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7 Consistency projector

Definition (Initial trajectory problem (ITP)). Given past trajectory $x^0 : (-\infty, 0) \rightarrow \mathbb{R}^n$ find $x : \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$\left. \begin{aligned} x|_{(-\infty, 0)} &= x^0 \\ (E\dot{x})|_{[0, \infty)} &= (Ax + f)|_{[0, \infty)} \end{aligned} \right\} \quad (\text{ITP})$$

“Theorem”: Consider (ITP) with regular (E, A) and $f = 0$. Choose S, T invertible such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right).$$

Then any solution x of (ITP) satisfies

$$x(0+) = \Pi_{(E,A)} x(0-)$$

where

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

is the *consistency projector*.

Proof: Let $\begin{pmatrix} v \\ w \end{pmatrix} = T^{-1}x$ and $\begin{pmatrix} v^0 \\ w^0 \end{pmatrix} = T^{-1}x^0$, then x solves (ITP) with $f = 0 \Leftrightarrow \begin{pmatrix} v \\ w \end{pmatrix}$ solves

$$\left. \begin{aligned} v|_{(-\infty, 0)} &= v^0 \\ \dot{v}|_{[0, \infty)} &= (Jv)|_{[0, \infty)} \end{aligned} \right\} \quad (*)$$

and

$$\left. \begin{aligned} w|_{(-\infty, 0)} &= w^0 \\ (N\dot{w})|_{[0, \infty)} &= w|_{[0, \infty)} \end{aligned} \right\} \quad (**)$$

Since (*) is an ODE on $[0, \infty)$ we have

$$v(t) = e^{Jt} v(0-) \quad \forall t \geq 0$$

In particular, $v(0+) = v(0-)$ From Lecture 1 we know that (**) considered on $(0, \infty)$ implies

$$w(t) = 0 \quad \forall t > 0$$

In particular, $w(0+) = 0$ (independently of $w(0-)$)

Altogether we have

$$\begin{pmatrix} v(0+) \\ w(0+) \end{pmatrix} = \begin{pmatrix} v(0+) \\ 0 \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0-) \\ w(0-) \end{pmatrix}$$

hence

$$x(0+) = T \begin{pmatrix} v(0+) \\ w(0+) \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0-) \\ w(0-) \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1} x(0-) = \Pi_{(E,A)} x(0-)$$

Remarks.

- a) $\Pi_{(E,A)}$ does not depend on the specific choice of S and T .

b) At this point we haven't actually shown that (ITP) has a solution!

Theorem. *Let (E,A) be regular. In the correct distributional solution space the ITP has a unique solution for all f .*

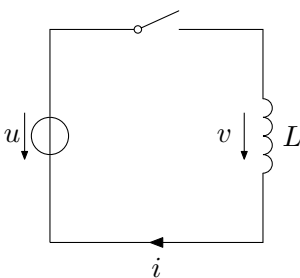
In particular, jumps and Dirac impulses at initial time are uniquely determined.

Attention: Choosing the right solution space is crucial and not immediately clear!

Here: Solution space = Space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

8 Switched DAEs: Definition

Recall example from Lecture 2:



Switch \rightarrow Different DAE models (=modes) depending on time-varying position of switch

Switching signal $\sigma : \mathbb{R} \rightarrow \{1, \dots, N\}$ picks mode number $\sigma(t)$ at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma(t)} \dot{x}(t) &= A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\ y(t) &= C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t) \end{aligned} \tag{swDAE}$$

Each mode might have different consistency spaces

\Rightarrow inconsistent initial values at each switch

\Rightarrow distributional solutions, i.e. $x \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$

Corollary. *Let*

$$\Sigma_0 := \left\{ \sigma : \mathbb{R} \rightarrow \{1, \dots, N\} \mid \sigma \text{ is piecewise constant and } \sigma|_{(-\infty, 0)} \text{ is constant} \right\}.$$

Consider (swDAE) with regular $(E_p, A_p) \forall p \in \{1, \dots, N\}$. Then

$$\forall u \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n \quad \forall \sigma \in \Sigma_0 \quad \exists \text{ solution } x \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$$

and $x(0-)$ uniquely determines x .

9 Impulse-freeness of switched DAEs

Question: When are all solutions of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ impulse free? (jumps are OK)

Lemma (Sufficient conditions).

- (E_p, A_p) all have index one (i.e. $N_p = 0$ in QWF)
 \Rightarrow (swDAE) impulse free

- all consistency spaces of (E_p, A_p) coincide (i.e. Wong limits \mathcal{V}_p^* are identical) \Rightarrow (swDAE) impulse free

Proof:

- Index-1-case: Consider nilpotent DAE-ITP:

$$\begin{aligned} (N\dot{w})_{[0,\infty)} &= w_{[0,\infty)} \\ \Rightarrow 0 &= w_{[0,\infty)} \\ \Rightarrow w[0] &:= w_{[0,0]} = 0 \end{aligned}$$

Hence an inconsistent initial value does not induce Dirac-impulse

- Same consistency space for all modes \Rightarrow no inconsistent initial values at switch \Rightarrow no jumps and no Dirac-impulse

Theorem. The switched DAE $E_\sigma \dot{x} = A_\sigma x$ is impulse free $\forall \sigma \in \Sigma_0$

$$\Leftrightarrow E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \dots, N\}$$

where $\Pi_p := \Pi_{(E_p, A_p)}$, $p \in \{1, \dots, N\}$ is the consistency projector.

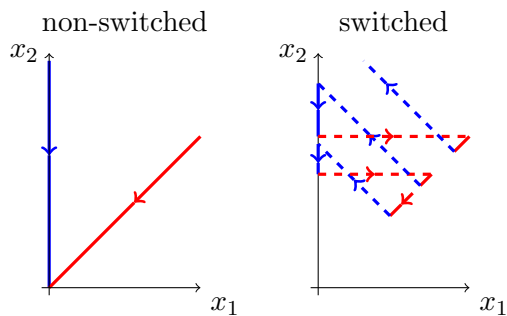
Remarks.

- a) Index 1 $\Leftrightarrow E_p(I - \Pi_p) = 0 \quad \forall p$
- b) Consistency spaces equal $\Leftrightarrow (I - \Pi_q)\Pi_p = 0 \quad \forall p, q$

10 Stability of switched DAEs

Examples.

a) $E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$



\rightarrow jumps destabilize

b) (E_1, A_1) as above, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

non-switched behavior exactly the same as above, but switched behavior now stable:

