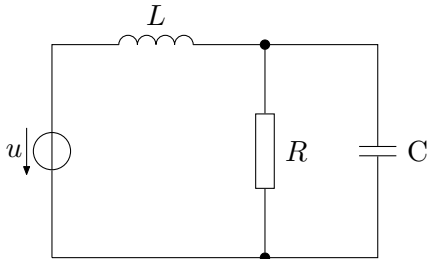


IF YOU HAVE ANY QUESTIONS CONCERNING THIS MATERIAL (IN PARTICULAR, SPECIFIC POINTERS TO LITERATURE), PLEASE DON'T HESITATE TO CONTACT ME VIA EMAIL: [trenn@mathematik.uni-kl.de](mailto:trenn@mathematik.uni-kl.de)

# 1 Solution Theory

## 1.1 Motivation: Modeling of electrical circuits



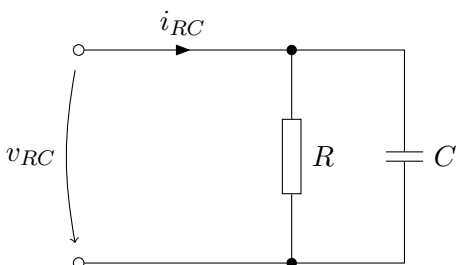
Basic components:

- Resistors:  $v_R(t) = Ri_R(t)$
- Capacitor:  $C \frac{d}{dt}v_C(t) = i_C(t)$
- Coil:  $L \frac{d}{dt}i_L(t) = v_L(t)$
- Voltage source:  $v_S(t) = u(t)$

All components have the same form:

$$\boxed{E\dot{x} = Ax + Bu} \quad E, A \in \mathbb{R}^{\ell \times n}, B \in \mathbb{R}^{\ell \times m}$$

- Resistor:  $x = \begin{pmatrix} v_R \\ i_R \end{pmatrix}, E = [0,0], A = [-1,R], B = []$
- Capacitor:  $x = \begin{pmatrix} v_C \\ i_C \end{pmatrix}, E = [C,0], A = [0,1], B = []$
- Inductor:  $x = \begin{pmatrix} v_L \\ i_L \end{pmatrix}, E = [0,L], A = [1,0], B = []$
- Voltage source  $x = \begin{pmatrix} v_C \\ i_C \end{pmatrix}, E = [0,0], A = [-1,0], B = [1]$



Connecting components: Component equations remain unchanged!

+ Kirchhoffs laws:

$$v_{RC} = v_R, \quad i_{RC} = i_R + i_C, \quad v_R + v_C = 0$$

Results again in  $E\dot{x} = Ax + Bu$  with  $x = (v_R, i_R, v_C, i_C, v_{RC}, i_{RC})$  and

$$E = \begin{bmatrix} 0 & 0 & & & & \\ & C & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & R & & & & \\ & & 0 & 1 & & \\ 1 & & & & -1 & \\ & -1 & & -1 & & \\ 1 & & 1 & & & \end{bmatrix}$$

Altogether:  $x = (v_R, i_R, v_C, i_C, v_L, i_L, v_S, i_S)$

$$E = \begin{bmatrix} 0 & 0 & & & & & & \\ & C & 0 & & & & & \\ & & & 0 & L & & & \\ & & & & & 0 & 0 & \\ & & & & & 0 & 0 & \\ & & & & & 0 & 0 & \\ & & & & & 0 & 0 & \\ & & & & & 0 & 0 & \end{bmatrix}, \quad A = \begin{bmatrix} -1 & R & & & & & & \\ & & 0 & 1 & & & & \\ & & & & 1 & 0 & & \\ 1 & & & & & & 1 & 0 \\ -1 & & & 1 & & & -1 & \\ & & & & & -1 & & 1 \\ & 1 & 1 & & 1 & & & \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

### 1.2 DAEs: What is different to ODEs

Example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\begin{array}{l} \dot{x}_2 = x_1 + f_1 \longrightarrow x_1 = -f_1 - \dot{f}_2 \\ 0 = x_2 + f_2 \longrightarrow x_2 = -f_2 \\ 0 = f_3 \end{array}$$

no restriction on  $x_3$

Observations:

- For fixed inhomogeneity, initial values cannot be chosen arbitrarily ( $x_1(0) = -f_1(0) - \dot{f}_2(0)$ ,  $x_2(0) = f_2(0)$ )
- For fixed inhomogeneity, solution not uniquely determined by initial value ( $x_3$  free)
- Inhomogeneity not arbitrary
  - structural restrictions ( $f_3 = 0$ )
  - differentiability restrictions ( $\dot{f}_2$  must be well defined)

### 1.3 Special DAE-cases

a) ODEs:

$$\dot{x} = Ax + f$$

- Initial values: arbitrary
- Solution uniquely determined by  $f$  and  $x(0)$
- Inhomogeneity constraints
  - no structural constraints
  - no differentiability constraints

b) nilpotent DAEs:

$$\begin{bmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} \dot{x} = x + f$$

$$\Leftrightarrow \begin{array}{lll} 0 = x_1 + f_1 & \longrightarrow & x_1 = -f_1 \\ \dot{x}_1 = x_2 + f_2 & \longrightarrow & x_2 = -f_2 - \dot{f}_1 \\ \vdots & \vdots & \vdots \\ \dot{x}_{n-1} = x_n + f_n & \longrightarrow & x_n = -\sum_{i=1}^n f_i^{(n-i)} \end{array}$$

