

# Basics on Differential-Algebraic Equations (DAEs)

Stephan Trenn

Technomathematics group, Dept. of Mathematics, University of Kaiserslautern

ICCAS 2014, Seoul, Korea  
October 23rd, 2014, Tutorial Session TA06

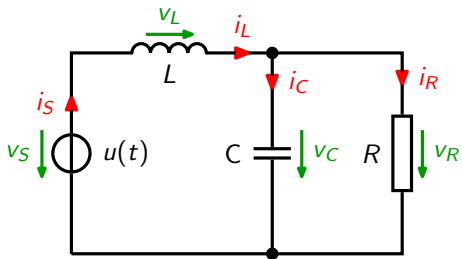


# Contents



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs
- 3 Special DAE-cases
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form
- 5 Wong sequences
- 6 Inconsistent initial values
  - Motivating example
  - Consistency projector
- 7 Switched DAEs
  - Definition and solution theory
  - Impulse-freeness
  - Stability

# Modeling of electrical circuits



## Basic circuit elements

- Resistor:**  $v_R(t) = R i_R(t)$
- Capacitor:**  $i_C(t) = C \frac{d}{dt} v_C(t)$
- Inductor:**  $v_L(t) = L \frac{d}{dt} i_L(t)$
- Voltage source:**  $v_S(t) = u(t)$

## DAEs

All components are given by a differential-algebraic equation (DAE)

$$E\dot{x} = Ax + Bu$$



# Contents



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs**
- 3 Special DAE-cases
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form
- 5 Wong sequences
- 6 Inconsistent initial values
  - Motivating example
  - Consistency projector
- 7 Switched DAEs
  - Definition and solution theory
  - Impulse-freeness
  - Stability

# Recall ODEs



Ordinary differential equations (ODEs):

$$\dot{x} = Ax + f$$

- Initial values: arbitrary
- Solution uniquely determined by  $f$  and  $x(0)$
- No inhomogeneity constraints

# Simple DAE example



DAE example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\begin{aligned} \dot{x}_2 = x_1 + f_1 & \longrightarrow x_1 = -f_1 - \dot{x}_2 \\ 0 = x_2 + f_2 & \longrightarrow x_2 = -f_2 \\ 0 = f_3 & \end{aligned}$$

no restriction on  $x_3$

# Conclusions from example



Solution of example:

$$x_1 = -f_1 - \dot{f}_2$$

$$x_2 = -f_2$$

$x_3$  free

$$f_3 = 0 \text{ necessary}$$

## Differences to ODEs

- For fixed inhomogeneity, **initial values cannot be chosen arbitrarily** ( $x_1(0) = -f_1(0) - \dot{f}_2(0)$ ,  $x_2(0) = f_2(0)$ )
- For fixed inhomogeneity, **solution not uniquely determined by initial value** ( $x_3$  free)
- **Inhomogeneity not arbitrary**
  - structural restrictions ( $f_3 = 0$ )
  - differentiability restrictions ( $\frac{d}{dt} f_2$  must be well defined)



# Contents



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs
- 3 Special DAE-cases**
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form
- 5 Wong sequences
- 6 Inconsistent initial values
  - Motivating example
  - Consistency projector
- 7 Switched DAEs
  - Definition and solution theory
  - Impulse-freeness
  - Stability

# Nilpotent DAEs



$$\begin{bmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} \dot{x} = x + f$$

$$\begin{aligned} \Leftrightarrow \quad 0 &= x_1 + f_1 &\longrightarrow & x_1 = -f_1 \\ \dot{x}_1 &= x_2 + f_2 &\longrightarrow & x_2 = -f_2 - \dot{f}_1 \\ \dot{x}_2 &= x_3 + f_3 &\longrightarrow & x_3 = -f_3 - \dot{f}_2 - \ddot{f}_1 \\ &\vdots && \vdots \\ \dot{x}_{n-1} &= x_n + f_n &\longrightarrow & x_n = -\sum_{i=1}^n f_i^{(n-i)} \end{aligned}$$

