

IF YOU HAVE ANY QUESTIONS CONCERNING THIS MATERIAL (IN PARTICULAR, SPECIFIC POINTERS TO LITERATURE), PLEASE DON'T HESITATE TO CONTACT ME VIA EMAIL: trenn@mathematik.uni-kl.de

4 Equivalence and Quasi-canonical forms

Fact 1: For any invertible matrix $S \in \mathbb{R}^{m \times m}$:

$$(x, u) \text{ solves } E\dot{x} = Ax + Bu \Leftrightarrow (x, u) \text{ solves } SE\dot{x} = SAx + SBu$$

Fact 2: For coordinate transformation $x = Tz$, $T \in \mathbb{R}^{n \times n}$ invertible:

$$(x, u) \text{ solves } E\dot{x} = Ax + Bu \Leftrightarrow (z, u) := (T^{-1}x, u) \text{ solves } ET\dot{z} = ATz + Bu$$

Together:

$$(x, u) \text{ solves } E\dot{x} = Ax + Bu \Leftrightarrow (z, u) := (T^{-1}x, u) \text{ solves } SET\dot{z} = SATz + SBu$$

Definition. $(E_1, A_1), (E_2, A_2)$ are called *equivalent* $:\Leftrightarrow (E_2, A_2) = (SE_1T, SA_1T)$
short:

$$(E_1, A_1) \cong (E_2, A_2)$$

Theorem (Quasi-Kronecker Form). For any $E, A \in \mathbb{R}^{\ell \times m}$, \exists invertible $S \in \mathbb{R}^{\ell \times \ell}$ and invertible $T \in \mathbb{R}^{n \times n}$:

$$(E, A) \stackrel{S, T}{\cong} \left(\begin{bmatrix} \boxed{E_U} & & & \\ & \boxed{I} & & \\ & & \boxed{N} & \\ & & & \boxed{E_O} \end{bmatrix}, \begin{bmatrix} \boxed{A_U} & & & \\ & \boxed{J} & & \\ & & \boxed{I} & \\ & & & \boxed{A_O} \end{bmatrix} \right)$$

where (E_U, A_U) consists of underdetermined blocks on the diagonal, N is nilpotent, and (E_O, A_O) consists of overdetermined diagonal blocks

Remark: 0×1 underdetermined blocks and 1×0 overdetermined blocks are possible

Example:

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \cong \left(\begin{bmatrix} \boxed{0} & \boxed{1} \\ \boxed{0} & \boxed{0} \end{bmatrix}, \begin{bmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{bmatrix} \right)$$

$$(E, A) \text{ from circuit} \cong \left(\begin{bmatrix} \boxed{I_{2 \times 2}} & & \\ & \boxed{0_{6 \times 6}} & \\ & & \end{bmatrix}, \begin{bmatrix} \boxed{0 \ 1/C} \\ \boxed{-1/L \ -1/RC} & & \\ & & \boxed{I_{6 \times 6}} \end{bmatrix} \right)$$

Corollary. $E\dot{x} = Ax + f$ has solution x for any sufficiently smooth f and each solution x is uniquely determined by $x(0)$ and f

\Leftrightarrow

$$(E, A) \cong \left(\begin{bmatrix} \boxed{I} & \boxed{0} \\ \boxed{0} & \boxed{N} \end{bmatrix}, \begin{bmatrix} \boxed{J} & \boxed{0} \\ \boxed{0} & \boxed{I} \end{bmatrix} \right), N \text{ nilpotent} \quad \boxed{\text{Quasi-Weierstrass-Form (QWF)}}$$

(E, A) is then called **regular** (Note: (E, A) regular $\Leftrightarrow \det(sE - A)$ is not the zero polynomial)

5 Wong sequences

Definition. Let $E, A \in \mathbb{R}^{m \times n}$. The corresponding Wong sequences of the pair (E, A) are:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), & i &= 0, 1, 2, 3, \dots \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{j+1} &:= E^{-1}A(\mathcal{W}_j), & j &= 0, 1, 2, 3, \dots \end{aligned}$$

Note: $M^{-1}\mathcal{S} := \{ x \mid Mx \in \mathcal{S} \}$ and $M\mathcal{S} := \{ Mx \mid x \in \mathcal{S} \}$

Clearly, $\exists i^*, j^* \in \mathbb{N}$

$$\begin{aligned} \mathcal{V}_0 \supset \mathcal{V}_1 \supset \dots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \dots \\ \mathcal{W}_0 \subset \mathcal{W}_1 \subset \dots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \dots \end{aligned}$$

Wong limits:

$$\boxed{\mathcal{V}^* := \bigcap_{i \in \mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}} \qquad \boxed{\mathcal{W}^* = \bigcup_{i \in \mathbb{N}} \mathcal{W}_i = \mathcal{W}_{j^*}}$$

Theorem. The following statements are equivalent for square $E, A \in \mathbb{R}^{n \times n}$:

- (i) (E, A) is regular
- (ii) $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$
- (iii) $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$

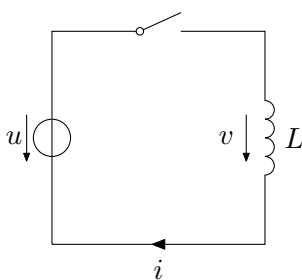
In particular, with $\text{im}V = \mathcal{V}^*$, $\text{im}W = \mathcal{W}^*$

$$(E, A) \text{ regular} \Rightarrow T := [V, W] \text{ and } S := [EV, AW]^{-1} \text{ invertible}$$

and S, T yield QWF:

$$(SET, SAT) = \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

6 Inconsistent initial values: Motivating example



DAE-model: $x = \begin{pmatrix} i \\ v \end{pmatrix}$

open switch: $0 = i,$ $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x$ nilpotent DAE
 inductivity law: $L \frac{d}{dt} i = v$

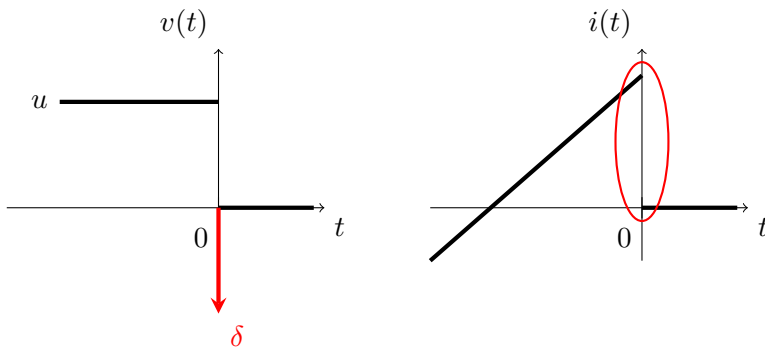
\Rightarrow unique solution $x(t) = 0 \forall t \in \mathbb{R}$

Now assume switch was opened at $t = 0$, i.e. DAE-model is only valid on $[0, \infty)$.

Different DAE-model for $t < 0$:

closed switch: $0 = v - u,$ $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u$
 inductivity law: $L \frac{d}{dt} i = v$

Solution (assume constant input u):



Observations:

- $x(0-) = \begin{bmatrix} i(0-) \\ v(0-) \end{bmatrix} \neq 0$ inconsistent for $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x$
- unique jump from $x(0-)$ to $x(0+)$ (consistent)
- derivative of jump = *Dirac impulse* appears in solution