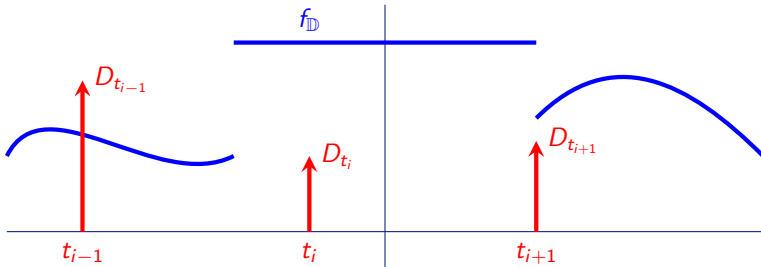


Piecewise-smooth distributions

define a more suitable, smaller space:

Definition (Piecewise-smooth distributions \mathbb{D}_{pwC^∞})

$$\mathbb{D}_{pwC^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t \in T} D_t \mid \begin{array}{l} f \in C_{pw}^\infty, \\ T \subseteq \mathbb{R} \text{ locally finite,} \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$





Properties of $\mathbb{D}_{pw}C^\infty$

- multiplication with C_{pw}^∞ -functions well defined (Fuchssteiner multiplication)
- left und right evaluation at $t \in \mathbb{R}$ possible: $D(t-), D(t+)$
- impulse at $t \in \mathbb{R}$: $D[t]$

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Application to (swDAE)

x solves (swDAE) $:\Leftrightarrow x \in (\mathbb{D}_{pw}C^\infty)^n$ and (swDAE) holds in $\mathbb{D}_{pw}C^\infty$

Theorem (Existence and uniqueness of solutions, T. 2009)

(E_p, A_p) regular $\forall p \Leftrightarrow$ (swDAE) uniquely solvable $\forall \sigma \forall x(0) \in \mathbb{R}^n$

Intermediate summary: problems and its solutions



(swDAE) $E_\sigma \dot{x} = A_\sigma x$

- 1 stability criteria for single DAEs $E_p \dot{x} = A_p x$
⇒ **Lyapunov functions**
- 2 **no classical solutions**
⇒ **allow jumps** in solutions
- 3 How does inconsistent initial value jump to consistent one?
⇒ **Consistency projectors** $\Pi_{(E_1, A_1)}, \dots, \Pi_{(E_N, A_N)}$
- 4 differentiation of jumps
⇒ space of **distributions** as solution space
- 5 **multiplication with non-smooth coefficients**
⇒ space of **piecewise-smooth distributions**
⇒ existence and uniqueness of solutions

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Asymptotic stability and impulse free solutions

Definition (Asymptotic stability of switched DAE)

(swDAE) asymptotically stable

: \Leftrightarrow x is **impulse free*** and $x(t_{\pm}) \rightarrow 0$ for $t \rightarrow \infty$

* i.e. $x[t] = 0 \forall t \in \mathbb{R}$; however jumps in x are still allowed

Let $\Pi_p := \Pi_{(E_p, A_p)}$ be the consistency projector of (E_p, A_p)

Impulse freeness condition

(IFC): $\forall p, q \in \{1, \dots, N\} : E_q(I - \Pi_q)\Pi_p = 0$

Theorem (T. 2009)

(IFC) \Leftrightarrow all solutions of $E_{\sigma}\dot{x} = A_{\sigma}x$ are impulse free $\forall \sigma$



Stability for arbitrary switching

Consider **(swDAE)** with:

($\exists \mathbf{V}_p$): $\forall p \in \{1, \dots, P\} \exists$ Lyapunov function V_p for (E_p, A_p)

i.e. each DAE $E_p \dot{x} = A_p x$ is asymptotically stable

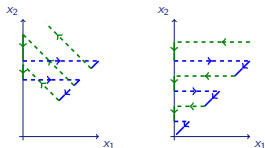
Lyapunov jump condition

(LJC): $\forall p, q = 1, \dots, N \forall x \in \mathcal{C}_{(E_p, A_p)} : V_q(\Pi_q x) \leq V_p(x)$

Theorem (Liberzon & T. 2009)

(IFC) \wedge ($\exists \mathbf{V}_p$) \wedge (LJC) \Rightarrow (swDAE) asymptotically stable $\forall \sigma$

Examples 1a and 1b fulfill **(IFC)** and **($\exists \mathbf{V}_p$)**,
but only 1b fulfills **(LJC)**





Slow switching

Consider the set of switching signals with **dwell time** $\tau > 0$:

$$\Sigma^\tau := \left\{ \sigma : \mathbb{R} \rightarrow \{1, \dots, N\} \left| \begin{array}{l} \forall \text{ switching times} \\ t_i \in \mathbb{R}, i \in \mathbb{Z} : \\ t_{i+1} - t_i \geq \tau \end{array} \right. \right\}.$$

Theorem (Liberzon & T. 2009)

$\exists \tau > 0$: **(IFC)** \wedge **($\exists \mathbf{V}_p$)** \Rightarrow **(swDAE)** asymptotically stable $\forall \sigma \in \Sigma^\tau$

