Averaging for non-homogeneous switched DAEs

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Contents

Averaging: Basic idea

$\sigma \rightarrow$ switched system $\quad \approx \quad$ non-switched averaged system

fast switching

Application

- Fast switches occurs at
  - Modulations (pulse width, amplitude, frequency)
  - „Sliding mode"-control
  - In general: fast digital controller
- Simplified analyses
  - Stability for sufficiently fast switching
  - In general: (approximate) desired behavior via suitable switching

Periodic switching signal

Switching signal

$\sigma : \mathbb{R} \to \{1, 2, \ldots, M\}$ has the following properties

- piecewise-constant and periodic with period $p > 0$
- duty cycles $d_1, d_2, \ldots, d_M \in [0, 1]$ with $d_1 + d_2 + \ldots + d_M = 1$

$\sigma, p \longrightarrow$ switched system $x_{\sigma,p}$ $\quad\approx\quad$ fast switching $\quad$ non-switched averaged system x_{av}

Desired approximation result

On any compact time interval it holds that

$$\|x_{\sigma,p} - x_{\mathrm{av}}\|_\infty = O(p)$$

Known results

$$\dot{x} = A_\sigma x + B_\sigma u, \quad x(0) = x_0$$

with averaged system

$$\dot{x}_{av} = A_{av} x_{av} + B_{av} u, \quad x_{av}(0) = x_0$$

where $A_{av} = \sum_{i=1}^M d_i A_i$ and $B_{av} = \sum_{i=1}^M d_i B_i$.

No further conditions required!

References

- Homogeneous case:
  Brocket & Wood 1974
- Inhomogenous case:
  Ezzine & Haddad 1989
- Numerous generalizations ...

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0$$

with average system

$$\dot{x}_{av} = \Pi_\cap A_{av}^{diff} \Pi_\cap x_{av}, \quad x_{av}(0^-) = \Pi_\cap x_0$$

where $A_{av}^{diff} = \sum_{i=1}^M d_i A_i^{diff}$.

Not always working! Additional assumptions needed on so called consistency projectors.

References

- Two modes:
  Iannelli, Pedicini, T. & Vasca 2013 ECC
- Arbitrarily many modes:
  Iannelli, Pedicini, T. & Vasca 2013 CDC

Switched DAEs with inhomogenity

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u$$

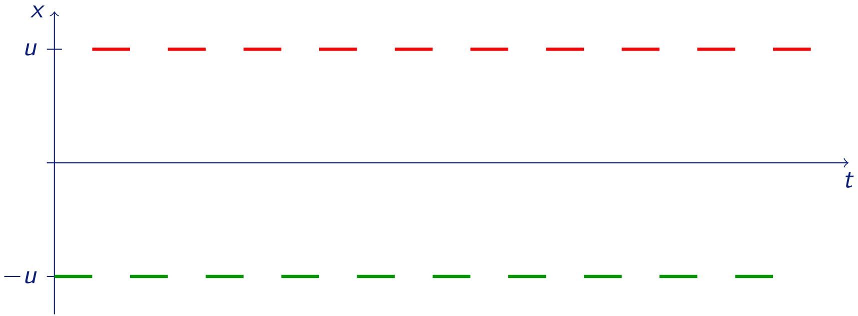
Canonical question

Averaging for $B_\sigma = 0 \quad \overset{?}{\Rightarrow} \quad$ Averaging for $B_\sigma \neq 0$

Trivial Counter Example

$$(E_1, A_1, B_1) = (0, 1, 1)$$
$$(E_2, A_2, B_2) = (0, 1, -1)$$

Solution of example with duty cycles $d_1 = d_2 = 0.5$:



Contents

Non-switched DAEs: Basic definitions

Theorem (Quasi-Weierstrass form, WEIERSTRASS 1868)

(E, A) *regular* $:\Leftrightarrow$ $\det(sE - A) \not\equiv 0$ $\Leftrightarrow$ $\exists S, T$ *invertible:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

Can easily obtained via Wong sequences (BERGER, ILCHMANN & T. 2012)

Definition (Consistency projector)

$$\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Definition (Differential and impulse projector)

$$\Pi^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \qquad \Pi^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$$

Explicit solution formula for DAEs

For $E\dot{x} = Ax + Bu$ with regular (E, A) let

$$A^{\text{diff}} := \Pi^{\text{diff}} A, \quad B^{\text{diff}} := \Pi^{\text{diff}} B, \quad E^{\text{imp}} := \Pi^{\text{imp}} E, \quad B^{\text{imp}} := \Pi^{\text{imp}} B.$$

