Nondecreasing Lyapunov functions

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Kaiserslautern

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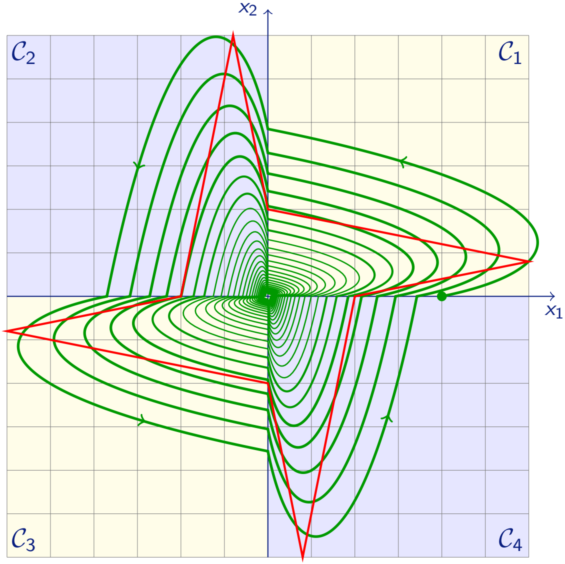
Motivating academic example



$$\dot{x} = A_{\sigma(x)}x$$

$$\sigma : \mathbb{R}^2 \backslash \{0\} \to \{1,2,3,4\}$$
$$\sigma^{-1}(i) = \mathcal{C}_i,$$

$$A_1 = A_3 = \begin{bmatrix} 1 & -5 \\ 0.2 & 1 \end{bmatrix}$$
$$A_2 = A_4 = \begin{bmatrix} 1 & -0.2 \\ 5 & 1 \end{bmatrix}$$

$$V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$$

Considered systems class

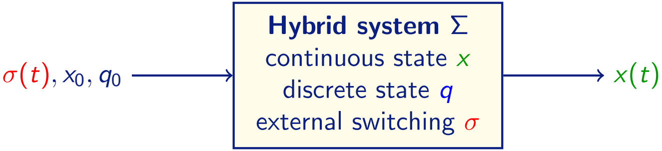
$$\left.\begin{aligned} \dot{x}(t) &= f_{q(t)}(x(t)), \quad \forall t \geq 0 \text{ with } q(t) = q(t^-), \\ x(t) &= g_{q(t^-),q(t)}(x(t^-)), \quad q(t) \neq q(t^-), \\ x(0^-) &= x_0 \in \mathbb{R}^n \\ q(t) &= h(q(t^-), x(t^-), \sigma(t)) \quad \forall t \geq 0, \\ q(0^-) &= q_0 \in \mathcal{Q}. \end{aligned}\right\} \qquad (\Sigma)$$

with its solution behavior

$$\mathcal{B} = \{\ x : [0,\infty) \to \mathbb{R}^n \ \mid \exists \text{ solution } (x, q, \sigma) \text{ of } \Sigma\ \}$$

- Includes time- as well as state-dependent switching
- Includes state-jumps
- Does *not consider* hybrid time domain

Considered systems class



$\sigma(t), x_0, q_0$

Hybrid system Σ

continuous state x

discrete state q

external switching σ

$x(t)$

with its solution behavior

$$\mathcal{B} = \{\ x : [0,\infty) \to \mathbb{R}^n\ \mid \exists \text{ solution } (x, q, \sigma) \text{ of } \Sigma\ \}$$

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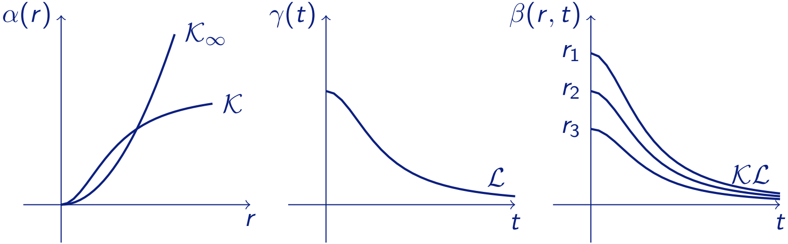
Stability and Lyapunov function definitions

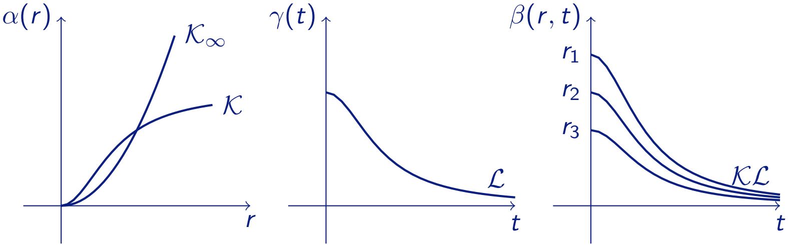


Definition (Asymptotic stability)

