Switched behaviors with impulses

A unifying framework

Stephan Trenn* and **Jan C. Willems****

* Technomathematics group, University of Kaiserslautern, Germany
** Department of Electrical Engineering, K.U. Leuven, Belgium

51st IEEE Conference on Decision and Control
Tuesday, December 11, 2012, 14:20–14:40, Maui, USA

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Contents

The usual modeling approach with inputs and outputs

Usual modeling using inputs and outputs:

Input u $\longrightarrow$ **System** $\dot{x} = Ax + Bu$, $y = Cx + Du$ $\longrightarrow$ **Output** y

Drawbacks of this approach:

- **Separating external signals as inputs and outputs**
  Example: Electrical circuit with "wires sticking out"
  Is the current or the voltage at the wires an input?
- **Algebraic constraints have to be eliminated**
  Example: First principles modeling of electrical circuit contains Kirchhoff laws as algebraic constraints

The behavioral approach

Behavioral approach $\leftrightarrow$ describe system by set of trajectories:

$$\begin{aligned}\mathfrak{B} &= \{\ w : \mathbb{R} \to \mathbb{R}^q \ \mid\ w \text{ fulfills system laws }\ \} \\ &= \{\ w \ \mid\ \mathcal{R}(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) = 0\ \}\end{aligned}$$

$$\boxed{\begin{array}{c}\textbf{System}\\ \mathcal{R}(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) = 0\end{array}} \longrightarrow \textbf{Signals } w$$

Kernel representation via matrix polynomials

Let $\mathcal{R}(s) \in \mathbb{R}^{p\times q}[s]$ be a polynomial with matrix coefficients:

$$\mathcal{R}(s) = R_0 + R_1 s + R_2 s^2 + \ldots R_d s^d, \quad R_0, R_1, \ldots, R_d \in \mathbb{R}^{p\times q}$$

The associated differential operator is given by

$$\mathcal{R}(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) = t \mapsto \Big(R_0 w(t) + R_1 \dot{w}(t) + R_2 \ddot{w}(t) + \ldots + R_d w^{(d)}(t)\Big)$$

Switched systems viewed as time-varying systems

Definition (Switched system)

System description changes suddenly at certain times
= time-varying system with "piecewise-constant" descriptions

Time-varying behaviors

Instead of $\mathcal{R}(s) \in \mathbb{R}^{p \times q}[s]$ consider $\mathcal{R}(s) \in \mathrm{map}(\mathbb{R} \to \mathbb{R}^{p \times q})[s]$, i.e. $\mathcal{R}(s)$ is a polynomial with matrix function coefficients:

$$\mathcal{R}(s) = R_0(\cdot) + R_1(\cdot)s + R_2(\cdot)s^2 + \ldots R_d(\cdot)s^d$$

and the associated differential operator is given by

$$\mathcal{R}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w)(t) = R_0(t)w(t) + R_1(t)\dot{w}(t) + R_2(t)\ddot{w}(t) + \ldots + R_d(t)w^{(d)}(t))$$

Kernel representation of time-varying behavior still:

$$\mathfrak{B} = \left\{ \ w \ \middle| \ \mathcal{R}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w) = 0 \ \right\}$$

Global kernel representation

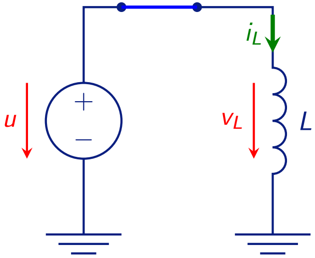
Global kernel representation

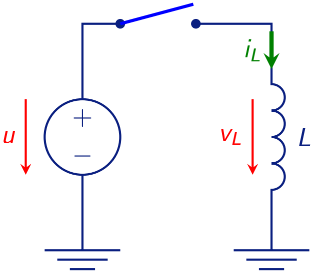
Here $\mathcal{R}(\frac{\mathrm{d}}{\mathrm{d}t})(w) = 0$ should hold on the whole time axis $\mathbb{R}$, in particular at the switching times!

Major difference to all previous approaches, where differential equations should only hold between the switches and the switching times are treated separately, see e.g.