# General nilpotent DAE



In general:  $N\dot{x} = x + f$  with  $N$  nilpotent, i.e.  $N^n = 0$

$$\overset{N}{\Rightarrow} \frac{d}{dt} N^2 \ddot{x} = N\dot{x} + N\dot{f} = x + f + N\dot{f}$$

$$\overset{N}{\Rightarrow} \frac{d}{dt} N^3 \ddot{\ddot{x}} = N^2 \ddot{x} + N^2 \ddot{f} = x + f + N\dot{f} + N^2 \ddot{f}$$

⋮

$$\overset{N}{\Rightarrow} \underbrace{N^n x^{(n)}}_{=0} = x + \sum_{i=0}^{n-1} N^i f^{(i)} \quad \Rightarrow \quad \boxed{x = - \sum_{i=0}^{n-1} N^i f^{(i)}}$$

## Properties

- Initial values: **fixed** by inhomogeneity
- Solution uniquely determined by  $f$
- Inhomogeneity constraints:
  - no structural constraints
  - **differentiability constraints**:  $\sum_{i=0}^{n-1} N^i f^{(i)}$  needs to be well defined

# Underdetermined DAEs



$$\begin{aligned}
 & \begin{matrix} n-1 \\ n \end{matrix} \begin{bmatrix} 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \\ & & & & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} x + f \\
 \Leftrightarrow & \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \end{pmatrix} = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} + f
 \end{aligned}$$

⇔ ODE with additional "input"  $x_n$

## Properties

- Initial values: arbitrary
- Solution **not uniquely determined** by  $x(0)$  and  $f$
- Inhomogeneity constraints: none

# Overdetermined DAEs



$$\begin{aligned}
 n+1 \quad & \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ 1 & & \ddots & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & 1 & 0 & \\ & & & & & \color{red}{1} \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ 0 & & \ddots & & & \\ & & & 1 & & \\ & & & 0 & 1 & \\ & & & & & \color{red}{0} \end{bmatrix} x + f \\
 \Leftrightarrow & \underbrace{\begin{bmatrix} 0 & & & & & \\ 1 & & \ddots & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & 1 & 0 & \end{bmatrix}}_{=:N} \dot{x} = x + \begin{pmatrix} f_1 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} \wedge \dot{x}_n = f_{n+1} \\
 \Leftrightarrow & x = - \sum_{i=0}^{n-1} N^i f^{(i)} \wedge \dot{x}_n = - \underbrace{\sum_{i=1}^n f_i^{n-i+1}}_{\Leftrightarrow \sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0} \stackrel{!}{=} f_{n+1}
 \end{aligned}$$

# Overdetermined DAEs properties



$$x = - \sum_{i=0}^{n-1} N^i f^{(i)} \wedge \sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0$$

## Properties

- Initial value: **fixed** by inhomogeneity
- Solution uniquely determined by  $f$
- Inhomogeneity constraints
  - **structural constraint**:  $\sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0$
  - **differentiability constraint**:  $f_i^{(n+1-i)}$  needs to be well defined

## No other cases

All DAEs are combinations of

ODEs, nilpotent DAEs, underdetermined DAEs, overdetermined DAEs

# Contents



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs
- 3 Special DAE-cases
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form**
- 5 Wong sequences
- 6 Inconsistent initial values
  - Motivating example
  - Consistency projector
- 7 Switched DAEs
  - Definition and solution theory
  - Impulse-freeness
  - Stability

# Equivalence



## Fact 1

For any invertible matrix  $S \in \mathbb{R}^{m \times m}$ :

$$(x, u) \text{ solves } E\dot{x} = Ax + Bu \Leftrightarrow (x, u) \text{ solves } SE\dot{x} = SAx + SBu$$

## Fact 2

For coordinate transformation  $T \in \mathbb{R}^{n \times n}$ :

$$(x, u) \text{ solves } E\dot{x} = Ax + Bu \stackrel{x=Tx}{\Leftrightarrow} (z, u) \text{ solves } ET\dot{z} = ATz + Bu$$