Σ is asymptotically stable $:\Leftrightarrow \exists\beta \in \mathcal{KL}\ \forall x(\cdot) \in \mathcal{B}\ \forall t \geq t_0$:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

Stability and Lyapunov function definitions



Definition (Lyapynov function)

$V : \mathbb{R}^n \to \mathbb{R}$ is called Lyapunov function $:\Leftrightarrow$

1. $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty : \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$
2. $\exists \beta \in \mathcal{KL}\ \forall x(\cdot) \in \mathcal{B} : \quad V\big(x(t)\big) \leq \beta\big(V(x(t_0)), t - t_0\big)$
3. $\forall v \geq 0 : \quad \beta(v, 0) = v$

Nondecreasing Lyapunov function: Motivation

$V : \mathbb{R}^n \to \mathbb{R}$ is called Lyapunov function $:\Leftrightarrow$

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Key observation

① + ② $\Rightarrow$ Asymptotic stability

$$\|x(t)\| \leq \alpha_1^{-1}\Big(\beta\big((\alpha_2(\|x(t)\|), t - t_0\big)\Big)$$

Nondecreasing Lyapunov function: Motivation

$V : \mathbb{R}^n \to \mathbb{R}$ is called Lyapunov function $:\Leftrightarrow$

① $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty : \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$

② $\exists \beta \in \mathcal{KL}\ \forall x(\cdot) \in \mathcal{B} : \quad V\big(x(t)\big) \leq \beta\big(V(x(t_0)), t - t_0\big)$

③ $\forall v \geq 0 : \quad \beta(v, 0) = v$

Key observation

① + ② $\Rightarrow$ Asymptotic stability

② + ③ $\Rightarrow$ V decreasing along solutions

$$V(x(t+\varepsilon)) \overset{②}{\leq} \beta(V(x(t)), \varepsilon) \overset{\mathcal{L}}{<} \beta(V(x(t)), 0) \overset{③}{=} V(x(t))$$

Nondecreasing Lyapunov function: Motivation

$V : \mathbb{R}^n \to \mathbb{R}$ is called Lyapunov function $:\Leftrightarrow$

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Key observation

❶ + ❷ $\Rightarrow$ Asymptotic stability

❷ + ❸ $\Rightarrow$ V decreasing along solutions

Remark

If V is a norm (in particular, convex) then ❶ is trivially fulfilled and ❷ is identical to asymptotic stability definition

Nondecreasing Lyapunov function: Motivation

nondecreasing

$V : \mathbb{R}^n \to \mathbb{R}$ is called Lyapunov function $:\Leftrightarrow$

① $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty : \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$

② $\exists \beta \in \mathcal{KL}\ \forall x(\cdot) \in \mathcal{B} : \quad V\big(x(t)\big) \leq \beta\big(V(x(t_0)), t - t_0\big)$

~~③ $\forall v \geq 0 : \quad \beta(v, 0) = v$~~

Key observation

① + ② $\Rightarrow$ Asymptotic stability

~~② + ③ $\Rightarrow$ V decreasing along solutions~~

Claim

Finding or constructing a **nondecreasing** Lyapunov function is easier.

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nondecreasing = not necessarily monotonically decreasing

A useful Lemma

Lemma

Assume $\widehat{V}:\mathbb{R}^n\to\mathbb{R}$ *satisfies*

1. $\exists\,\widehat{\alpha}_1,\widehat{\alpha}_2\in\mathcal{K}_\infty$:

$$\widehat{\alpha}_1(\|x\|)\leq\widehat{V}(x)\leq\widehat{\alpha}_2(\|x\|)\quad\forall x\in\mathbb{R}^n,$$

2. $\forall\,x(\cdot)\in\mathcal{B}\;\exists\,\widehat{x}(\cdot):\mathbb{R}\to\mathbb{R}^n\;\exists\,\widehat{\alpha}_3\in\mathcal{K}$:

$$\|x(t)\|\leq\widehat{\alpha}_3(\|\widehat{x}(t)\|)\quad\forall t\geq 0,$$

3. $\exists\,\widehat{\beta}\in\mathcal{KL}$:

$$\widehat{V}\big(\widehat{x}(t)\big)\leq\widehat{\beta}\big(\widehat{V}(\widehat{x}(t_0)),t-t_0\big)\quad\forall t\geq t_0.$$

