Stabilization of switched DAEs via fast switching

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Control task

Goal: Stabilization

Find σ such that $x(t) \to 0$ as $t \to \infty$.

Usual approach

State-depending switching $x \mapsto \sigma(x)$

Problem

State x may not be available for feedback control
$\to$ observer with estimation $\widehat{x}$
$\to$ non-matching switching signals $\sigma(x) \neq \sigma(\widehat{x})$, NO seperation principle

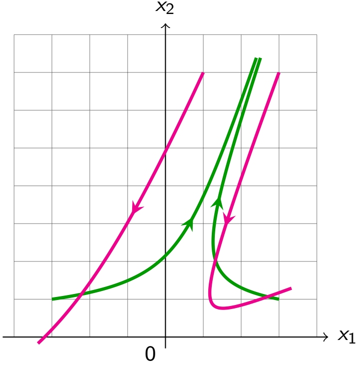
Alternative approach

Time-dependent switching $t \mapsto \sigma(t)$

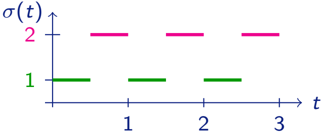
Example: Stabilization of switched ODEs

$$\dot{x} = A_\sigma x, \quad A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

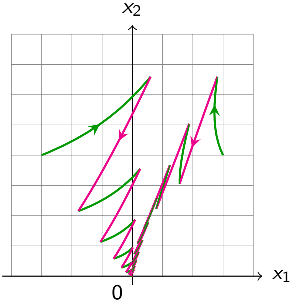
Unstable modes



Periodic switching signal:



$\Rightarrow$ Stability:



Why does the example work?

Convex combination

$$\tfrac{1}{2}A_1 + \tfrac{1}{2}A_2 = \tfrac{1}{2}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{Hurwitz!}$$

Classical averaging result

For switched ODE

$$\dot{x} = A_\sigma x, \quad \sigma(t) \in \{1, 2, \ldots, P\}$$

any convex combination

$$\dot{x} = A_{av} x, \quad A_{av} := \sum_{k=1}^{P} d_k A_k, \quad d_1, d_2, \ldots, d_P \in [0,1], \ \sum_{k=1}^{P} d_k = 1,$$

can be approximated arbitrarily well by sufficiently fast switching.

Corollary

$\exists$ *Hurwitz convex combination* $\Rightarrow$ *Stabilizable by fast (time-dependent) switching*

Switched DAEs

Switched linear DAE (differential algebraic equation)

(swDAE) $\boxed{E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)}$ or short $E_\sigma \dot{x} = A_\sigma x$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, P\}$
  - piecewise constant, right-continuous
  - locally finitely many jumps
- matrix pairs $(E_1, A_1), \ldots, (E_P, A_P)$
  - $E_k, A_k \in \mathbb{R}^{n\times n}$, $k = 1, \ldots, P$
  - (E_k, A_k) regular, i.e. $\det(sE_k - A_k) \not\equiv 0$

Main motivation

Modeling of electrical circuits

Special features

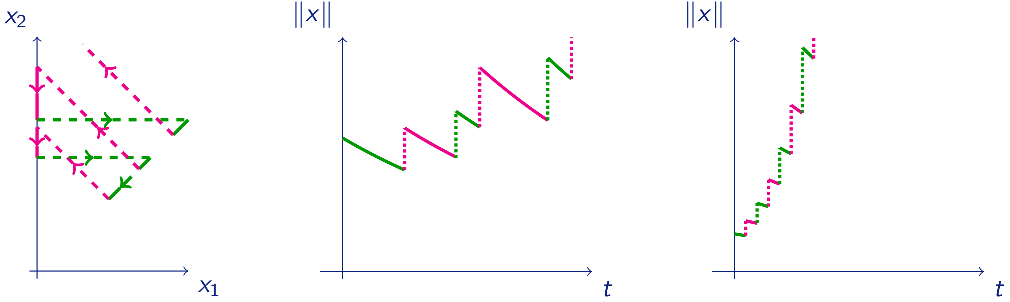
- Changing algebraic constraints
- Induced jumps
  $\to$ consistency projectors Π_p
- Dirac impulses possible

Example: Unbounded growth rate due to fast switching

Example: $E_\sigma \dot{x} = A_\sigma x$ with

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right), \qquad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

$$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$



For fast periodic switching: $x(t) \approx (\Pi_2 \Pi_1)^k x^0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}^k x^0 = \begin{bmatrix} 0 & 0 \\ 0 & 2^k \end{bmatrix} x^0, \quad k := \lfloor \frac{t}{p} \rfloor$

Some DAE notation

Theorem (Quasi-Weierstrass form, Weierstrass 1868)

(E,A) *regular* $:\Leftrightarrow \quad \det(sE-A)\neq 0 \quad \Leftrightarrow \quad \exists S,T$ *invertible:*

$$(SET,SAT)=\left(\begin{bmatrix} I & 0\\ 0 & N\end{bmatrix},\begin{bmatrix} J & 0\\ 0 & I\end{bmatrix}\right),\quad N \textit{ nilpotent}$$

Can easily obtained via Wong sequences (Berger, Ilchmann & T. 2012)

Definition (Consistency projector & Flow matrix)

$$\Pi := T\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}T^{-1} \qquad A^{\text{diff}} := T\begin{bmatrix} J & 0\\ 0 & 0\end{bmatrix}T^{-1}$$

Corollary (Explicit solution formula)

Solution of $E\dot{x}=Ax$ *on* $(0,\infty)$ *is given by:* $\boxed{x(t)=e^{A^{\text{diff}}t}\Pi x(0^-)}$