## Together

$$(x, u) \text{ solves } E\dot{x} = Ax + Bu \stackrel{x=Tx}{\Leftrightarrow} (z, u) \text{ solves } SET\dot{z} = SATz + SBu$$

## Definition

$(E_1, A_1), (E_2, A_2)$  equivalent  $:\Leftrightarrow (E_2, A_2) = (SE_1 T, SA_1 T)$ , short:

$$(E_1, A_1) \cong (E_2, A_2)$$



# Quasi-Kronecker form (QKF)



## Theorem (Quasi-Kronecker Form)

For any  $E, A \in \mathbb{R}^{\ell \times m}$

$$(E, A) \cong \left( \left[ \begin{array}{c} \boxed{E_U} \\ \quad \boxed{I} \\ \quad \quad \boxed{N} \\ \quad \quad \quad \boxed{E_O} \end{array} \right], \left[ \begin{array}{c} \boxed{A_U} \\ \quad \boxed{J} \\ \quad \quad \boxed{I} \\ \quad \quad \quad \boxed{A_O} \end{array} \right] \right)$$

where  $(E_U, A_U)$  consists of *underdetermined* blocks on the diagonal,  $N$  is *nilpotent*, and  $(E_O, A_O)$  consists of *overdetermined* diagonal blocks

# QKF Examples



## Remark

$0 \times 1$  and  $1 \times 0$  underdetermined/overdetermined blocks are possible

$$\text{Example: } \left( \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \cong \left( \begin{pmatrix} \text{---} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix} \end{pmatrix}, \begin{pmatrix} \text{---} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix} \end{pmatrix} \right)$$

$$(E, A) \text{ from circuit} \cong \left( \begin{pmatrix} \begin{bmatrix} I_{2 \times 2} \\ \text{---} \\ \text{---} \end{bmatrix} \\ \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \end{pmatrix}, \begin{pmatrix} \begin{bmatrix} 0 & 1/c \\ -1/L^{-1}/RC \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \\ \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \end{pmatrix} \right)$$

# Regularity



$$(E, A) \cong \left( \begin{bmatrix} \boxed{\cancel{E_0}} & & & \\ & \boxed{I} & & \\ & & \boxed{N} & \\ & & & \boxed{\cancel{E_0}} \end{bmatrix}, \begin{bmatrix} \boxed{\cancel{A_0}} & & & \\ & \boxed{J} & & \\ & & \boxed{I} & \\ & & & \boxed{\cancel{A_0}} \end{bmatrix} \right)$$

## Corollary (Quasi-Weierstrass-Form (QWF))

$E\dot{x} = Ax + f$  has solution  $x$  for any sufficiently smooth  $f$  and each solution  $x$  is uniquely determined by  $x(0)$  and  $f$

$$\Leftrightarrow (E, A) \cong \left( \begin{bmatrix} \boxed{I} & \boxed{0} \\ \boxed{0} & \boxed{N} \end{bmatrix}, \begin{bmatrix} \boxed{J} & \boxed{0} \\ \boxed{0} & \boxed{I} \end{bmatrix} \right)$$

quasi-Weierstrass form

$(E, A)$  is then called **regular** ( $\Leftrightarrow \det(sE - A)$  not the zero polynomial).

# Contents



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs
- 3 Special DAE-cases
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form
- 5 Wong sequences**
- 6 Inconsistent initial values
  - Motivating example
  - Consistency projector
- 7 Switched DAEs
  - Definition and solution theory
  - Impulse-freeness
  - Stability

# Definition of Wong sequences



## Definition

Let  $E, A \in \mathbb{R}^{m \times n}$ . The corresponding Wong sequences of the pair  $(E, A)$  are:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), & i &= 0, 1, 2, 3, \dots \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{j+1} &:= E^{-1}A(\mathcal{W}_j), & j &= 0, 1, 2, 3, \dots \end{aligned}$$

Note:  $M^{-1}\mathcal{S} := \{ x \mid Mx \in \mathcal{S} \}$  and  $M\mathcal{S} := \{ Mx \mid x \in \mathcal{S} \}$