- Geerts & Schumacher: "Impulsive-smooth behaviors in multimode systems", Automatica 1996
- Rocha, Willems, Rapisarda & Napp: "On the stability of switched behavioral systems", last year's CDC
- Bonilla & Malabre: "Description of switched systems by implicit representations", next talk

Example



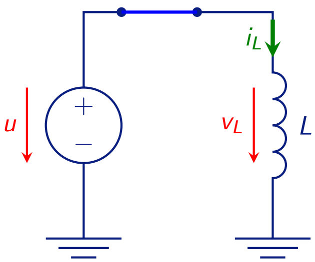


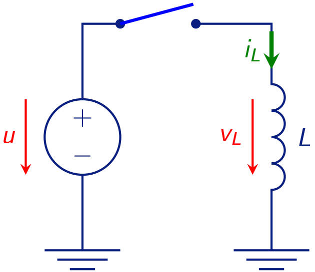
constant input: $\dot{u} = 0$

inductivity law: $L \frac{\mathrm{d}}{\mathrm{d}t} i_L = v_L$

switch dependent: $0 = v_L - u$ $\qquad$ $0 = i_L$

Example

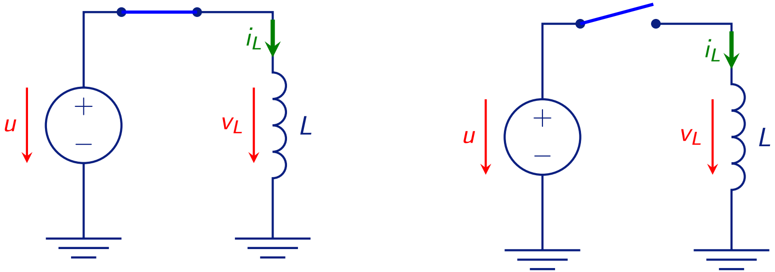




$$w = [u, i_L, v_L]^\top$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{w} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} w = 0 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{w} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} w = 0$$

Example

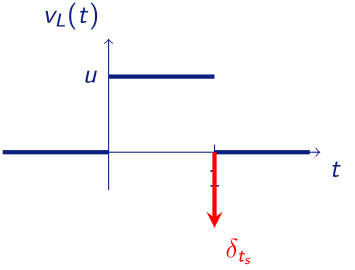


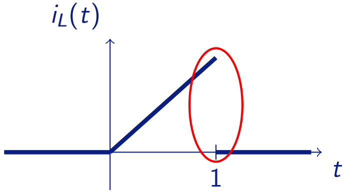
$$w = [u, i_L, v_L]^\top$$

switch closed on $[0,1)$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{w} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -\mathbb{1}_{[0,1)} & 1-\mathbb{1}_{[0,1)} & \mathbb{1}_{[0,1)} \end{bmatrix} w = 0$

Solution of example

$\dot{u} = 0 \quad \Leftrightarrow \quad u$ constant on whole time axis
Inductivity law $L\frac{\mathrm{d}}{\mathrm{d}t} i_L = v_L$ holds globally (switch independent)





Switch open on $(-\infty, 0)$: $i_L = 0 \quad \Rightarrow \quad v_L = 0$
Switch closed on $[0, 1)$: $v_L = u \quad \Rightarrow \quad i_L(t) = \frac{u}{L} t$
unique jump in w at $t = 0$
Switch open on $(1, \infty)$: $i_L = 0 \quad \Rightarrow \quad v_L = 0$
unique jump in w at $t = 1$ and Dirac impulse at $t = 1$

Requirements for switched behavior framework

Requirements extrapolated from example

- Solutions exhibit jumps
- Jumps are uniquely determined (no additional jump map is required)
- Solutions contain Dirac impulses
- Dirac impulses are also uniquely determined

Jumps and impulses can be handled by distributional solution space, however the definition

$$\mathcal{B} = \left\{ \, w \in \mathbb{D}^q \;\middle|\; \mathcal{R}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w) = 0 \, \right\}$$

requires multiplication of the distributions w, $\dot{w}, \ldots, w^{(d)}$ with piecewise-constant coefficient matrices!

Multiplication with non-smooth coefficients

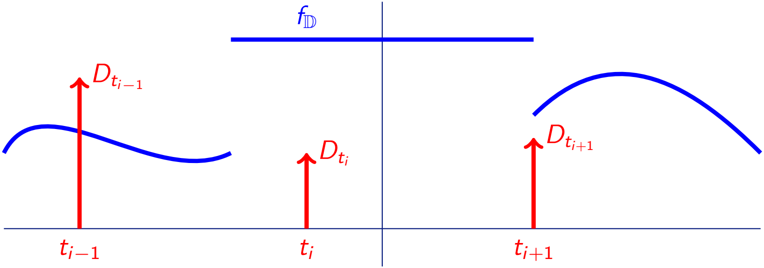
A general multiplication of distributions with non-smooth coefficient is not well defined!

Piecewise-smooth distributions

Way out: Consider smaller space of piecewise-smooth distributions

Definition (Piecewise smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$)

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T: D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Switched behavior well defined

Time-varying behavior with piecewise-smooth coefficient matrices:

$$\mathcal{B} = \left\{ \, w \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^q \;\middle|\; \mathcal{R}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w) = 0 \, \right\}$$

where $\mathcal{R}(s) \in (\mathcal{C}^\infty_{\mathrm{pw}})^{p \times q}[s]$ well defined.

Fuchssteiner multiplication

$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ even allows definition of multiplication of two distributions

$\Rightarrow$ we can consider general distributional behaviors:

$$\mathcal{B} = \left\{ \, w \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^q \;\middle|\; \mathcal{R}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w) = 0 \, \right\}$$

where $\mathcal{R}(s) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{p \times q}[s]$

Dirac impulses in coefficient matrices

Why should one need Dirac impulses in the coefficient matrices?

Impulsive systems

Definition (Impulsive system)

Let $t_0 < t_1 < t_2 < \ldots$ be the impact times. An impulsive system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && \text{for } t \in (t_k, t_{k+1}) \\ x(t_k+) &= J_k x(t_k-) && \text{for } k = 0, 1, 2, \ldots \end{aligned}$$

Theorem

For $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ and $J \in \mathbb{R}^{n\times n}$:

$$\dot{x} = (J - I)\delta_0 x \quad \Leftrightarrow \quad x(0+) = Jx(0-) \text{ and constant otherwise}$$

Corollary

x solves impulsive ODE $\Leftrightarrow$ x solves distributional ODE

$$\dot{x} = (A + \sum_k (J_k - I)\delta_{t_k})x + Bu =: \mathcal{A}x + Bu \text{ with } \mathcal{A} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$$

Special cases covered by this approach

$\mathfrak{B} = \{\ w \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^q \ \mid \mathcal{R}(\frac{\mathrm{d}}{\mathrm{d}t})(w) = 0\ \}$ where $\mathcal{R}(s) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{p\times q}$ includes:

- Switched ODEs $\dot{x} = A_\sigma x + B_\sigma u$ with
$$\mathcal{R}(s) = [A_\sigma\ \ B_\sigma] + [I\ \ 0]s$$
- Switched DAEs $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with
$$\mathcal{R}(s) = [A_\sigma\ \ B_\sigma] + [E_\sigma\ \ 0]s$$
- Systems with impulsive inputs (i.e. u contains Dirac impulses)
- Impulsive systems:
$$\mathcal{R}(s) = \left[A + \sum_k (J_k - I)\delta_{t_k},\ \ B\right] + [I\ \ 0]s$$
- Switched behaviors with glueing condition as in Rocha et al. 2011

Content

Switched behaviors (with impacts)

Definition (Switched behaviors (with impacts))

A distributional behavior given by $\mathcal{R}(s) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{p\times q}$ is called

- switched behavior $:\Leftrightarrow$ the coefficients of $\mathcal{R}(s)$ are piecewise-constant
- switched behavior with impacts $:\Leftrightarrow$ additionally Dirac impulses (and their derivatives) are allowed in the coefficient matrices

Note that switching signal is fixed, therefore write

$$\mathfrak{B}_\sigma = \left\{ \; w \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^q \;\middle|\; \mathcal{R}_\sigma\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w) = 0 \; \right\}$$

with corresponding k-th smooth mode

$$\mathfrak{B}_k = \left\{ \; w \in (\mathcal{C}^\infty)^q \;\middle|\; \mathcal{R}_k\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w) = 0 \; \right\}$$

Autonomy characterization

Definition (Autonomy)

A distributional behavior $\mathfrak{B}$ is autonomous $:\Leftrightarrow \forall w_1, w_2 \in \mathfrak{B}\ \forall t \in \mathbb{R}$:

$$(w_1)_{(-\infty,t)} = (w_2)_{(-\infty,t)} \quad \Rightarrow \quad w_1 = w_2$$

Theorem (Autonomy characterization)

Switched behavior $\mathfrak{B}^I_\sigma$ with impacts is autonomous $\forall\sigma$

$\Leftrightarrow$ Switched behavior $\mathfrak{B}_\sigma$ without impacts is autonomous $\forall\sigma$

$\Leftrightarrow$ Each smooth mode $\mathfrak{B}_k$ is autonomous

$\Leftrightarrow$ $\det \mathcal{R}_k(s) \not\equiv 0$ *for all modes k*

Uniquely defined jumps and impulses

Two kinds of jumps and impulses:

1. Canonical jumps and impulses given by mode equations
2. Arbitrary jumps and impulses given by impacts

Conclusions

- We have introduced the notion of distributional behaviors:

$$\mathfrak{B} = \left\{ \, w \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^q \;\middle|\; \mathcal{R}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)(w) = 0 \, \right\}, \quad \mathcal{R}(s) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{p\times q}[s]$$

  with solutions and coefficient matrices in the space of piecewise-smooth distributions
- Encompasses
  - Switched ODEs and DAEs
  - Impulsive systems
  - Switched behaviors with glueing conditions
- A first theoretical result: Autonomy characterization for switched behaviors with impacts
- Many open questions: Controllability, observability, latent variable elimination, ...