The Mironchenko-Wirth-Wulff Approach

Key observation

$$e^{A^{\text{diff}}t}\Pi \approx e^{A^\varepsilon t} \quad \text{with } A^\varepsilon := T\begin{bmatrix} J & 0 \\ 0 & -\frac{1}{\varepsilon}I \end{bmatrix} T^{-1}$$

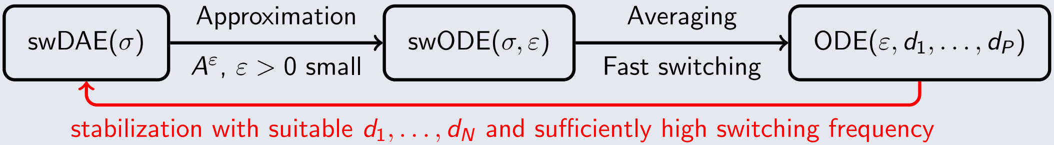
Hence

$$E_\sigma \dot{x} = A_\sigma x \quad \approx \quad \dot{x} = A^\varepsilon_\sigma x$$

Theorem (Mironchenko, Wirth & Wulff 2013)

σ stabilizes $\dot{x} = A^\varepsilon_\sigma x$ $\quad \forall \varepsilon \in (0, \varepsilon_0)$ $\quad \Rightarrow \quad$ σ stabilizes $E_\sigma \dot{x} = A_\sigma x$

Overall stabilization strategy



Discussion of the MWW-approach

No further assumptions needed for individual approximations

swDAE(σ) → swODE(σ, ε) $x_\sigma(t^-) - x_\sigma^\varepsilon(t) \to 0$ as $\varepsilon \to 0$

swODE(σ, ε) → ODE($\varepsilon, d_1, \ldots, d_P$) $x_\sigma^\varepsilon(t) - x_{av}^\varepsilon(t) \to 0$ as $p \to 0$

Problem

For fixed $\varepsilon > 0$ it is possible that $x_\sigma(t^-) - x_\sigma^\varepsilon(t) \to \infty$ as $p \to 0$

Underlying problem

Consistency projectors not explicitly considered:

- Destabilizing effect for fast switching
- Non-existence of averaged model

Direct approach

Directly utilize averaging results for switched DAEs

$E_\sigma \dot{x} = A_\sigma x$ —[Averaging / Fast switching]→ $\text{ODE}(d_1, \ldots, d_P)$

Assumptions

- (E_k, A_k) regular with Π_k, A_k^{diff}
- $\sigma : \mathbb{R} \to \{1, 2, \ldots, P\}$ periodic with
  - period $p > 0$
  - duty cycles $d_1, \ldots, d_P \in (0, 1)$
- For $\Pi_\cap := \Pi_P \Pi_{P-1} \cdots \Pi_1$:

$$\begin{aligned} \forall k : \quad & \operatorname{im} \Pi_k \supseteq \operatorname{im} \Pi_\cap \\ \forall k : \quad & \ker \Pi_k \subseteq \ker \Pi_\cap \end{aligned} \qquad \text{(PA)}$$

Theorem (Mostacciuolo, T., Vasca 2016)

Averaged system:

$$\dot{x}_{av} = \Pi_\cap A_{av}^{\text{diff}} \Pi_\cap x, \quad x(0) = \Pi_\cap x_0$$

where $A_{av}^{\text{diff}} := \sum_{k=1}^{P} d_k A_k^{\text{diff}}$. *If (PA) then*

$$\| x_{\sigma,p} - x_{av} \|_\infty = O(p)$$

on every compact interval in $(0, \infty)$

Stabilization via fast switching

Corollary

Averaged system is exponentially stable for some $d_1, \ldots, d_P$
$\Rightarrow \exists p > 0$ *sufficiently small:* $E_\sigma \dot{x} = A_\sigma x$ *exponentially stable*

Key steps of proof:

1. More precise $O(p)$-bound for all $T > 0$
$$\|x_{\sigma,p}(T^-) - x_{av}(T)\| \leq C(T) \cdot \|x(0^-)\| \cdot p$$
2. Chose $T > 0$ such that
$$\|x_{av}(T)\| < \|x_{av}(T/2)\|$$
3. Chose $p > 0$ sufficiently small such that
$$x_{\sigma,p}(T^-) \approx x_{av}(T) \quad \text{and} \quad x_{\sigma,p}(T/2^-) \approx x_{av}(T/2)$$
so that we can conclude
$$\|x_{\sigma,p}(T)\| < \|x_{\sigma,p}(T/2)\|$$
4. Conclude exponential stability.

Summary

Task

Stabilize $E_\sigma \dot{x} = A_\sigma x$ via time-depending switching rule

MWW-Approach

- Indirect via approximation of (swDAE) by (swODE$_\varepsilon$)
- Need switching signal independent of ε
- Don't need averaged system

A. Mironchenko, F. Wirth and K. Wulff: Stabilization of switched linear differential-algebraic equations via time-dependent switching signals, **Proc. IEEE CDC 2013**.

A. Mironchenko, F. Wirth, and K. Wulff: Stabilization of switched linear differential algebraic equations and periodic switching, **IEEE Transactions Automatic Control 2015**.

Averaging approach

- Directly use the averaging approach
- Projector assumption to ensure existence of averaged sytem
- Find Hurwitz convex combination

E. Mostacciuolo, S. Trenn, F. Vasca: Averaging for Switched DAEs: Convergence, Partial Averaging and Stability, *submitted for publication*.