Distributional averaging of switched DAEs with two modes

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Switched differential algebraic equations

Switched DAE:

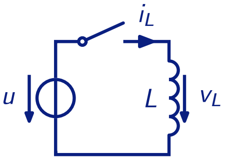
$$E_\sigma \dot{x} = A_\sigma x$$

Major differences to switched ODEs

Due to changing constraints, we see

- Induced state jumps
- Dirac impulses in the state variables

Circuit example:



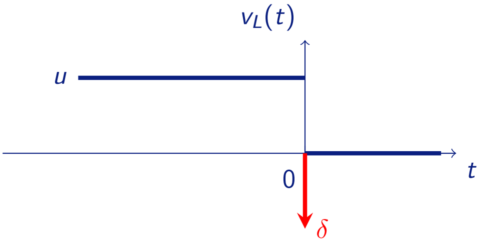
Mode 1 (switch closed):

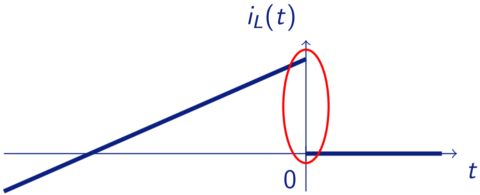
$$\begin{aligned} \tfrac{d}{dt} u &= 0 \\ L \tfrac{d}{dt} i_L &= v_L \\ 0 &= v_L - u \end{aligned}$$

Mode 2 (switch open):

$$\begin{aligned} \tfrac{d}{dt} u &= 0 \\ L \tfrac{d}{dt} i_L &= v_L \\ 0 &= i_L \end{aligned}$$

Solution of example (switch at $t = 0$ from mode 1 to mode 2):





Averaging: Basic idea

Application

- Fast switches occurs at
  - Pulse width modulation
  - „Sliding mode“-control
  - In general: fast digital controller
- Simplified analyses
  - Stability for sufficiently fast switching
  - In general: (approximate) desired behavior via suitable switching

Periodic switching signal

Switching signal

$\sigma : \mathbb{R} \to \{1, 2, \ldots, M\}$ has the following properties

- piecewise-constant and periodic with period $p > 0$
- duty cycles $d_1, d_2, \ldots, d_M \in [0, 1]$ with $d_1 + d_2 + \ldots + d_M = 1$

Desired approximation result

On any compact time interval it holds that

$$x_{\sigma,p} \to x_{\text{av}} \quad \text{as} \quad p \to 0$$

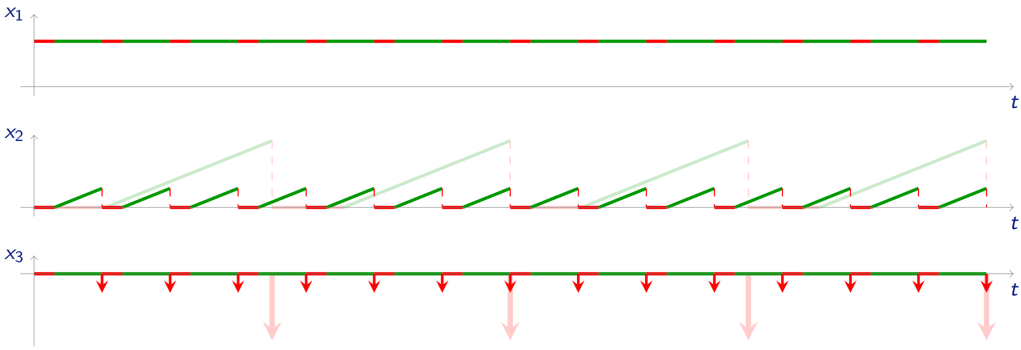
Averaging and Dirac impulses: Example

Mode 1

$$\dot{x}_1 = 0, \quad 0 = x_2, \quad \dot{x}_2 = x_3$$

Mode 2

$$\dot{x}_1 = 0, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = 0$$



Dirac impulses vanish?

