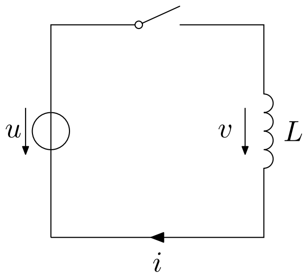
If you have any questions concerning this material (in particular, specific pointers to literature), please don't hesitate to contact me via email: `trenn@mathematik.uni-kl.de`

3 Inconsistent initial values and distributional solutions

3.1 Motivating example



DAE-model: $x = \begin{pmatrix} i \\ v \end{pmatrix}$

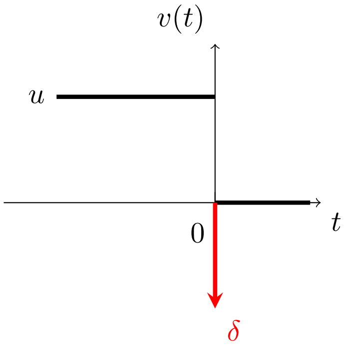
$$\begin{aligned} \text{open switch:} \quad & 0 = i, \\ \text{inductivity law:} \quad & L\tfrac{d}{dt} i = v \end{aligned} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x \quad \text{nilpotent DAE}$$

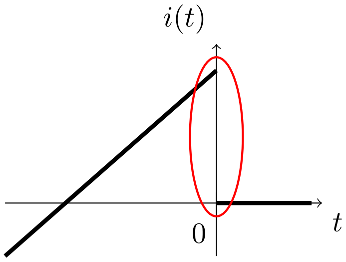
$\Rightarrow$ unique solution $x(t) = 0 \ \forall t \in \mathbb{R}$

Now assume switch was opened at $t = 0$, i.e. DAE-model is only valid on $[0,\infty)$.
Different DAE-model for $t < 0$:

$$\begin{aligned} \text{closed switch:} \quad & 0 = i, \\ \text{inductivity law:} \quad & L\tfrac{d}{dt} i = v \end{aligned} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Solution (assume constant input u):





Observations:

- $x(0-) = \begin{bmatrix} i(0-) \\ v(0-) \end{bmatrix} \neq 0$ inconsistent for $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x$
- unique jump from $x(0-)$ to $x(0+)$ (consistent)
- derivative of jump = *Dirac impulse* appears in solution

3.2 Consistency projector

Definition 1 (Initial trajectory problem (ITP))**.** Given pase trajectory $x^0 : (-\infty,0) \to \mathbb{R}^n$ find $x : \mathbb{R} \to \mathbb{R}^n$ such that

$$\left.\begin{aligned} x|_{(-\infty,0)} &= x^0 \\ (E\dot{x})|_{[0,\infty)} &= (Ax + f)|_{[0,\infty)} \end{aligned}\right\} \tag{ITP}$$

"Theorem": Consider (ITP) with regular (E,A) and $f = 0$. Choose S,T invertible such that

$$(SET,SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right).$$

Then any solution x of (ITP) satisfies

$$x(0+) = \Pi_{(E,A)} x(0-)$$

where

$$\boxed{\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}}$$

is the *consistency projector.*

Proof: Let $\begin{pmatrix} v \\ w \end{pmatrix} = T^{-1}x$ and $\begin{pmatrix} v^0 \\ w^0 \end{pmatrix} = T^{-1}x^0$, then x solves (ITP) with $f = 0 \quad \Leftrightarrow \quad \begin{pmatrix} v \\ w \end{pmatrix}$ solves

$$\left.\begin{aligned} v\big|_{(-\infty,0)} &= v^0 \\ \dot{v}_{[0,\infty)} &= (Jv)_{[0,\infty)} \end{aligned}\right\} \qquad (*)$$

and

$$\left.\begin{aligned} w\big|_{(-\infty,0)} &= w^0 \\ (N\dot{w})_{[0,\infty)} &= w_{[0,\infty)} \end{aligned}\right\} \qquad (**)$$

Since $(*)$ is an ODE on $[0,\infty)$ we have

$$v(t) = e^{Jt}v(0-) \quad \forall t \geq 0$$

In particular, $\boxed{v(0+) = v(0-)}$ From Lecture 1 we know that $(**)$ considered on $(0,\infty)$ implies

$$w(t) = 0 \quad \forall t > 0$$

In particular, $\boxed{w(0+) = 0}$ (independently of $w(0-)$)

Altogether we have

$$\begin{pmatrix} v(0+) \\ w(0+) \end{pmatrix} = \begin{pmatrix} v(0+) \\ 0 \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0-) \\ w(0-) \end{pmatrix}$$

hence

$$x(0+) = T \begin{pmatrix} v(0+) \\ w(0+) \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0-) \\ w(0-) \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}x(0-) = \Pi_{(E,A)}x(0-)$$

