The joint spectral radius for semigroups generated by switched differential algebraic equations

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Switched DAEs

Linear switched DAE (differential algebraic equation)

(swDAE) $E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$ or short $E_\sigma \dot{x} = A_\sigma x$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathtt{P}\}$
  - piecewise constant, right-continuous
  - locally finitely many jumps (no Zeno behavior)
- matrix pairs $(E_1, A_1), \ldots, (E_\mathtt{P}, A_\mathtt{P})$
  - $E_p, A_p \in \mathbb{R}^{n \times n}$, $p = 1, \ldots, \mathtt{P}$
  - (E_p, A_p) regular, i.e. $\det(E_p s - A_p) \not\equiv 0$
  - impulse-free solutions (but jumps are allowed!)

Question

Growth rate and extremal norms for $E_\sigma \dot{x} = A_\sigma x \ \forall \sigma$

Solution formula

Theorem (A^{diff} and Π, Tanwani & T. 2010)

Let (E, A) be regular and consider

$$E\dot{x} = Ax \quad \text{on } [0, \infty)$$

$\Rightarrow \exists$ *unique consistency projector* Π *and unique flow matrix* A^{diff}*:*

$$\boxed{x(0) = \Pi x(0-)} \qquad \boxed{\dot{x} = A^{\text{diff}} x} \quad \text{on} \quad (0, \infty)$$

Furthermore, $A^{\text{diff}}\Pi = \Pi A^{\text{diff}}$.

Corollary (Solution formula for switched DAE)

Any solution of the switched DAE $E_\sigma \dot{x} = A_\sigma x$ *has the form*

$$x(t) = e^{A_k^{\text{diff}}(t-t_k)}\Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})}\Pi_{k-1} \cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x(t_0-)$$

Switched ODEs with jumps

Corollary

x solves $E_\sigma \dot{x} = A_\sigma x$ on $[0,\infty)$ $\Leftrightarrow$ x solves switched ODE with jumps

$$\dot{x} = A^{\text{diff}}_{p_i} x \text{ on } [t_i, t_{i+1})$$
$$x(t_i) = \Pi_{p_i} x(t_i-), \quad i \in \mathbb{N}$$

where $0 = t_0, t_1, \ldots,$ are the switching times of σ and $\sigma\big|_{[t_i,t_{i+1})} \equiv p_i$

Impulse freeness assumption

Above solution characterization only valid when switched DAE produces no Dirac impulses in x.

Theorem (Impulse freeness characterization, T. 2009)

$E_\sigma \dot{x} = A_\sigma x$ has only impulse free solutions $\forall \sigma$ $\Leftrightarrow$

$$\forall p, q \in \{1, \ldots, \mathrm{P}\}: \quad E_q(I - \Pi_q)\Pi_p = 0$$

Evolution operator

Consider in the following switched ODE with jumps

$$\begin{aligned} \dot{x} &= A_i x \text{ on } [t_i, t_{i+1}) \\ x(t_i) &= \Pi_i x(t_i-), \quad i \in \mathbb{N} \end{aligned}$$

where $0 = t_0 < t_1 < t_2 < \ldots$ and

$$(A_i, \Pi_i) \in \mathcal{M} \subseteq \left\{ \ (A, \Pi) \ \middle| \ A\Pi = \Pi A, \ \Pi = \Pi^2 \ \right\} \text{ compact}$$

Solutions:

$$x(t) = e^{A_k(t-t_k)}\Pi_k e^{A_{k-1}(t_k - t_{k-1})}\Pi_{k-1} \cdots e^{A_1(t_2-t_1)}\Pi_1 e^{A_0(t_1-t_0)}\Pi_0 x(t_0-)$$

Definition (Set of all evolutions with fixed time span $t \geq 0$)

$$\mathcal{S}_t := \left\{ \prod_{i=0}^{k} e^{A_i \tau_i}\Pi_i \ \middle| \ (A_i, \Pi_i) \in \mathcal{M}, \ \sum_{i=0}^{k} \tau_i = t, \ \tau_i > 0, \tau_k \geq 0 \ \right\}$$