Fact 1

Impulse-free part of solution converges
$\Rightarrow$ Jump heights converge to zero

Fact 2

Dirac impulse magnitude proportional to jump heights.

Hope

Dirac impulses don't play a role in the limit of averaging process.

WRONG!

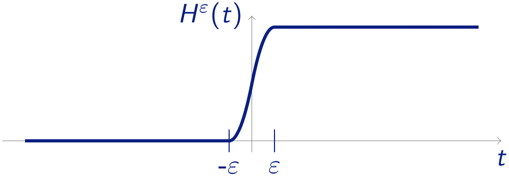
In the example we have: $x_3 = -\sum_{k=1}^{\infty} d_2 p x_1^0 \delta_{kp}$.

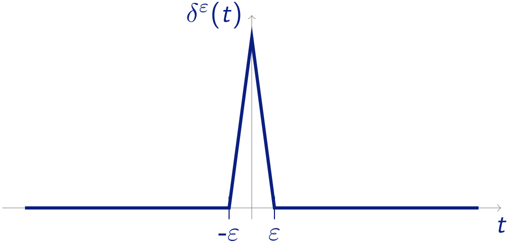
Accumulation of Dirac impulses

Magnitude of Dirac impulses are proportional to period p, BUT number of Dirac impulses is proportional to $1/p$

Relevance in reality?

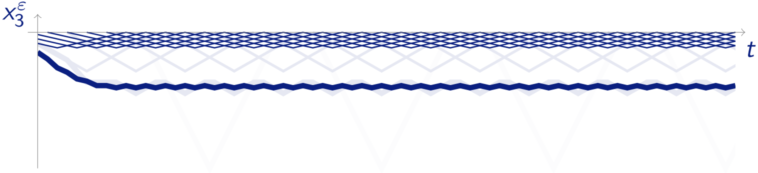
Consider a differentiable approximation H^ε of the Heaviside step function and its derivative δ^ε:





Approximation of x_3

$$x_3^\varepsilon = -\sum_{k=1}^{\infty} d_2 p x_1^0 \delta_{kp}^{\varepsilon}$$



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Distributions: Basic definitions

Test functions

$$\mathcal{C}_0^\infty := \left\{ \varphi : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{l}\varphi \text{ is smooth with} \\ \text{compact support}\end{array} \right\}$$

Distributions

$$\mathbb{D} := \left\{ D : \mathcal{C}^\infty \to \mathbb{R} \;\middle|\; \begin{array}{l} D \text{ is linear and} \\ \text{continuous}\end{array} \right\}$$

Lemma (Generalized functions)

For any locally integrable function $\alpha : \mathbb{R} \to \mathbb{R}$:

$$\alpha_{\mathbb{D}} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \int_{\mathbb{R}} \alpha\varphi \quad \in \mathbb{D}$$

Lemma (Dirac impulse)

For any $t_0 \in \mathbb{R}$ we have

$$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0) \quad \in \mathbb{D}$$

Definition (Piecewise-smooth distributions)

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ D = D^f + D[\cdot] \in \mathbb{D} \;\middle|\; \begin{array}{l} D_f = \alpha_{\mathbb{D}},\ \alpha \in \mathcal{C}^\infty_{\text{pw}}, \\ D[\cdot] = \sum_{t\in T} D_t,\ \ T \text{ is discrete},\ \ D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \ldots\} \end{array} \right\}$$

Convergence of distributions

Definition (Convergence of distributions)

$$D_n \to_{\mathbb{D}} D \text{ as } n \to \infty \quad :\Leftrightarrow \quad \forall \varphi \in \mathcal{C}_0^\infty: \ D_n(\varphi) \to_{\mathbb{R}} D(\varphi) \text{ as } n \to \infty$$

Recall example: $x_3 = -\sum_{k=1}^\infty d_2 p x_0^1 \delta_{kp}$, let $\varphi \in \mathcal{C}_0^\infty$ with $\operatorname{supp} \varphi \in [0, T]$ then

$$\begin{aligned} x_3(\varphi) &= -\sum_{k=1}^{\infty} d_2 p x_0^1 \delta_{kp}(\varphi) \\ &= -d_2 x_0^1 \sum_{k=1}^{\lfloor T/p \rfloor} p\varphi(kp) \\ &\to -d_2 x_0^1 \int_0^T \varphi \\ &= (-d_2 x_0^1)_{\mathbb{D}}(\varphi) \end{aligned}$$

Hence $x_3 \to_{\mathbb{D}} -d_2 x_0^1$

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Some DAE notation

Theorem (Quasi-Weierstrass form, WEIERSTRASS 1868)

(E, A) *regular* $:\Leftrightarrow \quad \det(sE - A) \not\equiv 0 \quad \Leftrightarrow \quad \exists S, T$ *invertible:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

Can easily obtained via Wong sequences (BERGER, ILCHMANN & T. 2012)

Definition (Consistency projector)

$$\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Definition (A^{diff} and E^{imp})

$$A^{\text{diff}} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad E^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}$$

Averaging result

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0 \tag{swDAE}$$

In the following we consider (swDAE) with two modes and switching period $p > 0$.

Theorem (Averaging result of impulse-free part, IANNELLI, PEDICINI, T. & VASCA 2013**)**

Consider (swDAE) with regular matrix pairs (E_1, A_1) and (E_2, A_2). Assume

$$\Pi_1 \Pi_2 = \Pi_2 \Pi_1 =: \Pi_\cap$$

and let the averaged system be given as

$$\dot{x}_{av} = \Pi_\cap A_{av}^{\text{diff}} \Pi_\cap x_{av}, \quad , x_{av}(0) = \Pi_\cap x_0$$

where $A_{av}^{\text{diff}} = d_1 A_1^{\text{diff}} + d_2 A_2^{\text{diff}}$. Then

$$x - x[\cdot] \to x_{av} \quad \text{uniformly on any compact interval as } p \to 0.$$

Averaging result

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0 \qquad \text{(swDAE)}$$

In the following we consider (swDAE) with two modes and switching period $p > 0$.

Theorem (Distributional averaging)

Consider (swDAE) with regular matrix pairs (E_1, A_1) and (E_2, A_2). Assume

$$\Pi_1 \Pi_2 = \Pi_2 \Pi_1 =: \Pi_\cap$$

and let the averaged system be given as

$$\dot{x}_{\text{av}} = \Pi_\cap A_{\text{av}}^{\text{diff}} \Pi_\cap x_{\text{av}}, \quad , x_{\text{av}}(0) = \Pi_\cap x_0$$

where $A_{\text{av}}^{\text{diff}} = d_1 A_1^{\text{diff}} + d_2 A_2^{\text{diff}}$. Then

$$x \to_{\mathbb{D}} (I - E_{\text{av}}^{\text{imp}}) x_{\text{av}\mathbb{D}} \quad \text{on any compact interval as } p \to 0,$$

where $E_{\text{av}}^{\text{imp}} := \sum_{i=0}^{n-2} (d_1 (E_2^{\text{imp}})^{i+1} A_1^{\text{diff}} + d_2 (E_1^{\text{imp}})^{i+1} A_2^{\text{diff}})(A_{\text{av}}^{\text{diff}})^i$.

Summary

$$\boxed{E_\sigma \dot{x} = A_\sigma x} \qquad \boxed{\dot{x}_{av} = A_{av} x_{av}} \qquad \boxed{x \to_{\mathbb{D}} (I - E_{av}^{imp}) x_{av}}$$

- First result on averaging for distributional solutions
- Dirac impulses vanish in the limit but cannot be neglected!
  - Convergence towards a smooth trajectory (without jumps and Dirac impulses)
  - Difference from impulse-free limit
- Practical relevance illustrated by considering approximations of Dirac impulses
- Future challenges:
  - Generalization to more than two modes (not trivial!)
  - Weakening of commutativity assumption of consistency projectors
  - Consideration of inhomogeneous switched DAEs