Clearly,  $\exists i^*, j^* \in \mathbb{N}$

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \dots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \dots$$

$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \dots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \dots$$

Wong limits:

$$\mathcal{V}^* := \bigcap_{i \in \mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}$$

$$\mathcal{W}^* = \bigcup_{i \in \mathbb{N}} \mathcal{W}_i = \mathcal{W}_{j^*}$$

# Wong sequences and the QWF



## Theorem

The following statements are equivalent for square  $E, A \in \mathbb{R}^{n \times n}$ :

- (i)  $(E, A)$  is regular
- (ii)  $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$
- (iii)  $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$

In particular, with  $\text{im } V = \mathcal{V}^*$ ,  $\text{im } W = \mathcal{W}^*$

$(E, A)$  regular  $\Rightarrow T := [V, W]$  and  $S := [EV, AW]^{-1}$  invertible

and  $S, T$  yield QWF:

$$(SET, SAT) = \left( \begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

# Calculation of Wong sequences



## Remark

Wong sequences can easily be calculated with Matlab even when the matrices still contain symbolic entries (like “R”, “L”, “C”).

```
function V=getPreImage(A,S)
% returns a basis of the preimage of A of the linear space spanned by
% the columns of S, i.e.  $\text{im } V = \{ x \mid Ax \in \text{im } S \}$ 

[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
else
    error('Both matrices must have same number of rows');
end;
```

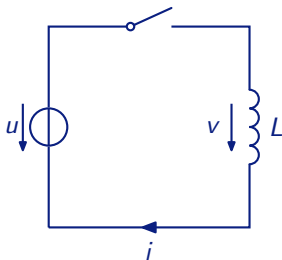
# Contents



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs
- 3 Special DAE-cases
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form
- 5 Wong sequences
- 6 Inconsistent initial values**
  - Motivating example
  - Consistency projector
- 7 Switched DAEs
  - Definition and solution theory
  - Impulse-freeness
  - Stability



# Circuit example



open switch:  $0 = i,$   
 inductivity law:  $L \frac{d}{dt} i = v$

## Nilpotent DAE model

$$\begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix} \dot{x} = x, \quad x = \begin{pmatrix} i \\ v \end{pmatrix}$$

⇒ unique solution  $x(t) = 0 \forall t$  for which switch is **open**

Now assume switch was closed for  $t < 0$

⇒ Different DAE-model for  $t < 0$

⇒ **Inconsistent initial values** for above DAE

# Solution of circuit example



$$t < 0$$

$$v = u$$

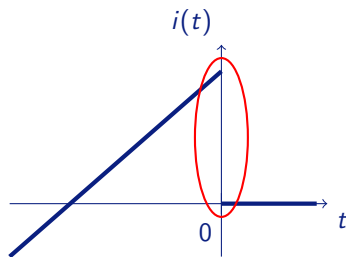
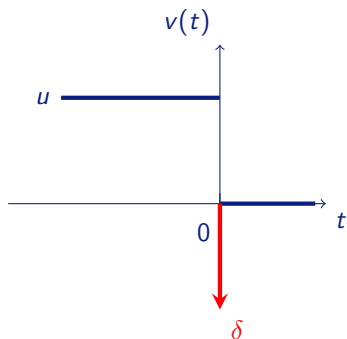
$$L \frac{d}{dt} i = v$$

$$t \geq 0$$

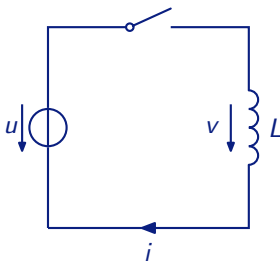
$$i = 0$$

$$v = L \frac{d}{dt} i$$

Solution (assume constant input  $u$ ):



# Observations



## Observations

- $x(0^-) \neq 0$  inconsistent for  $\begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix} \dot{x} = x$
- unique jump from  $x(0^-)$  to  $x(0^+)$
- derivative of jump = Dirac impulse appears in solution

