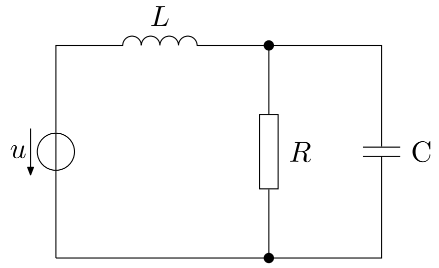
If you have any questions concerning this material (in particular, specific pointers to literature), please don't hesitate to contact me via email: trenn@mathematik.uni-kl.de

1 Solution Theory

1.1 Motivation: Modeling of electrical circuits



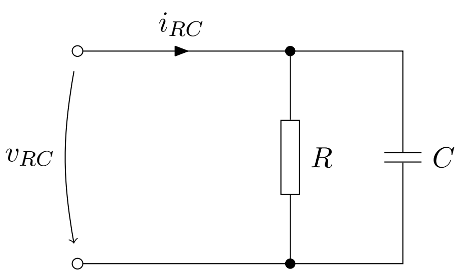
Basic components:

- Resistors: $v_R(t) = Ri_R(t)$
- Capacitor: $C\frac{d}{dt}v_C(t) = i_C(t)$
- Coil: $L\frac{d}{dt}i_L(t) = v_L(t)$
- Voltage source: $v_S(t) = u(t)$

All components have the same form:

$$\boxed{E\dot{x} = Ax + Bu} \quad E{,}A \in \mathbb{R}^{\ell\times n},\ B \in \mathbb{R}^{\ell\times m}$$

- Resistor: $x = \begin{pmatrix} v_R \\ i_R \end{pmatrix}$, $E = [0{,}0]$, $A = [-1{,}R]$, $B = []$
- Capacitor: $x = \begin{pmatrix} v_C \\ i_C \end{pmatrix}$, $E = [C{,}0]$, $A = [0{,}1]$, $B = []$
- Inductor: $x = \begin{pmatrix} v_C \\ i_C \end{pmatrix}$, $E = [0{,}L]$, $A = [1{,}0]$, $B = []$
- Voltage source $x = \begin{pmatrix} v_C \\ i_C \end{pmatrix}$, $E = [0{,}0]$, $A = [-1{,}0]$, $B = [1]$



Connecting components: Component equations remain unchanged!
+ Kirchhoffs laws:

$$v_{RC} = v_R, \quad i_{RC} = i_R + i_C, \quad v_R + v_C = 0$$

Results again in $E\dot{x} = Ax + Bu$ with $x = (v_R, i_R, v_C, i_C, v_{RC}, i_{RC})$ and

$$E = \begin{bmatrix} 0 & 0 & & & & \\ & & C & 0 & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & R & & & & \\ & & 0 & 1 & & \\ 1 & & & & -1 & \\ & -1 & & -1 & & 1 \\ 1 & & 1 & & & \end{bmatrix}$$

Altogether: $x = (v_R, i_R, v_C, i_C, v_L, i_L, v_S, i_S)$

$$E = \begin{bmatrix} 0 & 0 & & & & & & \\ & & C & 0 & & & & \\ & & & & 0 & L & & \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & R & & & & & & \\ & & 0 & 1 & & & & \\ & & & & 1 & 0 & & \\ & & & & & & 1 & 0 \\ 1 & & & & 1 & & -1 & \\ -1 & & 1 & & & & & \\ & & & & & -1 & & 1 \\ & 1 & & 1 & & 1 & & \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1.2 DAEs: What is different to ODEs

Example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\begin{aligned} \dot{x}_2 &= x_1 + f_1 \longrightarrow x_1 = -f_1 - \dot{f}_2 \\ 0 &= x_2 + f_2 \longrightarrow x_2 = -f_2 \nearrow \\ 0 &= f_3 \end{aligned}$$

no restriction on x_3

Observations:

- For fixed inhomogeneity, initial values cannot be chosen arbitrarily ($x_1(0) = -f_1(0) - \dot{f}_2(0)$, $x_2(0) = f_2(0)$)
- For fixed inhomogeneity, solution not uniquely determined by initial value (x_3 free)
- Inhomogeneity not arbitrary
  - structural restrictions ($f_3 = 0$)
  - differentiability restrictions ($\dot{f}_2$ must be well defined)

1.3 Special DAE-cases

a) ODEs:

$$\dot{x} = Ax + f$$

- Initial values: arbitrary
- Solution uniquely determined by f and $x(0)$
- Inhomogeneity constraints
  - no structural constraints
  - no differentiability constraints

b) nilpotent DAEs:

$$\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \dot{x} = x + f$$

$$\begin{aligned} \Leftrightarrow \quad 0 &= x_1 + f_1 & \longrightarrow \quad & x_1 = -f_1 \\ \dot{x}_1 &= x_2 + f_2 & \longrightarrow \quad & x_2 = -f_2 - \dot{f}_1 \\ &\vdots & \vdots \quad & \vdots \\ \dot{x}_{n-1} &= x_n + f_n & \longrightarrow \quad & x_n = -\sum_{i=1}^{n} f_i^{(n-i)} \end{aligned}$$

In general:

$$N\dot{x} = x + f \quad \text{with } N \text{ nilpotent, i.e. } N^n = 0$$

$$\overset{N\frac{d}{dt}}{\Rightarrow} N^2\ddot{x} = N\dot{x} + N\dot{f} = x + f + N\dot{f}$$

$$\overset{N\frac{d}{dt}}{\Rightarrow} \cdots \overset{N\frac{d}{dt}}{\Rightarrow} 0 = N^n x^{(n)} = x + \sum_{i=0}^{n-1} N^i f^{(i)}$$

$$\Rightarrow \; x = -\sum_{i=0}^{n-1} N^i f^{(i)}$$

is unique solution of $N\dot{x} = x + f$

- Initial values: *fixed* by inhomogeneity
- Solution uniquely determined by f
- Inhomogeneity constraints:
  - no structural constraints
  - differentiability constraints: $(N^i f)^{(i)}$ needs to be well defined

c) underdetermined DAEs

$$n-1 \overset{n}{\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}} \dot{x} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} x + f$$

$$\Leftrightarrow \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \end{pmatrix} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} + f$$

$\Leftrightarrow$ ODE with additional “input” x_n

- Initial values: arbitrary
- Solution *not uniquely* determined by $x(0)$ and f
- Inhomogeneity constraints: none

d) overdetermined DAEs

$$\underbrace{\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix}}_{n+1 \times n} \dot{x} = \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} x + f$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}}_{N} \dot{x} = x + \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \ \wedge \ \dot{x}_n = f_{n+1}$$

$$\Leftrightarrow x = -\sum_{i=0}^{n-1} N^i f^{(i)} \ \wedge \ \underbrace{\dot{x}_n = -\sum_{i=1}^{n} f_i^{n-i+1} \overset{!}{=} f_{n+1}}_{\Leftrightarrow \sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0}$$

- Initial valus: fixed by inhomogeneity
- Solution uniquely determined by f
- Inhomogeneity constraints
  - structural constraint: $\sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0$
  - differentiability constraint: f_i^{n+1-i} needs to be well defined

We will see: There are *no other cases*!

1.4 Solution behavior, equivalence and normal forms

Solution *behavior* of $E\dot{x} = Ax + f$

$$\mathfrak{B}_{[E,A,I]} := \left\{ (x,f) \ \middle|\ x \in \mathcal{C}^1(\mathbb{R} \to \mathbb{R}^n),\ f : \mathbb{R} \to \mathbb{R}^m,\ E\dot{x} = Ax + f \right\}$$

Fact 1: For any invertible matrix $S \in \mathbb{R}^{m\times m}$:

$$(x,f) \in \mathfrak{B}_{[E,A,I]} \ \Leftrightarrow \ (x,Sf) \in \mathfrak{B}_{[SE,SA,I]}$$

Fact 2: For coordinate transformation $x = Tz$, $T \in \mathbb{R}^{n\times n}$ invertible:

$$(x,f) \in \mathfrak{B}_{[E,A,I]} \ \Leftrightarrow \ (T^{-1}x,f) \in \mathfrak{B}_{[ET,AT,I]}$$

Together:

$$(x,f) \in \mathfrak{B}_{[E,A,I]} \ \Leftrightarrow \ (T^{-1},Sf) \in \mathfrak{B}_{[SET,SAT,I]}$$

Definition 1. (E_1,A_1), (E_2,A_2) are called equivalent
$:\Leftrightarrow$ $(E_2,A_2) = (SE_1T, SA_1T)$
short:

$$(E_1,A_1) \cong (E_2,A_2) \quad \text{or} \quad (E_1,A_1) \overset{S,T}{\cong} (E_2,A_2)$$

Theorem 1 (Quasi-Kronecker Form). *For* any $E,A \in \mathbb{R}^{\ell \times m}$, $\exists$ *invertible* $S \in \mathbb{R}^{\ell \times \ell}$ *and invertible* $T \in \mathbb{R}^{n \times n}$*:*

$$(E,A) \overset{S,T}{\cong} \left(\begin{bmatrix} \boxed{E_U} & & & \\ & \boxed{I} & & \\ & & \boxed{N} & \\ & & & \boxed{E_O} \end{bmatrix}, \begin{bmatrix} \boxed{A_U} & & & \\ & \boxed{J} & & \\ & & \boxed{I} & \\ & & & \boxed{A_O} \end{bmatrix} \right)$$

where (E_U,A_U) *consists of underdetermined blocks on the diagonal,* N *is nilpotent, and* (E_O,A_O) *consists of overdetermined diagonal bolcks*

Example:

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \cong \left(\begin{bmatrix} \overline{} & 0 & 1 \\ & 0 & 0 \\ & & | \end{bmatrix}, \begin{bmatrix} \overline{} & 1 & 0 \\ & 0 & 1 \\ & & | \end{bmatrix} \right)$$

Corollary 1. $E\dot{x} = Ax + f$ *has solution* x *for any sufficiently smooth* f *and each solution* x *is uniquely determined by* $x(0)$ *and* f

$\Leftrightarrow$

$(E,A) \cong \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$, N *nilpotent*

(E,A) *is then called regular.*