

The Bang Bang Funnel Controller

Stephan Trenn

joint work with Daniel Liberzon (Univ. Illinois)

AG Technomathematik, TU Kaiserslautern

Research Seminar, GIPSA-Lab, Grenoble, June 6th, 2014

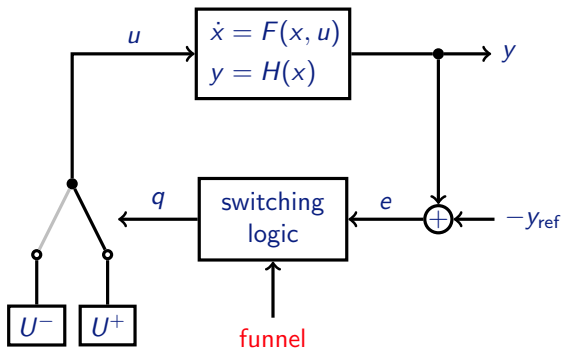


Contents



- 1 Introduction
- 2 Relative degree one
- 3 Higher relative degree
 - Relative degree two
 - Generalization to arbitrary relative degree
 - Hierarchical switching logic
 - Feasibility assumptions
- 4 Simulation

Tracking control: Closed Loop



reference signal $y_{ref} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ sufficiently smooth

The funnel

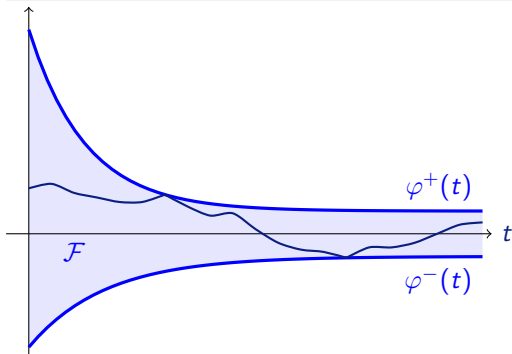


Control goal

Error $e := y - y_{\text{ref}}$ remains within *funnel*

$$\mathcal{F} = \mathcal{F}(\varphi^-, \varphi^+) := \{ (t, e) \mid \varphi^-(t) \leq e \leq \varphi^+(t) \}$$

where $\varphi_{\pm} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ sufficiently smooth



- time-varying error bound
- transient behavior
- practical convergence ($|e(t)| < \lambda$ for $t \gg 0$)

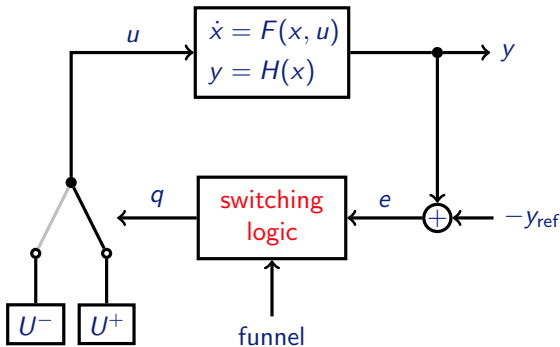


The bang-bang funnel controller

Continuous funnel controller: Introduced by Ilchmann et al. 2002

New approach

Achieve Control objectives with **bang-bang control**, i.e. $u(t) \in \{U^-, U^+\}$





Relative degree one

Definition (relative degree one)

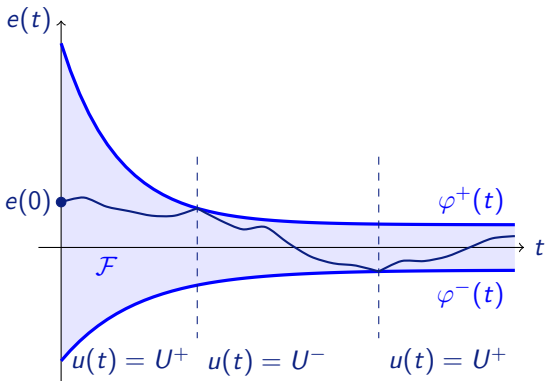
$$\begin{array}{l} \dot{x} = F(x, u) \\ y = H(x) \end{array} \quad \cong \quad \begin{array}{l} \dot{y} = f(y, z) + \overbrace{g(y, z)}^{>0} u \\ \dot{z} = h(y, z) \end{array}$$

- structural assumption
- f, g, h unknown to controller
- feasibility assumptions (later) formulated in terms of f, g, h

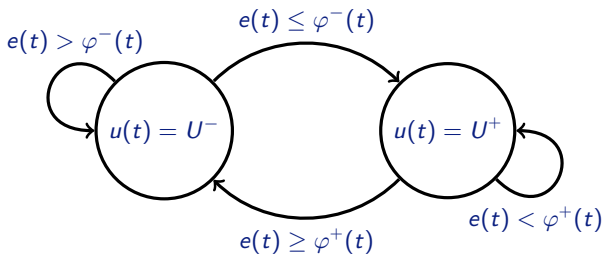
Crucial property

$$\begin{array}{l} u(t) \ll 0 \Rightarrow \dot{y}(t) \ll 0 \\ u(t) \gg 0 \Rightarrow \dot{y}(t) \gg 0 \end{array}$$

Switching logic: illustration



Switching logic: formal definition



Too simple?

⇒ feasibility assumptions



Feasibility assumptions

$$\begin{aligned} \dot{y} &= f(y, z) + g(y, z)u, & y_0 &\in \mathbb{R} \\ \dot{z} &= h(y, z), & z_0 &\in Z_0 \subseteq \mathbb{R}^{n-1} \end{aligned}$$

$$Z_t := \left\{ z(t) \left| \begin{array}{l} z : [0, t] \rightarrow \mathbb{R}^{n-1} \text{ solves } \dot{z} = h(y, z) \text{ for a} \\ z^0 \in Z_0 \text{ and for a } y : [0, t] \rightarrow \mathbb{R} \\ \text{with } \varphi^-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi^+(\tau) \\ \forall \tau \in [0, t] \end{array} \right. \right\}.$$

Feasibility assumptions

$$\forall t \geq 0 \quad \forall z_t \in Z_t : \quad \begin{aligned} U^- &\leq \frac{\dot{\varphi}^+(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi^+(t), z_t)}{g(y_{\text{ref}}(t) + \varphi^+(t), z_t)} \\ U^+ &\geq \frac{\dot{\varphi}^-(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi^-(t), z_t)}{g(y_{\text{ref}}(t) + \varphi^-(t), z_t)} \end{aligned}$$



Main result for relative-degree-one case

Theorem (Bang-bang funnel controller, Liberzon & T. 2010)

relative degree one & funnel & simple switching logic & feasibility

⇒

bang-bang funnel controller works:

- *existence and uniqueness of a global solution*
- *error remains within the funnel*
- *no zero behavior*

Necessary system knowledge

- for controller implementation
 - relative degree one
 - signals: error $e(t)$ and funnel boundaries $\varphi^\pm(t)$
- to check feasibility assumptions:
 - bounds for zero dynamics
 - bounds for f and g
 - bounds for y_{ref} and \dot{y}_{ref}
 - bounds for funnel boundaries

Contents



- 1 Introduction
- 2 Relative degree one
- 3 Higher relative degree**
 - Relative degree two
 - Generalization to arbitrary relative degree
 - Hierarchical switching logic
 - Feasibility assumptions
- 4 Simulation



Relative degree two

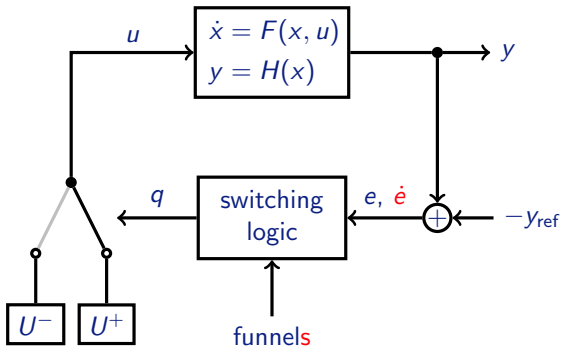
Definition (relative degree two)

$$\begin{array}{l} \dot{x} = F(x, u) \\ y = H(x) \end{array} \quad \cong \quad \begin{array}{l} \ddot{y} = f(y, \dot{y}, z) + \overbrace{g(y, \dot{y}, z)}^{>0} u \\ \dot{z} = h(y, \dot{y}, z) \end{array}$$

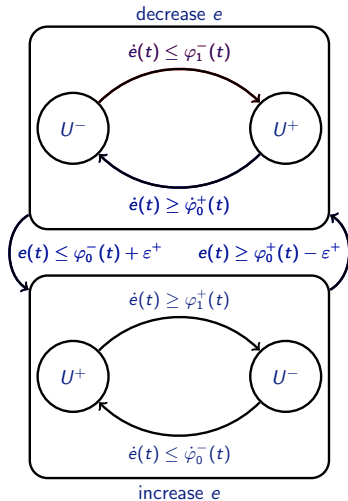
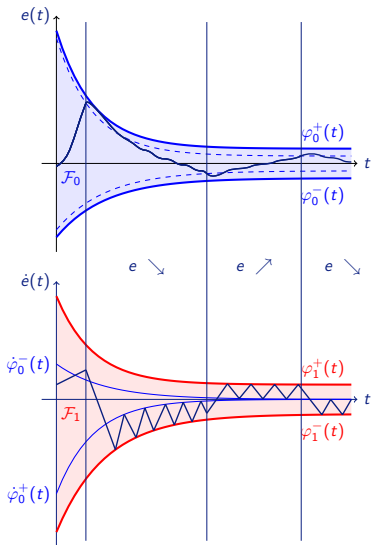
Crucial property

$$\begin{array}{l} u(t) \ll 0 \Rightarrow \ddot{y}(t) \ll 0 \\ u(t) \gg 0 \Rightarrow \ddot{y}(t) \gg 0 \end{array}$$

Tracking control: Closed loop



Switching logic



Relative degree r Definition (relative degree r)

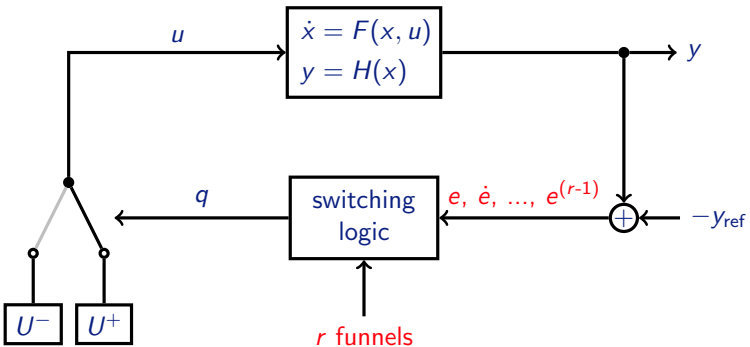
$$\begin{aligned} \dot{x} = F(x, u) &\cong y^{(r)} = f(y, \dot{y}, \dots, y^{(r-1)}, z) + \overbrace{g(y, \dots, y^{(r-1)}, z)}^{>0} u \\ y = H(x) &\quad \dot{z} = h(y, \dot{y}, \dots, y^{(r-1)}, z) \end{aligned}$$

Crucial property

$$u(t) \ll 0 \Rightarrow y^{(r)}(t) \ll 0$$

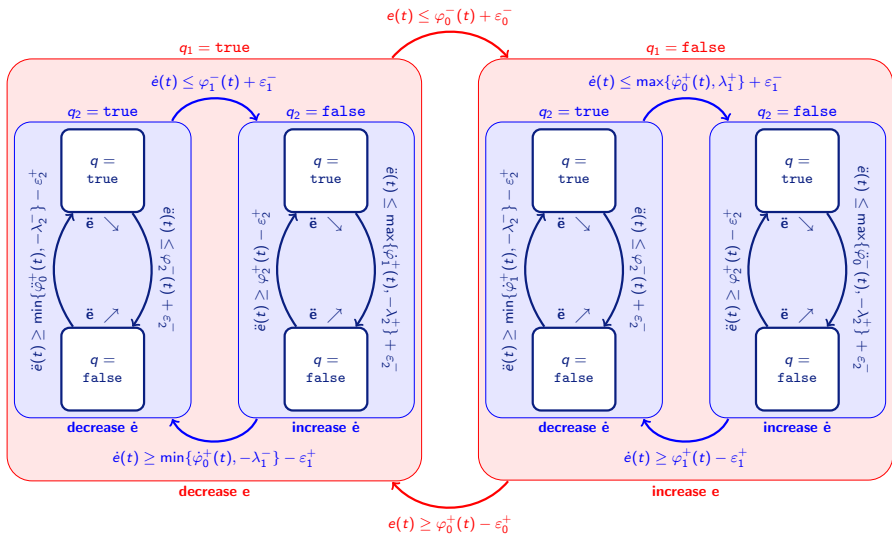
$$u(t) \gg 0 \Rightarrow y^{(r)}(t) \gg 0$$

Tracking control: Closed loop





Recursive Approach, Example $r = 3$

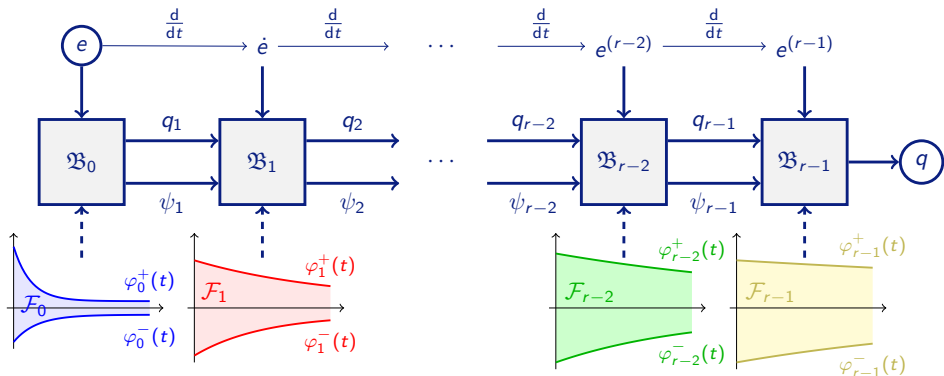


Contents



- 1 Introduction
- 2 Relative degree one
- 3 Higher relative degree**
 - Relative degree two
 - Generalization to arbitrary relative degree
 - Hierarchical switching logic**
 - Feasibility assumptions
- 4 Simulation

Hierarchical switching logic

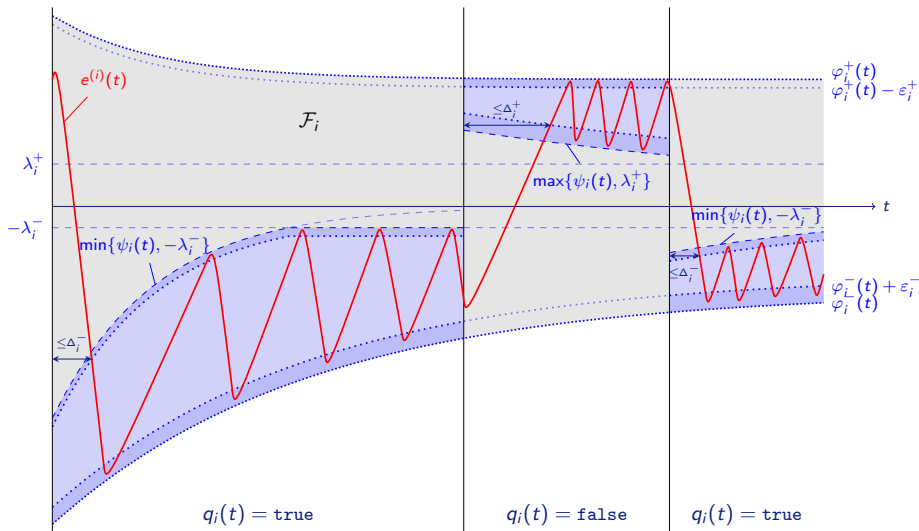


$q_i = \text{true} \Rightarrow \text{Goal: } e^{(i)}(t) < \min\{\psi_i(t), -\lambda_i^-\}$

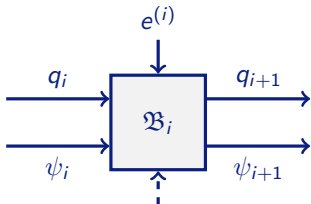
$q_i = \text{false} \Rightarrow \text{Goal: } e^{(i)}(t) > \max\{\psi_i(t), \lambda_i^+\}$



Desired behavior of block \mathfrak{B}_i

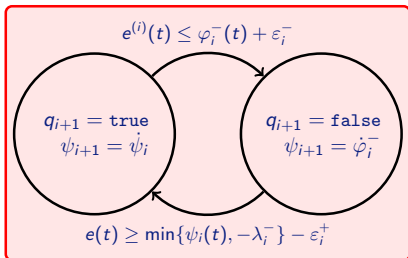


Definition of switching logic



$$\varphi_i^+, \varphi_i^-, \varepsilon_i^+, \varepsilon_i^-, \lambda_i^+, \lambda_i^-$$

$q_i = \text{true}$



$$q_i = \text{true} \quad \Rightarrow \quad \text{Goal: } e^{(i)} < \min\{\psi_i, -\lambda_i^-\}$$

$$q_i = \text{false} \quad \Rightarrow \quad \text{Goal: } e^{(i)} > \max\{\psi_i, \lambda_i^+\}$$

$$\text{where } \psi_i \in \{\dot{\varphi}_{i-1}^\pm, \ddot{\varphi}_{i-2}^\pm, \dots, (\varphi_0^\pm)^{(i)}\}$$

$q_i = \text{false}$

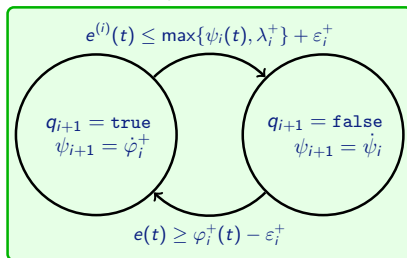
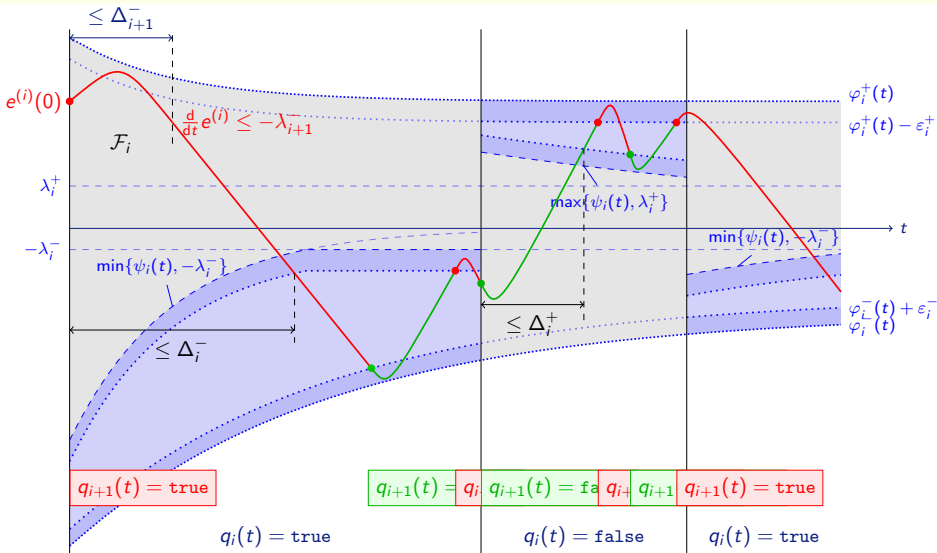