# Initial trajectory problem



## Definition (Initial trajectory problem (ITP))

Given past trajectory  $x^0 : (-\infty, 0) \rightarrow \mathbb{R}^n$  find  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  such that

$$\left. \begin{aligned} x|_{(-\infty, 0)} &= x^0 \\ (E\dot{x})|_{[0, \infty)} &= (Ax + f)|_{[0, \infty)} \end{aligned} \right\} \quad (\text{ITP})$$

## “Theorem” (Unique jump rule)

Consider (ITP) with  $f = 0$  and regular  $(E, A)$  with QWF

$$(SET, SAT) = \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right).$$

Then any solution  $x$  of (ITP) satisfies

$$\boxed{x(0^+) = \Pi_{(E,A)} x(0^-)} \quad \text{where } \Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

is the consistency projector.

# Proof of unique jump rule “Theorem”



Let  $\begin{pmatrix} v \\ w \end{pmatrix} = T^{-1}x$ , then  $x$  solves (ITP) with  $f = 0 \iff \begin{pmatrix} v \\ w \end{pmatrix}$  solves

$$\begin{cases} v_{(-\infty, 0)} = v^0 \\ \dot{v}_{[0, \infty)} = (Jv)_{[0, \infty)} \end{cases}$$

and

$$\begin{cases} w_{(-\infty, 0)} = w^0 \\ (N\dot{w})_{[0, \infty)} = w_{[0, \infty)} \end{cases}$$

## ODE

$$v(t) = e^{Jt} v(0^-) \quad \forall t \geq 0$$

In particular,  $v(0^+) = v(0^-)$

## Nilpotent DAE

$$w(t) = 0 \quad \forall t > 0$$

In particular,  $w(0^+) = 0$

Altogether we have

$$\begin{pmatrix} v(0^+) \\ w(0^+) \end{pmatrix} = \begin{pmatrix} v(0^-) \\ 0 \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0^-) \\ w(0^-) \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}x(0^-)$$

hence

$$x(0^+) = T \begin{pmatrix} v(0^+) \\ w(0^+) \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}x(0^-) = \Pi_{(E,A)}x(0^-)$$

# Existence of solution



## Remarks

- a)  $\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$  does not depend on the specific choice of  $T$ .
- b) At this point we haven't actually shown that (ITP) has a solution!

## Theorem

Let  $(E, A)$  be regular. In the correct *distributional solution space* the ITP has a *unique solution* for all  $f$ .

In particular, jump and Dirac impulses at  $t = 0$  are uniquely determined.

## Attention

Choosing the right solution space is crucial and not immediately clear!

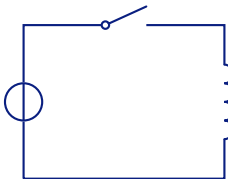
Here: Solution space = *piecewise-smooth distributions*  $\mathbb{D}_{\text{pw}C^\infty}$

# Contents



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs
- 3 Special DAE-cases
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form
- 5 Wong sequences
- 6 Inconsistent initial values
  - Motivating example
  - Consistency projector
- 7 Switched DAEs**
  - Definition and solution theory
  - Impulse-freeness
  - Stability

# Definition



Switch  $\rightarrow$  Different DAE models (=modes) depending on **time-varying** position of switch

## Definition (Switched DAE)

Switching signal  $\sigma : \mathbb{R} \rightarrow \{1, \dots, N\}$  picks mode at each time  $t \in \mathbb{R}$ :

$$\begin{aligned} E_{\sigma(t)} \dot{x}(t) &= A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\ y(t) &= C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t) \end{aligned} \quad (\text{swDAE})$$

## Attention

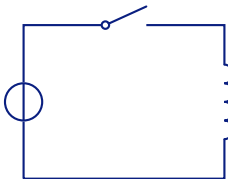
Each mode might have different consistency spaces

$\Rightarrow$  inconsistent initial values at each switch

$\Rightarrow$  **distributional solutions**, i.e.  $x \in \mathbb{D}_{\text{pw}}^n \mathcal{C}^\infty$ ,  $u \in \mathbb{D}_{\text{pw}}^m \mathcal{C}^\infty$ ,  $y \in \mathbb{D}_{\text{pw}}^p \mathcal{C}^\infty$