Theorem (Explicit DAE solution formula, T. 2012)

Every solution x of $E\dot{x} = Ax + Bu$ with regular (E, A) is given by

$$x(t) = e^{A^{\text{diff}} t} \Pi x_0^- + \int_0^t e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s) \, \mathrm{d}s - \sum_{\ell=0}^{n-1} (E^{\text{imp}})^\ell B^{\text{imp}} u^{(\ell)}(t), \quad x_0^- \in \mathbb{R}^n$$

Corollary ($B^{\text{imp}} = 0$ case)

If $B^{\text{imp}} = 0$, then x solves $E\dot{x} = Ax + Bu$ if, and only if, x solves

$$\dot{x} = A^{\text{diff}} x + B^{\text{diff}} u, \quad x(0) = \Pi x_0^-, \quad x_0^- \in \mathbb{R}^n$$

Solution behavior of switched DAEs

Consider the switched DAE $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with regular matrix pairs (E_i, A_i).

Distributional solutions

Existence and uniqueness of solutions is guaranteed, however

- only within a distributional solution framework
- in particular, Dirac impulses may occur in x

Here we are only interested in the impulse-free part $x - x[\cdot]$ of the (distributional) solution x.
The effects of Dirac impulses for averaging are discussed this evening 17:40 here.

Theorem (Switched DAEs and switched ODEs with jumps)

Assume $B_i^{\text{imp}} = 0$. Then x solves switched DAE $\Leftrightarrow$ $x - x[\cdot]$ solves

$$\begin{aligned} \dot{x}(t) &= A^{\text{diff}}_{\sigma(t)} x(t) + B^{\text{diff}}_{\sigma(t)} u(t), && \forall t \notin \{\, t_k \mid t_k \text{ is } k\text{-th switching time of } \sigma \,\} \\ x(t_k^+) &= \Pi_{\sigma(t_k^+)} x(t_k^-), && k = 0, 1, 2, \ldots, \end{aligned}$$

i.e.

$$\boxed{x \text{ solves switched DAE} \quad \Leftrightarrow \quad x - x[\cdot] \text{ solves switched ODE with jumps}}$$

Contents

Known averaging result

Theorem (Homogeneous case, Ianelli, Pedicini, T. & Vasca 2013)

Consider homogeneous switched DAE $E_\sigma \dot{x} = A_\sigma x$ with regular matrix pairs (E_i, A_i).
If $\Pi_i \Pi_j = \Pi_j \Pi_i$ then the averaged system is given by

$$\dot{x}_{\text{av}} = \Pi_\cap A_{\text{av}}^{\text{diff}} \Pi_\cap x_{\text{av}}, \quad x_{\text{av}}(0) = \Pi_\cap x_0^-$$

where

$$\Pi_\cap = \Pi_M \Pi_{M-1} \cdots \Pi_1, \quad A_{\text{av}}^{\text{diff}} := d_1 A_1^{\text{diff}} + d_2 A_2^{\text{diff}} + \cdots + d_M A_M^{\text{diff}},$$

i.e. on every compact interval contained in $(0, \infty)$ we have

$$\|x_{\sigma,p} - x_{\text{av}}\|_\infty = O(p).$$

Condition on consistency projector can be relaxed (Mostacciuolo, T. & Vasca 2016) to the assumption that $\forall i \in \{1, 2, \ldots, M\}$

$$\operatorname{im} \Pi_\cap \subseteq \operatorname{im} \Pi_i, \qquad \ker \Pi_\cap \supseteq \ker \Pi_i$$

Main result

We have seen that $B_i^{\text{imp}} = 0$ is necessary for the relationship

$$x \text{ solves switched DAE} \quad \Leftrightarrow \quad x - x[\cdot] \text{ solves switched ODE with jumps}$$

It is also sufficient to ensure averaging:

Theorem (Averaging for inhomogeneous switched DAEs)

Consider switched DAE $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with regular (E_i, A_i), p-periodic switching signal σ and Lipschitz continuous u. Assume furthermore

- $B_i^{\text{imp}} = 0 \quad \forall i \in \{1, \ldots, M\}$,
- $\Pi_i \Pi_j = \Pi_j \Pi_i \quad \forall i, j \in \{1, \ldots, M\}$.