Semi group property

Lemma (Semi group)

The set

$$\mathcal{S} := \bigcup_{t>0} \mathcal{S}_t$$

is a semi group with

$$\mathcal{S}_{s+t} = \mathcal{S}_s \mathcal{S}_t := \{\ \Phi_s \Phi_t \ \mid \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t\ \}$$

Need commutativity to show "$\subseteq$":

$$e^{A\tau}\Pi = e^{A(\tau-\tau')} e^{A\tau'}\Pi\Pi = e^{A(\tau-\tau')}\Pi e^{A\tau'}\Pi$$

for any $(A, \Pi) \in \mathcal{M}$ and $0 < \tau' < \tau$

Exponential growth bound

Definition (Exponential growth bound)

For $t > 0$ the *exponential growth bound* of $E_\sigma \dot{x} = A_\sigma x$ is

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\}$$

Definition implies for all solutions x of $E_\sigma \dot{x} = A_\sigma x$:

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\| \, \|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t)\, t} \|x(0-)\|$$

Difference to switched ODEs without jumps

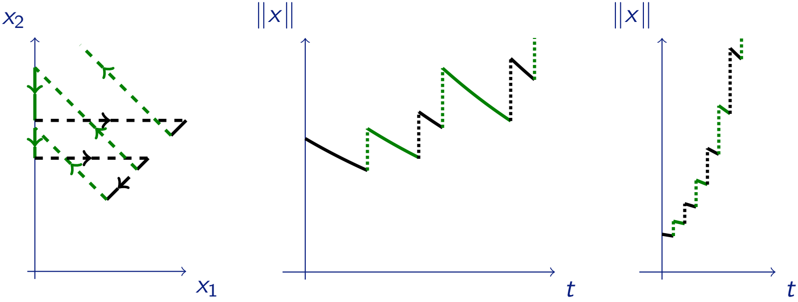
$\lambda_t(\mathcal{S}_t) = \pm\infty$ is possible!

All jumps are trivial, i.e. $\Pi_p = 0 \quad \Rightarrow \quad \lambda_t(\mathcal{S}_t) = -\infty$

Infinite exponential growth bound

Example:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right) \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



For small dwell times: $\Phi_t \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Existence of exponential growth rate

Theorem (Boundedness of $\mathcal{S}_t$)

$\mathcal{S}_t$ is bounded $\Leftrightarrow$ the set of jump projectors is product bounded

Reminder:

$$\mathcal{S}_t := \left\{ \prod_{i=0}^{k} e^{A_i \tau_i} \Pi_i \;\middle|\; (A_i, \Pi_i) \in \mathcal{M},\; \sum_{i=0}^{k} \tau_i = \Delta t,\; \tau_i > 0,\; \tau_k \geq 0 \right\}$$

Theorem (Exponential growth rate well defined)

Let the jump projectors be product bounded and not all be trivial, then the (upper) Lyapunov exponent

$$\lambda(\mathcal{S}) := \lim_{t\to\infty} \lambda_t(\mathcal{S}_t) = \lim_{t\to\infty} \sup_{\Phi_t \in \mathcal{S}_t} \frac{\|\Phi_t\|}{t}$$

of the semi-group $\mathcal{S}$ is well defined and finite.

Connection to the generalized spectral radius

Oberservation: x solves switched ODE $\Leftrightarrow$

$$x(t+1) \in \{ \ \Phi x(t) \ | \ \Phi \in \mathcal{S}_1 \ \}$$

Definition (Generalized spectral radius)

For $k \in \mathbb{N}$ define the *discrete growth rate*

$$\rho_k(\mathcal{S}_1) := \sup_{\Phi_i \in \mathcal{S}_1} \|\Phi_k \Phi_{k-1} \cdots \Phi_1\|^{1/k}.$$

The generalized spectral radius is

$$\rho(\mathcal{S}_1) := \lim_{k\to\infty} \rho_k(\mathcal{S}_1).$$

Clearly, $\ln \rho_k(\mathcal{S}_1) = \sup_{\Phi\in\mathcal{S}_k} \frac{\ln \|\Phi\|}{k} = \lambda_k(\mathcal{S}_k)$ and therefore

$$\boxed{\lambda(\mathcal{S}) = \ln \rho(\mathcal{S}_1)}$$

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Converse Lyapunov theorem for switched DAEs

Consider again

$$E_\sigma \dot{x} = A_\sigma x \qquad \textbf{(swDAE)}$$

with corresponding semigroup $\mathcal{S}_t$.