In general:

$$N\dot{x} = x + f \quad \text{with } N \text{ nilpotent, i.e. } N^n = 0$$

$$\stackrel{N \frac{d}{dt}}{\Rightarrow} N^2 \ddot{x} = N\dot{x} + N\dot{f} = x + f + N\dot{f}$$

$$\stackrel{N \frac{d}{dt}}{\Rightarrow} \dots \stackrel{N \frac{d}{dt}}{\Rightarrow} 0 = N^n x^{(n)} = x + \sum_{i=0}^{n-1} N^i f^{(i)}$$

$$\Rightarrow x = -\sum_{i=0}^{n-1} N^i f^{(i)}$$

is unique solution of  $N\dot{x} = x + f$

- Initial values: *fixed* by inhomogeneity
- Solution uniquely determined by  $f$
- Inhomogeneity constraints:
  - no structural constraints
  - differentiability constraints:  $(N^i f)^{(i)}$  needs to be well defined

c) underdetermined DAEs

$${}_{n-1} \begin{bmatrix} 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ & & & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ & & & 0 & 1 \end{bmatrix} x + f$$

$$\Leftrightarrow \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \end{pmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ & & & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} + f$$

$\Leftrightarrow$  ODE with additional "input"  $x_n$

- Initial values: arbitrary
- Solution *not uniquely* determined by  $x(0)$  and  $f$
- Inhomogeneity constraints: none

d) overdetermined DAEs

$$\begin{aligned}
 n+1 \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & & & & & \\ 0 & \ddots & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & & & & & 0 \end{bmatrix} x + f \\
 \Leftrightarrow \underbrace{\begin{bmatrix} 0 & & & & & \\ 1 & \ddots & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 \end{bmatrix}}_N \dot{x} = x + \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \wedge \dot{x}_n = f_{n+1} \\
 \Leftrightarrow x = - \sum_{i=0}^{n-1} N^i f^{(i)} \wedge \dot{x}_n = - \underbrace{\sum_{i=1}^n f_i^{n-i+1}}_{\Leftrightarrow \sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0} \stackrel{!}{=} f_{n+1}
 \end{aligned}$$

- Initial value: fixed by inhomogeneity
- Solution uniquely determined by  $f$
- Inhomogeneity constraints
  - structural constraint:  $\sum_{i=1}^{n+1} f_i^{(n+1-i)} = 0$
  - differentiability constraint:  $f_i^{n+1-i}$  needs to be well defined

We will see: There are *no other cases!*

### 1.4 Solution behavior, equivalence and normal forms

Solution *behavior* of  $E\dot{x} = Ax + f$

$$\mathfrak{B}_{[E,A,I]} := \{ (x,f) \mid x \in C^1(\mathbb{R} \rightarrow \mathbb{R}^n), f : \mathbb{R} \rightarrow \mathbb{R}^m, E\dot{x} = Ax + f \}$$

Fact 1: For any invertible matrix  $S \in \mathbb{R}^{m \times m}$ :

$$(x,f) \in \mathfrak{B}_{[E,A,I]} \Leftrightarrow (x,Sf) \in \mathfrak{B}_{[SE,SA,I]}$$

Fact 2: For coordinate transformation  $x = Tz$ ,  $T \in \mathbb{R}^{n \times n}$  invertible:

$$(x,f) \in \mathfrak{B}_{[E,A,I]} \Leftrightarrow (T^{-1}x,f) \in \mathfrak{B}_{[ET,AT,I]}$$

Together:

$$(x,f) \in \mathfrak{B}_{[E,A,I]} \Leftrightarrow (T^{-1},Sf) \in \mathfrak{B}_{[SET,SAT,I]}$$

**Definition 1.**  $(E_1,A_1), (E_2,A_2)$  are called equivalent

$$:\Leftrightarrow (E_2,A_2) = (SE_1T, SA_1T)$$

short:

$$(E_1,A_1) \cong (E_2,A_2) \quad \text{or} \quad (E_1,A_1) \stackrel{S,T}{\cong} (E_2,A_2)$$

**Theorem 1** (Quasi-Kronecker Form). For any  $E, A \in \mathbb{R}^{\ell \times m}$ ,  $\exists$  invertible  $S \in \mathbb{R}^{\ell \times \ell}$  and invertible  $T \in \mathbb{R}^{n \times n}$ :

$$(E, A) \stackrel{S, T}{\cong} \left( \begin{bmatrix} \boxed{E_U} & & & \\ & \boxed{I} & & \\ & & \boxed{N} & \\ & & & \boxed{E_O} \end{bmatrix}, \begin{bmatrix} \boxed{A_U} & & & \\ & \boxed{J} & & \\ & & \boxed{I} & \\ & & & \boxed{A_O} \end{bmatrix} \right)$$

where  $(E_U, A_U)$  consists of underdetermined blocks on the diagonal,  $N$  is nilpotent, and  $(E_O, A_O)$  consists of overdetermined diagonal blocks

Example:

$$\left( \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \cong \left( \begin{bmatrix} & 0 & 1 \\ & 0 & 0 \\ & & | \end{bmatrix}, \begin{bmatrix} & 1 & 0 \\ & 0 & 1 \\ & & | \end{bmatrix} \right)$$

**Corollary 1.**  $E\dot{x} = Ax + f$  has solution  $x$  for any sufficiently smooth  $f$  and each solution  $x$  is uniquely determined by  $x(0)$  and  $f$

$\Leftrightarrow$

$$(E, A) \cong \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), N \text{ nilpotent}$$

$(E, A)$  is then called regular.