Illustration of switching logic of block \mathfrak{B}_i



Contents



- 1 Introduction
- 2 Relative degree one
- 3 Higher relative degree**
 - Relative degree two
 - Generalization to arbitrary relative degree
 - Hierarchical switching logic
 - Feasibility assumptions
- 4 Simulation



Feasibility assumptions (F1)-(F8)

„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F1): Relative degree r

$$\begin{aligned} \dot{x} = F(x, u) &\cong y^{(r)} = f(y, \dot{y}, \dots, y^{(r-1)}, z) + \overbrace{g(y, \dots, y^{(r-1)}, z)}^{>0} u \\ y = H(x) &\quad \dot{z} = h(y, \dot{y}, \dots, y^{(r-1)}, z), \quad z_0 \in Z_0 \subseteq \mathbb{R}^{n-r} \end{aligned}$$

and no finite escape of zero dynamics

Feasibility assumptions (F1)-(F8)



„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F2): reference signal

y_{ref} is r times (weakly) differentiable

Because of relative-degree assumption: y is by definition r times (weakly) differentiable, hence $e = y - y_{\text{ref}}$ is also r times (weakly) differentiable.

Feasibility assumptions (F1)-(F8)



„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F3): Initial error

$$e^{(i)}(0) \in [\varphi_i^-(0) + \varepsilon_i^+, \varphi_i^-(0) - \varepsilon_i^+]$$

Feasibility assumptions (F1)-(F8)



„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F4): Funnel boundaries

- φ_i^\pm are $r - i$ times (weakly) differentiable
- φ_i^\pm and its derivatives are bounded



Feasibility assumptions (F1)-(F8)

„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F5): Funnels suitable

- $\varphi_0^+(t) - \varepsilon_0^+ > \varphi_0^-(t) + \varepsilon_0^-$
- For $\psi^\pm \in \{\dot{\varphi}_{i-1}^\pm, \ddot{\varphi}_{i-2}^\pm, \dots, (\varphi_0^\pm)^{(i)}\}$ and $i = 1, 2, \dots, r - 1$:
 - $\varphi_i^+(t) - \varepsilon_i^+ > \max\{\psi^-(t), \lambda_i^+\} + \varepsilon_i^-$
 - $\min\{\psi^+(t), \lambda_i^-\} - \varepsilon_i^+ > \varphi_i^-(t) + \varepsilon_i^-$



Feasibility assumptions (F1)-(F8)

„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F6): Settling times

For $i = 0, 1, \dots, r - 1$ and $\Delta_r^\pm \geq 0$ exists Δ_i^\pm with

$$\Delta_i^\pm \geq \Delta_{i+1}^\pm + \frac{\|\varphi_i^+\|_\infty + \|\varphi_i^-\|_\infty}{\lambda_{i+1}^\pm}$$

“restrictive” condition only together with the following assumption ...



Feasibility assumptions (F1)-(F8)

„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F7): Safety distance

For $i = 0, 1, \dots, r - 2$ and $\psi^\pm \in \{\varphi_i^\pm, \dot{\varphi}_{i-1}^\pm, \dots, (\varphi_0^\pm)^{(i)}\}$

$$\varepsilon_i^\pm > \Delta_{i+2}^\pm \|\dot{\psi}^\pm - \varphi_{i+1}^\pm\|_\infty + \frac{(\|\dot{\psi}^\pm\|_\infty + \varphi_{i+1}^\pm\|_\infty)^2}{2\lambda_{i+2}^\mp}$$

Remark: λ_r^\pm is additional parameter for analysis and satisfies ...

