Switched differential algebraic equations: Jumps and impulses

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Seminar at *LAMIH*, University of Valenciennes, France, 20.09.2012

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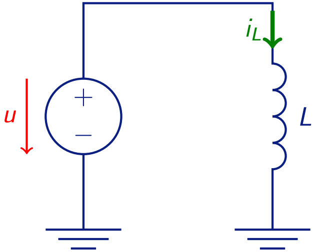
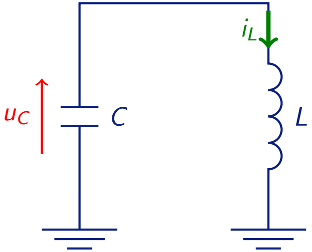
Contents

Standard modeling of circuits



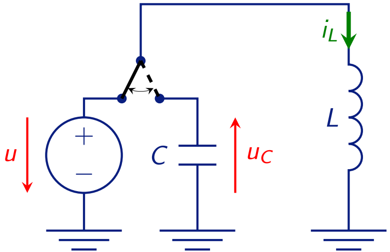


$$\frac{\mathrm{d}}{\mathrm{d}t} i_L = \frac{1}{L} u$$

$$\frac{\mathrm{d}}{\mathrm{d}t} i_L = -\frac{1}{L} u_C$$
$$\frac{\mathrm{d}}{\mathrm{d}t} u_C = \frac{1}{C} i_L$$

General form: $\dot{x} = Ax + Bu$

Switched ODE?



Mode 1: $\frac{\mathrm{d}}{\mathrm{d}t} i_L = \frac{1}{L} u$

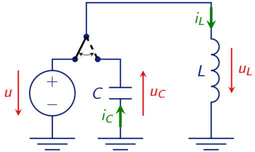
Mode 2: $\frac{\mathrm{d}}{\mathrm{d}t} i_L = -\frac{1}{L} u_C$

$\frac{\mathrm{d}}{\mathrm{d}t} u_C = \frac{1}{C} i_L$

No switched ODE

Not possible to write as $\boxed{\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t)}$.

Include algebraic equations in description



With $x := (i_L, u_L, i_C, u_C)$ write each mode as:

$$E_p \dot{x} = A_p x + B_p u$$

Algebraic equations $\Rightarrow$ E_p singular

Mode 1: $L \frac{d}{dt} i_L = u_L,\ C \frac{d}{dt} u_C = i_C,\ 0 = u_L - u,\ 0 = i_C$

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} u$$

Mode 2: $L \frac{d}{dt} i_L = u_L,\ C \frac{d}{dt} u_C = i_C,\ 0 = i_L - i_C,\ 0 = u_L + u_C$

Switched DAEs

DAE = Differential algebraic equation

Switched DAE

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \qquad \textbf{(swDAE)}$$

or short $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$

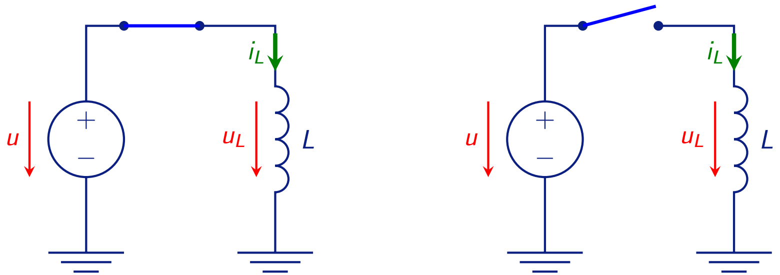
with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \dots, \mathrm{p}\}$
  - piecewise constant
  - locally finitely many jumps
- modes $(E_1, A_1, B_1), \dots, (E_\mathrm{p}, A_\mathrm{p}, B_\mathrm{p})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \dots, \mathrm{p}$
  - $B_p : \mathbb{R}^{n\times m}$, $p = 1, \dots, \mathrm{p}$
- input $u : \mathbb{R} \to \mathbb{R}^m$

Question

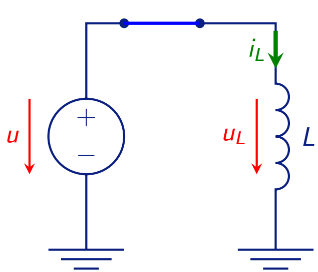
Existence and nature of solutions?

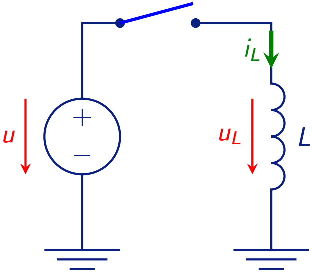
Impulse example



inductivity law:	$L\frac{\mathrm{d}}{\mathrm{d}t}i_L = u_L$	
switch dependent:	$0 = u_L - u$	$0 = i_L$

Impulse example





$x = [i_L, u_L]^\top$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$x = [i_L, u_L]^\top$

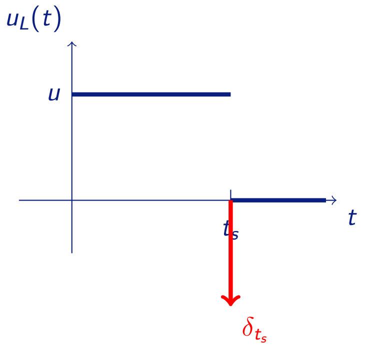
$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

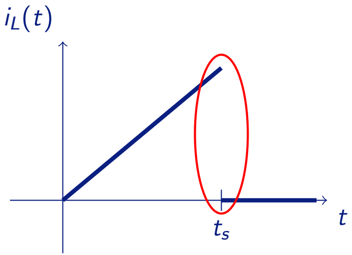
Solution of example

$L\frac{\mathrm{d}}{\mathrm{d}t}i_L = u_L,\qquad 0 = u_L - u$ or $0 = i_L$

Assume: u constant, $i_L(0) = 0$

switch at $t_s > 0$: $\sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$





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Distribution theorie - basic ideas

Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ_0 is "derivative" of Heaviside step function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1. Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
2. Axiomatic: Space of all "derivatives" of continuous functions (J. Sebastião e Silva 1954)

Distributions - formal

Definition (Test functions)

$\mathcal{C}_0^\infty := \{\ \varphi : \mathbb{R} \to \mathbb{R}\ |\ \varphi$ is smooth with compact support $\}$

Definition (Distributions)

$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R}\ |\ D$ is linear and continuous $\}$

Definition (Regular distributions)

$f \in L_{1,\text{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_{\mathbb{D}} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)\mathrm{d}t \in \mathbb{D}$

Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

Dirac Impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$

Multiplication with functions

Definition (Multiplication with smooth functions)

$\alpha \in \mathcal{C}^\infty: \quad (\alpha D)(\varphi) := D(\alpha\varphi)$

(swDAE) $\quad E_\sigma \dot{x} = A_\sigma x + B_\sigma u$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma, B_\sigma \notin \mathcal{C}^\infty$

Multiplication cannot be defined for general distributions!

Dilemma

Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients

Distributions

- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

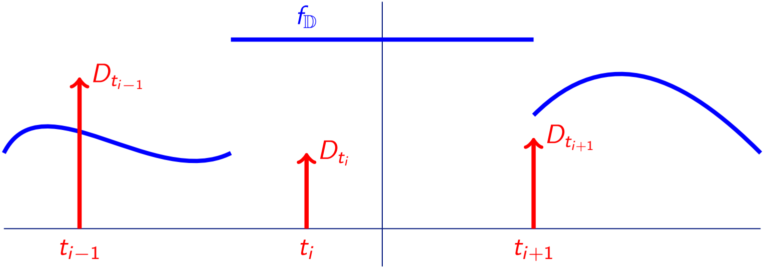
Space of distributions too big.

Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$)

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

- $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- Multiplication with $\mathcal{C}^\infty_{\text{pw}}$-functions well defined
- Left and right sided evaluation at $t \in \mathbb{R}$: $D(t-), D(t+)$
- Impulse at $t \in \mathbb{R}$: $D[t]$

(swDAE) $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with input $u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$

Application to (swDAE)

x solves **(swDAE)** $:\Leftrightarrow$ $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ and **(swDAE)** holds in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

Relevant questions

Consider $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with regular matrix pairs (E_p, A_p).

- Existence of solutions?
- Uniqueness of solutions?
- Inconsistent initial value problems?
- Jumps and impulses in solutions?
- Conditions for impulse free solutions?

Theorem (Existence and uniqueness, T. 2009)

$\forall x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \ \forall t_0 \in \mathbb{R} \ \forall u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m \ \exists! x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$:

$$x_{(-\infty,t_0)} = {x^0}_{(-\infty,t_0)}$$
$$(E_\sigma \dot{x})_{[t_0,\infty)} = (A_\sigma x + B_\sigma u)_{[t_0,\infty)}$$

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Regularity: Definition and characterization

Definition (Regularity)

(E, A) regular $\quad :\Leftrightarrow \quad \det(sE - A) \not\equiv 0$

Theorem (Characterizations of regularity)

The following statements are equivalent:

- *(E, A) is regular.*
- *$\exists S, T \in \mathbb{R}^{n\times n}$ invertible which yield quasi-Weierstrass form*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

  where N is a nilpotent matrix.
- *$\forall$ smooth f $\exists$ classical solution x of $E\dot{x} = Ax + f$ which is uniquely given by $x(t_0)$ for any $t_0 \in \mathbb{R}$.*

Wong sequences and the quasi-Weierstrass form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

Theorem (Armentano '86, Berger, Ilchmann, T. '12)

For regular (E, A) define the Wong sequences

$$\mathcal{V}^{i+1} := A^{-1}(E\mathcal{V}^i), \qquad \mathcal{V}^0 := \mathbb{R}^n,$$
$$\mathcal{W}^{i+1} := E^{-1}(A\mathcal{W}^i), \qquad \mathcal{W}^0 := \{0\}.$$

Then $\mathcal{V}^i \stackrel{finite}{\to} \mathcal{V}^*$ and $\mathcal{W}^i \stackrel{finite}{\to} \mathcal{W}^*$. Choose V, W such that $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$ than

$$T := [V, W], \quad S := [EV, AW]^{-1}$$

*yield (**QWF**).*

Matlab Code for calculating quasi-Weierstrass form

Calculating a basis of the pre-image $A^{-1}(\operatorname{im} S)$:

```
function V=getPreImage(A,S)
[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
end;
```

Calculating V with $\operatorname{im} V = \mathcal{V}^*$:

```
function V = getVspace(E,A)
[m,n]=size(E);
if (m==n) & [m,n]==size(A)
    V=eye(n,n);
    oldsize=n+1; newsize=n; finished=0;
    while (newsize~=oldsize);
       EV=colspace(E*V);
       V=getPreImage(A,EV);
       oldsize=newsize; newsize=rank(V);
    end;
end;
```

Calculating W with $\operatorname{im} W = \mathcal{W}^*$ analogously.