# Definition



Switch  $\rightarrow$  Different DAE models (=modes) depending on **time-varying** position of switch

## Definition (Switched DAE)

Switching signal  $\sigma : \mathbb{R} \rightarrow \{1, \dots, N\}$  picks mode at each time  $t \in \mathbb{R}$ :

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (\text{swDAE})$$

## Attention

Each mode might have different consistency spaces

$\Rightarrow$  inconsistent initial values at each switch

$\Rightarrow$  **distributional solutions**, i.e.  $x \in \mathbb{D}_{\text{pwc}}^n \mathcal{C}^\infty$ ,  $u \in \mathbb{D}_{\text{pwc}}^m \mathcal{C}^\infty$ ,  $y \in \mathbb{D}_{\text{pwc}}^p \mathcal{C}^\infty$

# Existence and uniqueness of solutions for (swDAE)



$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \tag{swDAE}$$

$$\Sigma_0 := \left\{ \sigma : \mathbb{R} \rightarrow \{1, \dots, N\} \left| \begin{array}{l} \sigma \text{ is piecewise constant and} \\ \sigma|_{(-\infty, 0)} \text{ is constant} \end{array} \right. \right\}.$$

## Corollary (from previous section)

Consider (swDAE) with regular  $(E_p, A_p) \forall p \in \{1, \dots, N\}$ . Then

$$\forall u \in \mathbb{D}_{\text{pwc}}^m \forall \sigma \in \Sigma_0 \exists \text{ solution } x \in \mathbb{D}_{\text{pwc}}^n$$

and  $x(0-)$  *uniquely* determines  $x$ .

# Sufficient conditions for impulse-freeness



## Question

When are **all solutions** of homogenous (swDAE)  $E_\sigma \dot{x} = A_\sigma x$  **impulse free**?

Note: Jumps are OK.

## Lemma (Sufficient conditions)

- $(E_p, A_p)$  all have **index one** (i.e.  $N_p = 0$  in QWF)  
 $\Rightarrow$  (swDAE) impulse free
- all **consistency spaces** of  $(E_p, A_p)$  **coincide** (i.e. Wong limits  $\mathcal{V}_p^*$  are identical)  
 $\Rightarrow$  (swDAE) impulse free

# Sketch of proof



- Index-1-case: Consider nilpotent DAE-ITP:

$$\begin{aligned} (N\dot{w})_{[0,\infty)} &= w_{[0,\infty)} \\ \Rightarrow 0 &= w_{[0,\infty)} \\ \Rightarrow w[0] := w_{[0,0]} &= 0 \end{aligned}$$

Hence an inconsistent initial value **does not induce Dirac-impulse**

- Same consistency space for all modes
  - $\Rightarrow$  no inconsistent initial values at switch
  - $\Rightarrow$  no jumps and no Dirac-impulses

# Characterization of impulse-freeness



## Theorem (Impulse-freeness)

The switched DAE  $E_\sigma \dot{x} = A_\sigma x$  is *impulse free*  $\forall \sigma \in \Sigma_0$

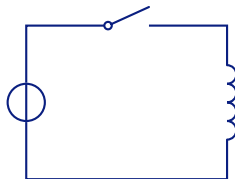
$$\Leftrightarrow E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \dots, N\}$$

where  $\Pi_p := \Pi_{(E_p, A_p)}$ ,  $p \in \{1, \dots, N\}$  is the consistency projector.