Feasibility assumptions (F1)-(F8)



„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Feasibility assumption (F8): Last change rate

$$\lambda_r^+ > \max \{ \dot{\varphi}_{r-1}^-, \ddot{\varphi}_{r-2}^-, \dots, (\varphi_0^-)^{(r)} \}$$
$$-\lambda_r^- < \min \{ \dot{\varphi}_{r-1}^+, \ddot{\varphi}_{r-2}^+, \dots, (\varphi_0^+)^{(r)} \}$$

Feasibility assumptions (F1)-(F8)



„Ingredients“ for bang-bang funnel controller

- Existence and knowledge of relative degree
- Error e and its derivatives (up to $(r - 1)$ -st)
- Funnel boundaries φ_i^\pm and its derivatives
- Safety distances ε_i^\pm and minimal change rates λ_i^\pm
- For analysis: Settling times Δ_i^\pm

Remarks:

- Assumptions **(F1)-(F8)** independent of U^+ and U^-
- Also independent of system parameters
- For (almost) arbitrary given funnel \mathcal{F}_0 the remaining funnels can be constructed such that **(F3)-(F8)** are satisfied



Feasibility assumption (F9)

$$\begin{aligned} \dot{x} = F(x, u) &\cong y^{(r)} = f(y, \dot{y}, \dots, y^{(r-1)}, z) + g(y, \dots, y^{(r-1)}, z)u \\ y = H(x) &\quad \dot{z} = h(y, \dot{y}, \dots, y^{(r-1)}, z), \quad z_0 \in Z_0 \subseteq \mathbb{R}^{n-r} \end{aligned}$$

Feasibility assumption (F9): Input “strong” enough

$$U^+ \geq \frac{\lambda_r^+ + y_{\text{ref}}^{(r)}(t) - f(y_t^0, y_t^1, \dots, y_t^{r-1}, z_t)}{g(y_t^0, y_t^1, \dots, y_t^{r-1}, z_t)}$$

$$U^- \leq \frac{-\lambda_r^- + y_{\text{ref}}^{(r)}(t) - f(y_t^0, y_t^1, \dots, y_t^{r-1}, z_t)}{g(y_t^0, y_t^1, \dots, y_t^{r-1}, z_t)}$$

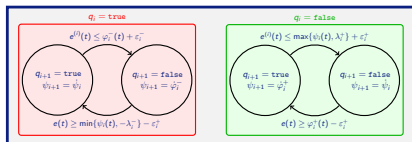
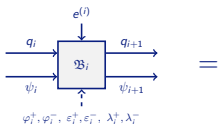
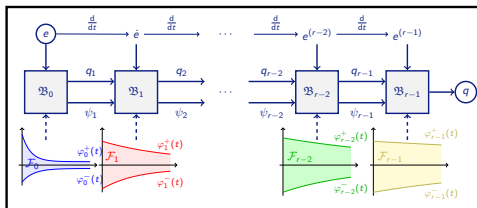
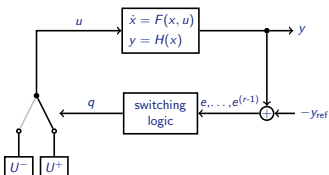
for all $t \geq 0$, $(y_t^0, y_t^1, \dots, y_t^{r-1}) \in \Phi_t^{y_{\text{ref}}}$, $z_t \in Z_t^{y_{\text{ref}}}$

$$\Phi_t^{y_{\text{ref}}} := \left\{ (y_0, \dots, y_{r-1}) \mid \forall i: y_i - y_{\text{ref}}^{(i)}(t) \in [\varphi_i^-(t), \varphi_i^+(t)] \right\},$$

$$Z_t^{y_{\text{ref}}} := \left\{ z(t) \mid \begin{array}{l} z \text{ solves } \dot{z} = h(y, \dot{y}, \dots, y^{(r-1)}, z), z(0) = z_0 \in Z_0, \\ y \in \mathcal{C}^{r-1} \text{ with } (y(\tau), \dots, y^{(r-1)}(\tau)) \in \Phi_\tau^{y_{\text{ref}}}, \tau \in [0, t] \end{array} \right\}$$