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Consistency projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \qquad \textbf{(QWF)}$$

Definition (Consistency projector)

Let (E, A) be regular with (**QWF**), consistency projector:

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Theorem

x solves $E_\sigma \dot{x} = A_\sigma x \quad \Rightarrow \quad$ for all switching times $t \in \mathbb{R}$:

$$x(t+) = \Pi_{(E_q, A_q)} x(t-), \quad q := \sigma(t+)$$

Differential projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

Definition (Differential projector)

Let (E, A) be regular with **(QWF)**, differential projector:

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A$

Theorem (Tanwani & T. 2010)

x solves $E_\sigma \dot{x} = A_\sigma x \quad \Rightarrow \qquad$ for non-switching times $t \in \mathbb{R}$:

$$\dot{x}(t) = A^{\text{diff}}_{\sigma(t)} x(t)$$

Impulse projector

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \qquad \textbf{(QWF)}$$

Definition (Impulse projector)

Let (E, A) be regular with **(QWF)**, impulse projector:

$$\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$$

$E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E$

Theorem (Tanwani & T. 2009)

x solves $E_\sigma \dot{x} = A_\sigma x \quad \Rightarrow \quad \forall t \in \mathbb{R}:$

$$x[t] = \sum_{i=0}^{n-2} (E^{\text{imp}}_{\sigma(t+)})^{i+1} (x(t+) - x(t-)) \delta_t^{(i)}$$

Impulse freeness

Consider $E_\sigma \dot{x} = A_\sigma x$

Theorem (Impulse freeness, T. 2009)

$$\forall p, q \in \{1, \ldots, \mathtt{p}\}: \; E_q(\Pi_{(E_q,A_q)} - I)\Pi_{(E_p,A_p)} = 0 \quad \Rightarrow \quad x[t] = 0 \; \forall t$$

Weaker than the usual index one (a.k.a. impulse-freeness) assumption.

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Solution formula, inhomogeneous non-switched case

Consider $E\dot{x} = Ax + f$

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \qquad \textbf{(QWF)}$$

$$\Pi_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}, \qquad \Pi^{\text{diff}}_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S, \qquad \Pi^{\text{imp}}_{(E,A)} := T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] S,$$

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A, \qquad E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E$$

Theorem (Explicit solution formula, non-switched, T. 2012)

x solves $E\dot{x} = Ax + f$ $\quad \Leftrightarrow \quad \exists c \in \mathbb{R}^n\ \forall t \in \mathbb{R}:$

$$x(t) = e^{A^{\text{diff}} t} \Pi_{(E,A)} c + \int_0^t e^{A^{\text{diff}}(t-s)} \Pi^{\text{diff}}_{(E,A)} f(s) \mathrm{d}s - \sum_{i=0}^{n-1} (E^{\text{imp}})^i \Pi^{\text{imp}}_{(E,A)} f^{(i)}(t)$$

Jumps and impulses for switched DAE

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \qquad \textbf{(swDAE)}$$

$$B_q^{\text{imp}} := \Pi_{(E_q,A_q)}^{\text{imp}} B_q, \quad q \in \{1,\ldots,\mathrm{p}\},\ u[\cdot] = 0$$

Corollary (Jumps and impulses)

x solves (**swDAE**) $\Rightarrow \quad \forall t \in \mathbb{R}:$

$$x(t+) = \Pi_{(E_q,A_q)} x(t-) - \sum_{i=0}^{n-1} (E_q^{\text{imp}})^i B_q^{\text{imp}} u^{(i)}(t+),$$

$$x[t] = -\sum_{i=0}^{n-1} (E_q^{\text{imp}})^{i+1} (I - \Pi_{(E_q,A_q)}) x(t-)\ \delta_t^{(i)} \qquad q := \sigma(t+)$$

$$- \sum_{i=0}^{n-1} (E_q^{\text{imp}})^{i+1} \sum_{j=0}^{i} B_q^{\text{imp}} u^{(i-j)}(t+)\ \delta_t^{(j)}$$

Conclusions

- DAEs natural for modeling electrical circuits
- Switches induce jumps and impulses $\Rightarrow$ Distributional solutions
  - General distributions not suitable
  - Smaller space: Piecewise-smooth distributions
- Regularity $\Leftrightarrow$ Existence & uniqueness of solutions
- Unique consistency jumps
- Condition for impulse-freeness
- Explicit solution formulas