Reminder:

(IFC): $\forall p, q \in \{1, \dots, N\}$: $E_q(I - \Pi_q)\Pi_p = 0$

Examples 1a and 1b both fulfill **(IFC)** and **($\exists \mathbf{V}_p$)**

\Rightarrow both examples are asymptotically stable for **slow switching**



Generalization to nonlinear switched DAEs

Previous results can be generalized to nonlinear switched DAEs:

$$E_\sigma(x)\dot{x} = f_\sigma(x)$$

Then **(IFC)** has to be replaced by

$$\forall p, q \in \{1, \dots, P\} \quad \forall x_0^- \in \mathfrak{C}_p \quad \exists \text{ unique } x_0^+ \in \mathfrak{C}_q : x_0^+ - x_0^- \in \ker E_q(x_0^+)$$

where \mathfrak{C}_p is the consistency manifold of $E_p(x)\dot{x} = f_p(x)$

See our recent Automatica paper “Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability”

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Commutativity and stability of switched ODEs



Theorem (Narendra and Balakrishnan 1994)

Consider switched ODE

$$\text{(swODE)} \quad \dot{x} = A_{\sigma}x$$

with A_p Hurwitz, $p \in \{1, 2, \dots, P\}$ and **commuting** A_p , i.e.

$$[A_p, A_q] := A_p A_q - A_q A_p = 0 \quad \forall p, q \in \{1, 2, \dots, P\} \quad (\text{C})$$

\Rightarrow **(swODE)** asymptotically stable $\forall \sigma$.

Proof idea: Consider switching times $t_0 < t_1 < \dots < t_k < t$ and $p_j := \sigma(t_j+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}(t-t_k)} e^{A_{p_{k-1}}(t_k-t_{k-1})} \dots e^{A_{p_1}(t_2-t_1)} e^{A_{p_0}(t_1-t_0)} x_0 \\ &\stackrel{(\text{C})}{=} e^{A_1 \Delta t_1} e^{A_2 \Delta t_2} \dots e^{A_p \Delta t_p} x_0 \end{aligned}$$

and $\Delta t_p \rightarrow \infty$ for at least one p and $t \rightarrow \infty$.

Generalization to (swDAE)



$$\text{(swDAE)} \quad E_\sigma \dot{x} = A_\sigma x$$

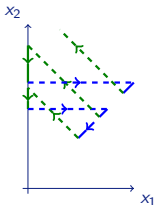
Generalization - Questions

- Which matrices have to commute?
- What about the jumps?

Example 1a:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right)$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

$[A_1, A_2] = 0$, but **unstable** for fast switching





The matrix A^{diff}

Let (E, A) regular with $(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$, N nilpotent

consistency projector: $\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Definition (differential “projector”)

$$\Pi_{(E,A)}^{\text{diff}} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

Lemma (Dynamics of DAE, Tanwani & T. 2010)

x solves $E\dot{x} = Ax \Rightarrow \dot{x} = \underbrace{\Pi_{(E,A)}^{\text{diff}}}_{=: A^{\text{diff}}} Ax$

Note: $A^{\text{diff}} = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$, hence $[A^{\text{diff}}, \Pi_{(E,A)}] = 0$

Commutativity condition



$$\text{(swDAE)} \quad E_\sigma \dot{x} = A_\sigma x$$

Theorem (Liberzon, T., Wirth 2011)

$$\text{(IFC)} \wedge (\exists \mathbf{V}_p) \wedge$$

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \dots, P\} \quad (\text{C})$$

\Rightarrow **(swDAE)** is asymptotically stable $\forall \sigma$.

(IFC) \wedge **($\exists \mathbf{V}_p$)** \wedge **(C)** $\Rightarrow \exists$ *common quadratic Lyapunov function* with

$$V(\Pi_p x) \leq V(x) \quad \forall x \quad \forall p$$

Remarkable: No explicit condition on jumps!



Proof idea

Proof idea:

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \dots, P\} \quad (\text{C})$$

implies

$$[\Pi_p, A_q^{\text{diff}}] = 0 \quad \wedge \quad [\Pi_p, \Pi_q] = 0.$$

Consider switching times $t_0 < t_1 < \dots < t_k < t$ and $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}^{\text{diff}}(t-t_k)} \Pi_{p_k} e^{A_{p_{k-1}}^{\text{diff}}(t_k-t_{k-1})} \Pi_{p_{k-1}} \dots e^{A_{p_1}^{\text{diff}}(t_2-t_1)} \Pi_{p_1} e^{A_{p_0}^{\text{diff}}(t_1-t_0)} \Pi_{p_0} x_0 \\ &\stackrel{(\text{C})}{=} e^{A_1^{\text{diff}} \Delta t_1} \Pi_1 e^{A_2^{\text{diff}} \Delta t_2} \Pi_2 \dots e^{A_P^{\text{diff}} \Delta t_P} \Pi_P x_0 \end{aligned}$$

and $\Delta t_p \rightarrow \infty$ for at least one p and $t \rightarrow \infty$.