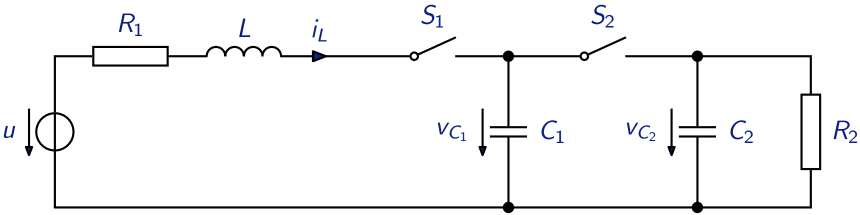
Then the average system is given by

$$\dot{x}_{\text{av}} = \Pi_\cap A_{\text{av}}^{\text{diff}} \Pi_\cap x_{\text{av}} + \Pi_\cap B_{\text{av}}^{\text{diff}} u, \quad x_{\text{av}}(0) = \Pi_\cap x_0^-$$

where $\Pi_\cap = \Pi_M \Pi_{M-1} \cdots \Pi_1$, $A_{\text{av}}^{\text{diff}} := d_1 A_1^{\text{diff}} + \ldots + d_M A_M^{\text{diff}}$ and $B_{\text{av}}^{\text{diff}} := d_1 B_1^{\text{diff}} + \ldots + d_M B_M^{\text{diff}}$, i.e. on every compact set contained in $(0, \infty)$ we have

$$\|x_{\sigma,p} - x_{\text{av}}\|_\infty = O(p).$$

Illustrative example



With $x = (v_{C_1},\ v_{C_2},\ i_L)^\top$ we have the following four DAE descriptions:

S_1 closed, S_2 open

$$E_1 = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{R_2} & 0 \\ 1 & 0 & -R_1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

S_1 closed, S_2 closed

$$E_2 = \begin{bmatrix} C_1 & C_2 & 0 \\ 0 & 0 & L \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & -\frac{1}{R_2} & 1 \\ -1 & 0 & -R_1 \\ 1 & -1 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

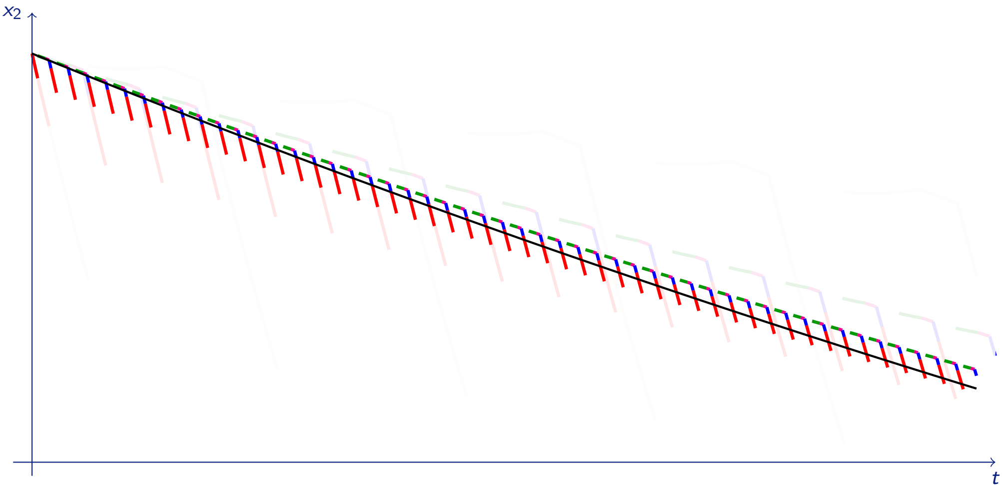
S_1 open, S_2 closed

$$E_3 = \begin{bmatrix} C_1 & C_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & -\frac{1}{R_2} & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

S_1 open, S_2 open

$$E_4 = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{R_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Simulation



Summary

- Considered averaging for switched DAEs

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \quad \to \quad \dot{x}_{\mathrm{av}} = A_{\mathrm{av}} x_{\mathrm{av}} + B_{\mathrm{av}} u, \; x_{\mathrm{av}} = \Pi_\cap x_0^-$$

- Key challenges:
  - Jumps in the solutions
  - Dirac impulses (not considered here)
- Key assumptions:
  - Commutativity of consistency projectors
  - Input doesn't effect algebraic constraints ($B_i^{\mathrm{imp}} = 0$)
- Possible extensions:
  - Role of Dirac impulses $\to$ Talk this evening 17:40
  - $B_i^{\mathrm{imp}} \neq 0$
  - Relax assumptions on projectors
  - Partial averaging
  - Nonperiodic switching signals
  - Stability analysis
  - Nonlinear case