Main result



Theorem (Liberzon & T. 2013)

- Feasibility (F1)-(F9) \Rightarrow *bang-bang funnel controller works*
- Almost arbitrary \mathcal{F}_0 + *BIBO* zero dynamics + boundedness of y_{ref} \Rightarrow *Feasibility* holds with *sufficiently large* U^+ and U^-

Contents



- 1 Introduction
- 2 Relative degree one
- 3 Higher relative degree
 - Relative degree two
 - Generalization to arbitrary relative degree
 - Hierarchical switching logic
 - Feasibility assumptions
- 4 Simulation



Simulation for $r = 4$

Example (academic), finite escape time for y possible:

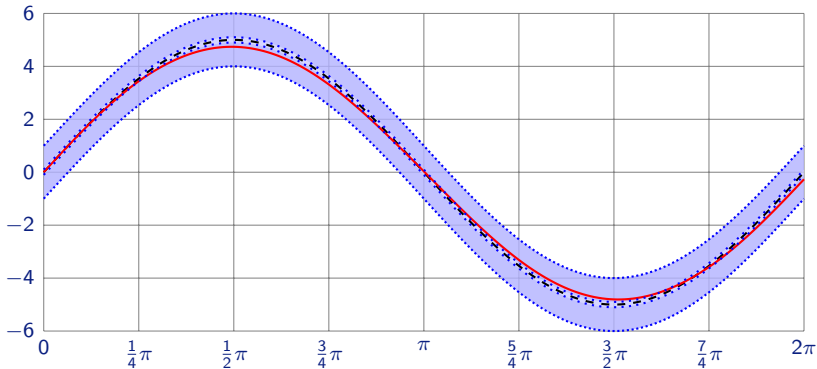
$$\begin{aligned}
 y^{(4)} &= z \ddot{y}^2 + e^z u, & y^{(i)}(0) &= y_{\text{ref}}^{(i)}(0), \quad i = 0, 1, 2, 3, \\
 \dot{z} &= z(a - z)(z + b) - cy, & z(0) &= 0, \\
 y_{\text{ref}}(t) &= 5 \sin(t)
 \end{aligned}$$

controller parameter (constant funnels):

$$\begin{array}{llll}
 \varphi_0^+ = -\varphi_0^- \equiv 1, & \varepsilon_0^+ = \varepsilon_0^- = 0.9, & & \Delta_0^+ = \Delta_0^- = \infty, \\
 \varphi_1^+ = -\varphi_1^- \equiv 0.5, & \varepsilon_1^+ = \varepsilon_1^- = 0.1, & \lambda_1^+ = \lambda_1^- = 0, & \Delta_1^+ = \Delta_1^- = \Delta_0^\pm / 2 = \infty, \\
 \varphi_2^+ = -\varphi_2^- \equiv 0.5, & \varepsilon_2^+ = \varepsilon_2^- = 0.1, & \lambda_2^+ = \lambda_2^- = 0.2, & \Delta_2^+ = \Delta_2^- = 0.4, \\
 \varphi_3^+ = -\varphi_3^- \equiv 4.5, & \varepsilon_3^+ = \varepsilon_3^- = 0.1, & \lambda_3^+ = \lambda_3^- = 4, & \Delta_3^+ = \Delta_3^- = 0.1, \\
 & & \lambda_4^+ = \lambda_4^- = 102, & \Delta_4^+ = \Delta_4^- = 0.0001.
 \end{array}$$

$$U^+ = -U^- = 254$$

Simulation results, reference tracking



Switching frequency: up to 1000 Hz
Total number of switchings: ca. 2200



Simulation results, error plots