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Evolution operator

$$x(t) = \underbrace{e^{A_k^{\text{diff}}(t-t_k)} \Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})} \Pi_{k-1} \cdots e^{A_1^{\text{diff}}(t_2-t_1)} \Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)} \Pi_0}_{=: \Phi^\sigma(t, t_0)} x(t_0)$$

Let $\mathcal{M} := \{ (A_p^{\text{diff}}, \Pi_p) \mid \text{corresponding to } (E_p, A_p), p = 1, \dots, p \}$.

Definition (Set of all evolution matrices with fixed time span $t > 0$)

$$\begin{aligned} \mathcal{S}_t &:= \{ \Phi^\sigma(t, 0) \mid \sigma \text{ arbitrary switching signal} \} \\ &= \left\{ \prod_{i=0}^k e^{A_i^{\text{diff}} \tau_i} \Pi_i \mid (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M}, \sum_{i=0}^k \tau_i = \Delta t, \tau_i > 0 \right\} \end{aligned}$$

Lemma (Semi group, T. & Wirth 2012)

The set $\mathcal{S} := \bigcup_{t>0} \mathcal{S}_t$ is a semi group with

$$\mathcal{S}_{s+t} = \mathcal{S}_s \mathcal{S}_t := \{ \Phi_s \Phi_t \mid \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t \}$$



Exponential growth bound

Definition (Exponential growth bound)

For $t > 0$ the *exponential growth bound* of $E_\sigma \dot{x} = A_\sigma x$ is

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\}$$

Definition implies for all solutions x of $E_\sigma \dot{x} = A_\sigma x$:

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\| \|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t) t} \|x(0-)\|$$

Difference to switched ODEs without jumps

$\lambda_t(\mathcal{S}_t) = \pm\infty$ is possible!

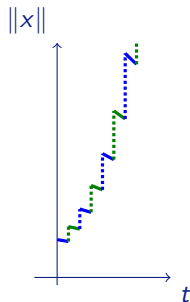
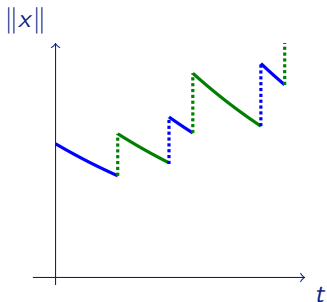
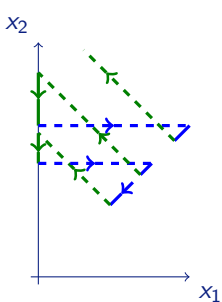
All jumps are trivial, i.e. $\Pi_p = 0 \Rightarrow \lambda_t(\mathcal{S}_t) = -\infty$



Infinite exponential growth bound

Example 1a revisited:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right) \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



For small dwell times: $\Phi_t \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$



Lyapunov exponent of a switched DAE

Theorem (Boundedness of \mathcal{S}_t , T. & Wirth 2012)

\mathcal{S}_t is *bounded* \Leftrightarrow the set of consistency *projectors* is *product bounded*

(swDAE)
$$E_\sigma \dot{x} = A_\sigma x$$

Theorem (Lyapunov exponent well defined, T. & Wirth 2012)

Let the consistency projectors be product bounded and not all be trivial, then the (*upper*) *Lyapunov exponent*

$$\lambda(\mathcal{S}) := \lim_{t \rightarrow \infty} \lambda_t(\mathcal{S}_t) = \lim_{t \rightarrow \infty} \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t}$$

of (swDAE) is well defined and *finite*.

Note that: (swDAE) uniformly exponentially stable

$$:\Leftrightarrow \exists M \geq 1, \mu > 0 : \|x(t)\| \leq M e^{-\mu t} \|x(0-)\| \quad \forall t \geq 0$$

$$\Rightarrow \lambda(\mathcal{S}) \leq -\mu < 0$$



Converse Lyapunov theorem for switched DAEs

For $\varepsilon > 0$ define “Lyapunov norm”

$$\|x\|_\varepsilon := \sup_{t>0} \sup_{\Phi_t \in \mathcal{S}_t} e^{-(\lambda(S)+\varepsilon)t} \|\Phi_t x\|$$

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Theorem (Converse Lyapunov theorem, T. & Wirth 2012)

(swDAE) is uniformly exponentially stable $\forall \sigma$

$\Rightarrow V = \|\cdot\|_\varepsilon$ is Lyapunov function for sufficiently small $\varepsilon > 0$

In particular: $V(\Pi x) \leq V(x)$ for all consistency projectors Π

Non-smooth Lyapunov function

$\|\cdot\|_\varepsilon$ in general non-smooth. Smoothification as in Yin, Sontag & Wang 1996 might violate jump condition!

Summary



(swDAE) $E_\sigma \dot{x} = A_\sigma x$

- solution theory
 - no classical solutions: jumps and impulses
 - impulse freeness condition (IFC)
 - jumps are still allowed
- stability conditions
 - multiple Lyapunov functions with jump condition (LJC)
 - slow switching
 - commutativity (quadratic Lyapunov function)
 - converse Lyapunov theorem



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