(swDAE) uniformly exponentially stable

$:\Leftrightarrow \quad \exists M \geq 1, \mu > 0 : \|x(t)\| \leq M e^{-\mu t} \|x(0-)\| \quad \forall t \geq 0$

$\Rightarrow \quad \lambda(\mathcal{S}) \leq -\mu < 0.$

Definition (Lyapunov norm)

For $\varepsilon > 0$ define

$$|\!|\!| x |\!|\!|_\varepsilon := \sup_{t>0} \sup_{\Phi_t \in \mathcal{S}_t} e^{-(\lambda(\mathcal{S})+\varepsilon)t} \|\Phi_t x\|$$

Theorem (Converse Lyapunov theorem, T. & Wirth 2012)

(swDAE) *is uniformly exponentially stable* $\forall \sigma$

$\Rightarrow$ $V = |\!|\!| \cdot |\!|\!|_\varepsilon$ *is Lyapunov function for sufficiently small* $\varepsilon > 0$

In particular: $V(\Pi x) \leq V(x)$ for all projectors Π

Barabanov norm

Definition (Barabanov norm)

$|||\cdot|||$ is called Barabanov norm for $\mathcal{S}$, iff

1. $|||\Phi_t x^0||| \leq e^{\lambda t}|||x^0|||, \quad \Phi_t \in \mathcal{S}_t$
2. $\forall x^0 \in \mathbb{R}^n \; \exists \overline{\Phi}_t \in \overline{\mathcal{S}_t} : \quad |||\overline{\Phi}_t x_0||| = e^{\lambda t}|||x_0|||$

In particular, ever Barabanov norm with $\lambda < 0$ defines a Lyapunov function

Theorem (Existence of Barabanov norm)

Assume $\mathcal{S}$ is irreducible, i.e. $\mathcal{S}\mathcal{M} \subseteq \mathcal{M}$ implies $\mathcal{M} = \emptyset$ or $\mathcal{M} = \mathbb{R}^n$. Then the following are equivalent:

1. *The consistency projectors are product bounded*
2. *The Lyapunov exponent $\lambda(\mathcal{S})$ is bounded*
3. *There exists a Barabanov norm with $\lambda = \lambda(\mathcal{S})$*

Construction of Barabanov norm

Construction of Barabanov norm similar as in (Wirth 2002, LAA):

$$S_\infty := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} e^{-\lambda(S)\,t} \mathcal{S}_t}$$

is a compact nontrivial semigroup, the limit semigroup.

$$|||x||| := \max\{\ \|Sx\|\ \mid S \in \mathcal{S}_\infty\ \}$$

is the sought Barabanov norm.

Conclusions

- Studied switched DAEs $E_\sigma \dot{x} = A_\sigma x$
- Key observation:

$$x(t) = e^{A_k^{\text{diff}}(t-t_k)}\Pi_k \cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x(t_0-)$$

- Flow set

$$\mathcal{S}_t := \left\{ \prod_{i=0}^{k} e^{A_i^{\text{diff}}\tau_i}\Pi_i \;\middle|\; (A_i^{\text{diff}}, \Pi_i) \in \mathcal{M},\ \sum_{i=0}^{k}\tau_i = \Delta t,\ \tau_i > 0 \right\}$$

- Product boudedness of consistency projectors necessary and sufficient for boundedness of $\mathcal{S}_t$
- Converse Lyapunov theorem
- Construction of Barabanov norm in irreducible case