## Remark

- Index-1-case  $\Rightarrow E_q(I - \Pi_q) = 0 \quad \forall q$
- Consistency spaces equal  $\Rightarrow (I - \Pi_q)\Pi_p = 0 \quad \forall p, q$

# Circuit example



$$(E_1, A_1) = \left( \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \quad (E_2, A_2) = \left( \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

$$\Pi_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_1(I - \Pi_1)\Pi_2 = \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix} \neq 0 \Rightarrow \text{impulses possible}$$

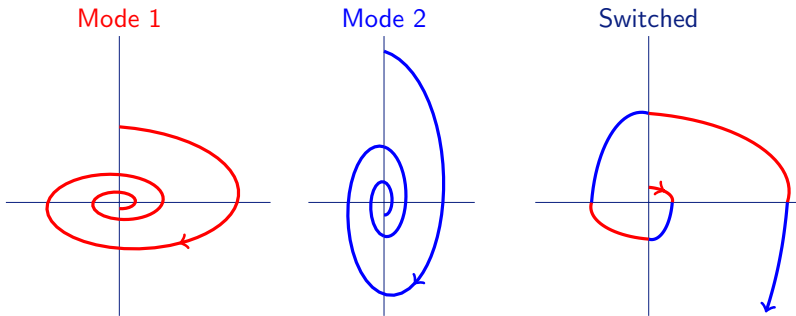
# Stability



## Question

All modes stable  $\stackrel{?}{\Rightarrow}$  Switched system stable?

Answer: **NO!** Already false for switched ODEs:

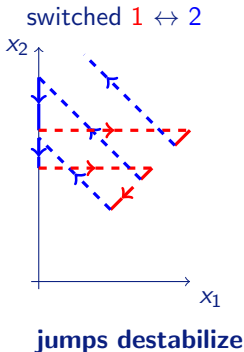
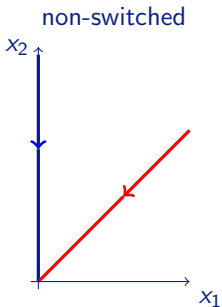


# Jumps and Stability: Example



$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$





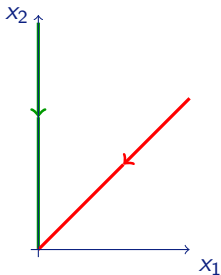
# Jumps and Stability: Example



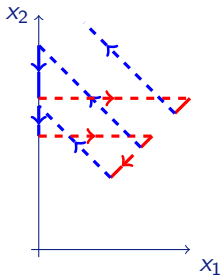
$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

non-switched

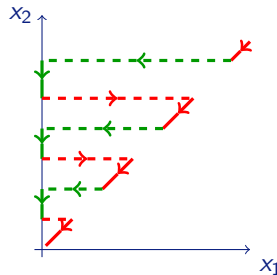


switched 1 ↔ 2



jumps destabilize

switched 1 ↔ 3






jumps do not destabilize

# Summary



- 1 Motivation: Modeling of electrical circuits
- 2 DAEs: Differences to ODEs
- 3 Special DAE-cases
  - Nilpotent DAEs
  - Underdetermined DAEs
  - Overdetermined DAEs
- 4 Equivalence and quasi-Kronecker form/quasi-Weierstrass form
- 5 Wong sequences
- 6 Inconsistent initial values
  - Motivating example
  - Consistency projector
- 7 Switched DAEs
  - Definition and solution theory
  - Impulse-freeness
  - Stability



-  **S. Trenn (2013): Solution concepts for linear DAEs: a survey.**  
Chapter 4 in: A. Ilchmann, T. Reis (eds.), *Surveys in Differential-Algebraic Equations I*, Springer Verlag.  
[doi:10.1007/978-3-642-34928-7\\_4](https://doi.org/10.1007/978-3-642-34928-7_4)
-  **S. Trenn (2012): Switched differential algebraic equations**  
Chapter 6 in: F. Vasca, L. Iannelli (eds.), *Dynamics and Control of Switched Electronic Systems*, Springer Verlag.  
[doi:10.1007/978-1-4471-2885-4\\_6](https://doi.org/10.1007/978-1-4471-2885-4_6)
-  **S. Trenn (2013): Stability of switched DAEs**  
Chapter 3 in: J. Daafouz, S. Tarbouriech, M. Sigalotti (eds.), *Hybrid Systems with Constraints*, Wiley.  
[doi:10.1002/9781118639856.ch3](https://doi.org/10.1002/9781118639856.ch3)

Preprints and slides available at [research.stephantrenn.de](http://research.stephantrenn.de)