Then $\widehat{V}$ *is a nondecreasing Lyapunov function.*

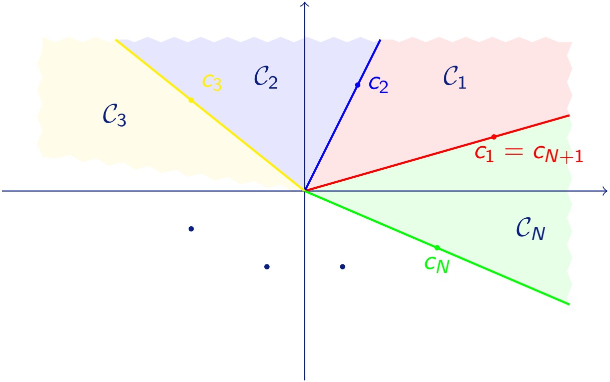
Contents

A generic example class

$$\boxed{\dot{x} = A_{\sigma(x)} x} \quad \text{in } \mathbb{R}^2$$

$\sigma : \mathbb{R}^2 \setminus \{0\} \to \{1, 2, \ldots, N\}$ with

① $\sigma^{-1}(i) = \mathcal{C}_i := \{\ \lambda c_i + \mu c_{i+1} \ \mid \lambda > 0, \mu \geq 0\ \}$

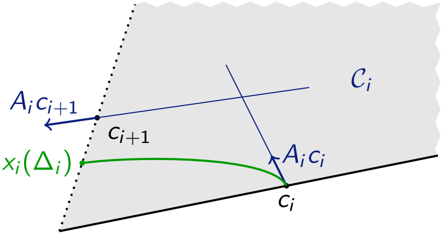


A generic example class

$$\boxed{\dot{x} = A_{\sigma(x)} x} \quad \text{in } \mathbb{R}^2$$

$\sigma : \mathbb{R}^2 \setminus \{0\} \to \{1, 2, \ldots, N\}$ with

1. $\sigma^{-1}(i) = \mathcal{C}_i := \{\ \lambda c_i + \mu c_{i+1}\ \mid \lambda > 0, \mu \geq 0\ \}$
2. Solution flows from left to right on boundaries of cones $\mathcal{C}_i$
3. Vectors with directions $A_i c_i$ and $-A_i c_{i+1}$ intersect in $\mathcal{C}_i$
4. Solution of $\dot{x}_i = A_i x_i$, $x_i(0) = c_i$, satisfies $\|x_i(\Delta_i)\| < \|c_{i+1}\|$



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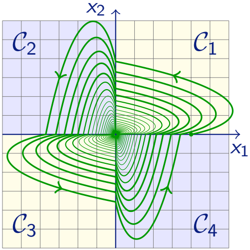
Lyapunov function difficult to find

Asymptotic stability clear, but construction of Lyapunov function difficult.

In fact, no piecewise quadratic Lyapunov function exists in general!

Nonexistence of piecewise quadratic Lyapunov function



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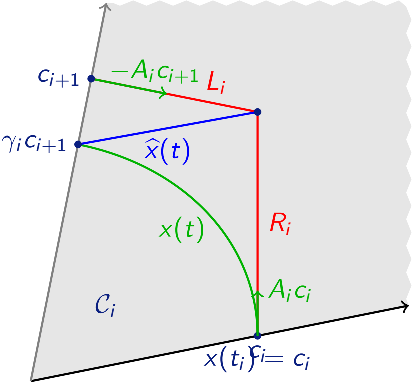
Lemma

For the above system there is no piecewise-quadratic Lyapunov function $V : \mathbb{R}^2 \to \mathbb{R}$ of the form $V(x) = x^\top P_i x$ for $x \in \mathcal{C}_i$.

Remark

A piecewise quadratic Lyapounov function can be constructed if one allows more "pieces". In fact, with the recent method of IERVOLINO, VASCA and IANNELLI (2014) 108 cones are sufficient.

Simple construction of nondecreasing Lyapunov function

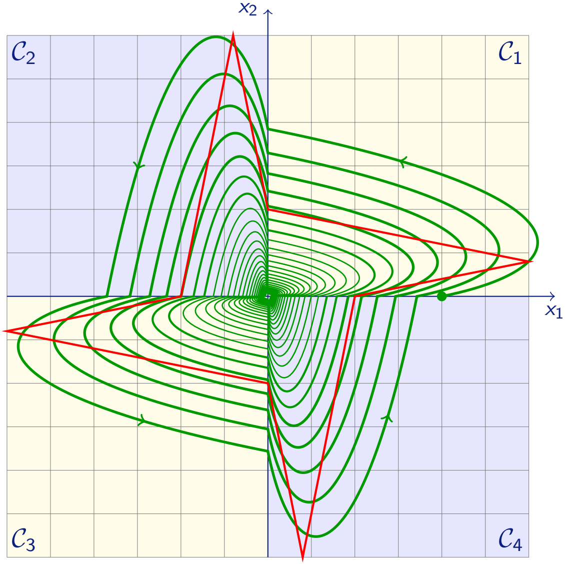


$\widehat{V} : \mathbb{R}^2 \to \mathbb{R}$ is the unique piecewise linear function with

$$\widehat{V}^{-1}(1) = R_1 \cup L_1 \cup R_2 \cup L_2 \dots R_N \cup L_N$$

Comparison function $\widehat{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ piecewise linear, not continuous

Summary



- Introduced and motivated the concept of nondecreasing Lyapunov function
- Applicable for large system class
- Presented explicit construction for generic example class