Remarks 1.

a) $\Pi_{(E,A)}$ does not depend on the specific choice of S and T: Assume

$$(S_1ET_1, S_1AT_1) = \left(\begin{bmatrix} I & \\ & N_1 \end{bmatrix}, \begin{bmatrix} J_1 & \\ & I \end{bmatrix}\right) \quad \text{and} \quad (S_2ET_2, S_2AT_2) = \left(\begin{bmatrix} I & \\ & N_2 \end{bmatrix}, \begin{bmatrix} J_2 & \\ & I \end{bmatrix}\right)$$

From the exercise we know $T_1 = T_2 \begin{bmatrix} P & \\ & Q \end{bmatrix}$ for some invertible P, Q, hence

$$T_1 \begin{bmatrix} I & \\ & 0 \end{bmatrix} T_1^{-1} = T_2 \begin{bmatrix} P & \\ & Q \end{bmatrix} \begin{bmatrix} I & \\ & 0 \end{bmatrix} \begin{bmatrix} P^{-1} & \\ & Q_1^{-1}T_2^{-1} \end{bmatrix} = T_2 \begin{bmatrix} I & \\ & 0 \end{bmatrix} T_2^{-1}$$

b) At this point we haven't actually shown that (ITP) has a solution!

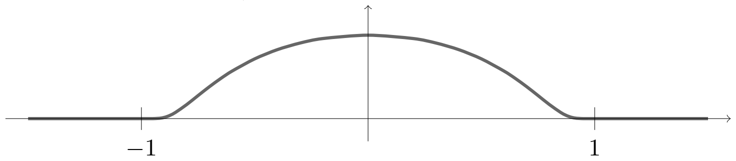
3.3 Basics of distribution theory

Definition 2 (Test functions).

$$\mathcal{C}_0^\infty := \{\ \varphi : \mathbb{R} \to \mathbb{R}\ \mid \varphi \in \mathcal{C}^\infty\ \wedge\ \operatorname{supp}\varphi \text{ is bounded }\}$$

where $\operatorname{supp}\varphi := \overline{\{\ t \in \mathbb{R}\ \mid \varphi(t) \neq 0\ \}}$

Example: $\varphi(t) = \begin{cases} e^{\frac{1}{t^2-1}}, & t \in (-1,1) \\ 0, & t \in \mathbb{R} \setminus (-1,1) \end{cases}$



$\operatorname{supp}\varphi = [-1,1]$

Lemma 1 (Topology on $\mathcal{C}_0^\infty$). *There is a topology (family of "open" sets) on $\mathcal{C}_0^\infty$ such that for any sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty$ it holds*

$$\varphi_n \overset{\text{in } \mathcal{C}_0^\infty}{\longrightarrow} 0 \quad \text{as } n \to \infty$$

$\Leftrightarrow$

1) $\exists$ *compact* $K \subseteq \mathbb{R}$*:* $\operatorname{supp}\varphi_n \subseteq K\ \forall n$ *and*

2) $\underbrace{\varphi_n^{(i)} \overset{\text{uniformly}}{\longrightarrow} 0}_{\text{i.e. } \underbrace{\|\varphi_n^{(i)}\|_\infty}_{:=\sup_{x\in\mathbb{R}} |\varphi_n^{(i)}(x)|} \overset{\text{in } \mathbb{R}}{\longrightarrow} 0} \quad \text{as } n \to \infty \quad \forall i \in \mathbb{N}$

Definition 3 (Distributions).

$$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R}\ \mid D \text{ linear and continuous }\}$$

Corollary 1 (Continuity test). *Let $D : \mathcal{C}_0^\infty \to \mathbb{R}$ be linear. Then $D \in \mathbb{D} \quad \Leftrightarrow$ $\forall (\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty$ with*

1) $\exists K$ *compact:* $\operatorname{supp}\varphi_n \subseteq K$

2) $\varphi_n^{(i)} \overset{\text{uniformly}}{\longrightarrow} 0\ \forall i \in \mathbb{N}$

it holds

$$D(\varphi_n) \overset{\text{in } \mathbb{R}}{\longrightarrow} 0 \quad \text{as} \quad n \to \infty$$

Lemma 2 (Generalized functions). $\mathcal{L}^1_{\text{loc}} := \left\{\ f : \mathbb{R} \to \mathbb{R}\ \middle|\ \forall \text{ compact } K :\ \int_K |f| < \infty\ \right\}$

$$\forall f \in \mathcal{L}^1_{\text{loc}}\ \forall \varphi \in \mathcal{C}_0^\infty : \quad f_{\mathbb{D}}(\varphi) := \int_{\mathbb{R}} f \cdot \varphi$$

Then

$$f_{\mathbb{D}} \in \mathbb{D}$$

Furthermore, for any $f_1, f_2 \in \mathcal{L}^1_{\text{loc}}$:

$$f_{1\mathbb{D}} = f_{2\mathbb{D}} \quad \Leftrightarrow \quad f_1 = f_2 \text{ almost everywhere}$$

Definition 4 (Dirac impulse).

$$\delta(\varphi) := \varphi(0) \quad \forall \varphi \in \mathcal{C}_0^\infty$$

Lemma 3. *For $D \in \mathbb{D}$ let*

$$\boxed{D'(\varphi) := -D(\varphi')}$$

then

1) $D' \in \mathbb{D}$

2) *If* $f \in \mathcal{C}^1 : \quad (f_{\mathbb{D}})' = (f')_{\mathbb{D}}$

3) *For* $\mathbb{1}_{[0,\infty)}(t) := \begin{cases} 1, \ t \in [0,\infty) \\ 0t \in (-\infty,0) \end{cases}$ *we have*

$$\delta = \left(\mathbb{1}_{[0,\infty_{\mathbb{D}}}\right)'$$

3.4 Distributional solutions

$$E\dot{X} = AX + f_{\mathbb{D}} \quad X \in \mathbb{D}^n, \ f \in \mathcal{L}^1_{\text{loc}}$$

Lemma 4. *Let (E,A) be regular, then*

1) $\forall f \in \mathcal{L}^1_{\text{loc}} \ \exists X \in \mathbb{D}^n$ *such that* $E\dot{X} = AX + f_{\mathbb{D}}$

2) *If* $f \in \mathcal{C}^\infty$ *then* $\forall X \in \mathbb{D}^n$ *with* $E\dot{X} = AX + f_{\mathbb{D}}$ $\exists x \in (\mathcal{C}^\infty)^n$*:*

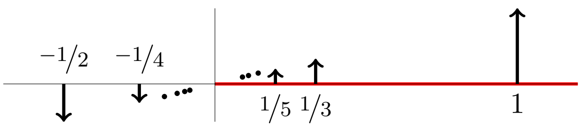
$$\boxed{X = x_{\mathbb{D}}}$$

ITP? Need restrictions to intervals!

Theorem 1. *It is* not possible *to define restriction for general distributions.*

Proof: For $d_n := \frac{(-1)^n}{n+1}$ let

$$D = \sum_{i=0}^{\infty} d_n \delta_{d_n} \overset{!}{\in} \mathbb{D}$$

where $\delta_t(\varphi) = \varphi(t)$. 

Then $D_{[0,\infty)} = \sum_{k=0}^{\infty} d_{2k}\delta_{d_{2k}}$. For any $\varphi \in \mathcal{C}_0^\infty$ with $\varphi(t) = 1$ on $[0,1]$ we have

$$D_{[0,\infty)}(\varphi) = \sum_{k=0}^{\infty} d_{2k} \underbrace{\delta_{d_{2k}}(\varphi)}_{=1} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \to \infty \quad ↯$$

Way out: Consider smaller space of *piecewiese smooth distributions*

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t \in T} D_t \ \middle| \ \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}}, T \subseteq \mathbb{R} \text{ discrete,} \\ \forall t \in T: \ D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \ldots\} \end{array} \right\}$$

where $f \in \mathcal{C}^\infty_{\text{pw}} \ :\Leftrightarrow \ \exists \alpha_i \in \mathcal{C}^\infty, i \in \mathbb{Z}, \exists\{\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2 \ldots\}$ ordered: $f = \sum_{i \in \mathbb{Z}} (\alpha_i)_{[t_i, t_{i+1})}$

Lemma 5.

- $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \quad \Rightarrow \quad D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- $\forall f \in \mathcal{C}^\infty_{\text{pw}}: \quad f_{\mathbb{D}} \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

- $\forall$ *intervals* $M \subseteq \mathbb{R}$ $\forall D = f_{\mathbb{D}} + \sum_{t\in T} D_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$:

$$D_M := (f_M)_{\mathbb{D}} + \sum_{t\in T\cap M} D_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$$

  well defined restriction.

Theorem 2. *Let* (E,A) *be regular,* $\forall x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ $\forall f \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$ $\exists! x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$:

$$\begin{aligned}(x^0)_{(-\infty,0)} &= x_{(-\infty,0)}\\ (E\dot{x})_{[0,\infty)} &= (Ax+f)_{[0,\infty)}\end{aligned}$$